SILOV TYPE C ALGEBRAS OVER A CONNECTED LOCALLY COMPACT ABELIAN GROUP

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A certain class of commutative Banach algebras of functions on a compact abelian group has been studied by G. E. Silov [6]. His algebras, which he calls homogeneous rings, are partially characterized by the property of containing arbitrary translates of elements. The most interesting examples are various algebras of complex functions on the circle or torus of any dimension with various differentiability properties and algebras of continuous functions on a compact abelian group which have absolutely convergent Fourier series. Silov's results have been extended by Mirkil [5] to algebras over non-abelian compact groups. We present here some results which generalize parts of the theory to translation closed algebras over connected locally compact abelian groups. The major problem in an extension in this direction centers about a replacement for the type of classical Fourier analysis for continuous functions on compact groups which has no satisfactory analog even in the abelian non-compact case. Our approach to this problem is to recapture *locally* some of the compact case when it becomes necessary. This approach makes it necessary to add to Silov's conditions various additional assumptions. Nevertheless, a considerable portion of the theory survives; enough, in fact, to include analogs of all the interesting examples from the compact case. In § 1 we present the basic construction on which the structure theorems of § 2 are based. In § 3 various examples are discussed. It will be assumed that the reader is familiar with the general theory of commutative regular Banach algebras. An account assuming an identity can be found in [6]. The results extend easily to algebras without identity. Such extensions can be found in [2], [3], [4], or, for certain non-commutative algebras, in [8].

1. In this section we describe a method of constructing a Banach algebra from the following ingredients:

(i) a connected locally compact abelian group G,

(ii) a primary commutative Banach algebra K with identity, maximal ideal Q, and norm $|\cdot|$, and

(iii) a homomorphism ω of the character group $\hat{G}$ of G into the coset of the identity in K modulo Q.

By well-known structure theorems [7, section 29] $G = E_p \times G_c$ where E_p is the p-dimensional vector group and G_c is compact abelian. From

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this it follows easily that G is σ-compact, i.e., G contains a sequence $\{C_n\}$ of compact neighborhoods of the identity 0 such that

(1) C_n is contained in C_{n+1} for all n and

(2) $G = \bigcup_{n=1}^{\infty} C_n$.

Such a sequence $\{C_n\}$ will be called a σ-covering of G. If f is a complex function defined on G and $\{C_n\}$ is a fixed σ-covering we denote by $[f]^{(n)}$ the function defined by

$$[f]^{(n)}(t) = f(t) , \qquad t \in C_n$$
$$[f]^{(n)}(t) = 0 , \qquad t \notin C_n .$$

Now suppose that for each $n = 1, 2, \cdots$ we have a linear combination of characters $\sum_{i=1}^{k_n} c_{in}\chi_{in}$, c_{in} complex, $\chi_{in} \in \hat{G}$. Form the sequence $\{f^{(n)}\}$ with $f^{(n)} = [\sum_{i=1}^{k_n} c_{in}\chi_{in}]^{(n)}$. Such a sequence will be called *ω-Cauchy* if it is Cauchy in the metric

$$N(f^{(n)} - f^{(m)}) = \sup_{t \in G} | \sum c_{in}[\chi_{in}]^{(n)}(t)\omega(\chi_{in}) - \sum c_{jm}[\chi_{jm}]^{(m)}(t)\omega(\chi_{jm}) | .$$

$N(f^{(n)})$ is defined in the obvious way, and it is clear that

$$| N(f^{(n)}) - N(f^{(m)}) | \leqq N(f^{(n)} - f^{(m)}) .$$

Thus the complex sequence $\{N(f^{(n)}\}$ is Cauchy if $\{f^{(n)}\}$ is ω-Cauchy. We define $\| \{f^{(n)}\} \|$ to be $\lim N(f^{(n)})$, $n \to \infty$. If $\{f^{(n)}\}$ and $\{g^{(n)}\}$ are ω-Cauchy then $\{(f - g)^{(n)}\}$ is also ω-Cauchy. $\{f^{(n)}\}$ and $\{g^{(n)}\}$ will be called equivalent if $\| \{(f - g)^{(n)}\} \| = 0$. *The resulting set of equivalence classes of ω-Cauchy sequences $\{f^{(n)}\}$ will be denoted by $K_\omega(G)$.* In $K_\omega(G)$ we introduce the obvious operations $\alpha\{f^{(n)}\}$, $\{f^{(n)}\} + \{g^{(n)}\}$ and $\{f^{(n)}\} \cdot \{g^{(n)}\}$. With the above norm $K_\omega(G)$ is clearly a normed complex algebra.

THEOREM 1.1. *$K_\omega(G)$ is a Banach algebra independent of the choice of the σ-covering $\{C_n\}$.*

We omit the details of the proof of this theorem. The second statement follows readily from remark (A) below, and a more or less standard diagonalization process shows that $K_\omega(G)$ is complete.

Two remarks on the structure of $K_\omega(G)$ are immediate.

(A) $K_\omega(G)$ is isomorphic and isometric to an algebra of continuous K-valued functions defined on G and vanishing at ∞, the norm being the usual sup norm. This can be seen as follows. Each element $\{f^{(n)}\}$ of $K_\omega(G)$ is a Cauchy sequence in the Banach algebra of all bounded K-valued functions on G with the sup norm. Assign to $\{f^{(n)}\}$ its limit $\tilde{f}$ in this algebra. $\tilde{f}(t)$ is necessarily continuous since any $t_0 \in G$ has a neighborhood within which $f^{(n)}(t)$ is continuous for all sufficiently large

n. $\tilde{f}(t) \to 0$ as $t \to \infty$ since each $f^{(n)}(t)$ has compact support. The mapping $\{f^{(n)}\} \to \tilde{f}$ is clearly a homomorphism. Moreover,

$$\begin{aligned} \| \{f^{(n)}\} \| = \lim_n N(f^{(n)}) &= \lim_n \sup_t | \sum c_{in}[\chi_{in}]^{(n)}(t)\omega(\chi_{in}) | \\ &= \sup_t \lim_n | \sum c_{in}[\chi_{in}]^{(n)}(t)\omega(\chi_{in}) | \\ &= \sup_t | \tilde{f}(t) | \; . \end{aligned}$$

so the correspondence is an isometry.

(B) Since $\omega(\chi)(Q) = 1$ for each $\chi \in \hat{G}$ we have $| \sum c_{in}[\chi_{in}]^{(n)}(t) | \leqq | \sum c_{in}[\chi_{in}]^{(n)}(t)\omega(\chi_{in}) |$. Thus each element of $K_\omega(G)$ determines uniquely a complex function $f(t)$ such that $\sup | f(t) | \leqq \| \{f^{(n)}\} \|$. The mapping $\{f^{(n)}\} \to f$ is a continuous homomorphism of $K_\omega(G)$ onto a subalgebra of $C_0(G)$, the Banach algebra of all continuous complex functions vanishing at ∞ on G. $K_\omega(G)$ will be said to be *radical* or to *separate points of* G accordingly as the corresponding subalgebra of $C_0(G)$ is zero or separates points of G.

In the sequel we shall denote a general element of $K_\omega(G)$ by $\tilde{f}$ as suggested by (A) and the image of this element in the corresponding subalgebra of $C_0(G)$ by f.

EXAMPLES. (1) Remark (B) and the Stone-Weierstrass theorem show that if ω is the trivial homomorphism sending each χ into the identity in K then $K_\omega(G) = C_0(G)$.

(2) Let $G = E_1$ and K be the Banach algebra with two generators 1, x with $x^2 = 0$. K is the set of all polynomials $\alpha_0 + \alpha_1 x$, α_i complex, with norm defined by $| \alpha_0 + \alpha_1 x | = | \alpha_0 | + | \alpha_1 |$. K is primary with Q the subalgebra generated by x. $\hat{G} = E_1$ and a general character is $\chi(t) = e^{i\lambda t}$, $\lambda \in E_1$. Define ω by $\omega(\chi) = \omega(\lambda) = 1 + i\lambda x$. ω is clearly a continuous homomorphism. A general element $\{f^{(n)}\}$ of $K_\omega(G)$, with $f^{(n)} = [\sum c_{pn}\chi_{pn}]^{(n)}$, is a function $\tilde{f}(t) = f(t) + g(t)x$ where

$$f(t) = \lim_n \sum_p c_{pn}\chi_{pn}(t) \; ,$$

$$g(t) = \lim_n \sum_p c_{pn}\chi'_{pn}(t)$$

and both limits are uniform in a neighborhood of each $t_0 \in E_1$. Thus $g(t) = f'(t)$ and both $f(t)$ and $f'(t)$ tend to 0 at ∞. $K_\omega(G)$ is the algebra $D_1(E_1)$ of Example 1, § 3. Various properties of $K_\omega(G)$ are immediate from standard theorems on Fourier series. We point out several which play roles in subsequent theorems of this section. The homomorphism $\tilde{f} \to f$ of remark (B) is clearly an isomorphism in this case. Moreover, if f is any complex continuously differentiable function on E_1 with compact support then $\tilde{f} \in K_\omega(G)$. This is obvious if we take for a σ-covering the collection of intervals $[-n, n]$ and look at the Fourier series for such a function on an arbitrary interval $[-n, n]$ containing the support of f. To obtain a sequence $\{f^{(n)}\}$ defining $\tilde{f}$ we need only take, for each sufficiently large n, a suitable partial sum of the Fourier series for f on $[-n, n]$. Thus $K_\omega(G)$ contains elements $\tilde{f}$ such that $f(t) = 1$ on an arbitrary compact subset of G and $f(t) = 0$ on a disjoint closed set. By Theorem 1.5 below G is the space of maximal regular ideals of $K_\omega(G)$ so $K_\omega(G)$ is a regular Banach algebra. In fact, by the definition of the norm $K_\omega(G)$ contains a *bounded* sequence $\{\tilde{f}_n\}$ for which $\tilde{f}_n(t) = 1$ on $[-n, n]$ and $\tilde{f}_n(t)$ has compact support. Such a sequence is an "approximate identity" in $K_\omega(G)$, i.e., $\lim \tilde{f}\tilde{f}_n = \tilde{f}$ for any $\tilde{f} \in K_\omega(G)$. Thus the elements with compact support are dense in $K_\omega(G)$. Finally, any element $\tilde{f}$ whose support is contained in $[-n, n]$ can be approximated uniformly on $[-n, n]$ by K-valued functions of the form $\Sigma c_p \omega(\chi_p)\chi_p(t)$ where each χ_p is constant on the subgroup $\{0, \pm n, \pm 2n, \cdots\}$, or, equivalently, each χ_p is an integral multiple of $2\pi/n$ (cf. condition (A) below). This, too, follows from a glance at the Fourier series for the image f on the interval $[-n, n]$.

LEMMA 1.2. *For any $K_\omega(G)$ we have the following:*

(a) *$f(t) = \tilde{f}(t)(Q)$ for any $\tilde{f} \in K_\omega(G)$,*

(b) *$K_\omega(G)$ is closed under multiplication by $\hat{G}$ in the sense that for $\tilde{f} \in K_\omega(g)$ and $\chi \in \hat{G}$ there exists an element $\chi\tilde{f} \in K_\omega(G)$ such that $[\chi\tilde{f}](t) = \chi(t)\omega(\chi)\tilde{f}(t)$ for all $t \in G$.*

(c) *$K_\omega(G)$ is closed under translation in the sense that for $\tilde{f} \in K_\omega(G)$ and $s \in G$ there exists an element $\tilde{f}_s \in K_\omega(G)$ such that $\tilde{f}_s(t) = \tilde{f}(t - s)$ for all $t \in G$.*

Proof. For each $t \in G$,

$$\begin{aligned}\tilde{f}(t)(Q) &= [\lim \Sigma_i c_{in}[\chi_{in}]^{(n)}(t)\omega(\chi_{in})](Q) \\ &= \lim \Sigma_i c_{in}[\chi_{in}]^{(n)}(t)\{\omega(\chi_{in}(Q)\} = f(t)\ ,\end{aligned}$$

since $\omega(\chi_{in})(Q) = 1$. This proves (a). (b) is clear: if $\tilde{f} \leftrightarrow \{f^{(n)}\}$ then $\chi\tilde{f} \leftrightarrow \{[\chi f]^{(n)}\}$. (c) would be equally trivial if it were true that $[\chi]^{(n)}(t - s) = \chi(-s)[\chi]^{(n)}(t)$ for all $t \in G$. Since this is not the case a slight extra argument is necessary. Let $\tilde{f} \in K_\omega(G)$ with

$$\sup_{t \in G} | \Sigma_i c_{in}[\chi_{in}]^{(n)}(t)\omega(\chi_{in}) - \tilde{f}(t) | \to 0 \quad \text{as} \quad n \to \infty .$$

For each n pick an integer n' in such a way that $n' \to \infty$ as $n \to \infty$ and $C_{n'} \supset C_n - s$ for all n. Then for any $t \in C_n$

$$[\chi_{in'}]^{(n')}(t-s) = [\chi_{in'}]^{(n)}(t)\cdot\chi_{in'}(-s)\ .$$

We may assume that $|\tilde{f}(t)| < \varepsilon$ for $t \notin C_n$, n sufficiently large, so it follows that

$$\sup_{t\in G} |\Sigma_i c_{in'}\chi_{in'}(-s)[\chi_{in'}]^{(n)}(t)\omega(\chi_{in'}) - \tilde{f}(t-s)| < \varepsilon$$

for sufficiently large n. This means that $\tilde{f}_s \in K_\omega(G)$.

LEMMA 1.3. *$K_\omega(G)$ is either radical or separates points of G.*

This follows immediately from Lemma 1.2, parts (b) and (c) together with the fact that $\hat{G}$ separates points of G. This lemma together with remark (B) yield the following lemma. Again we omit the details of the easy proof. We denote the structure space of maximal regular ideals of $K_\omega(G)$ by $\mathfrak{M}_K$.

LEMMA 1.4. *For $t \in G$ the set $M_t = \{\tilde{f} \in K_\omega(G) \mid f(t) = 0\}$ is a maximal regular ideal of $K_\omega(G)$. Given an arbitrary $\tilde{f} \in K_\omega(G)$ the image $\tilde{f}(M_t)$ of $\tilde{f}$ modulo the maximal regular ideal M_t is $f(t)$. If $K_\omega(G)$ is not radical then the mapping $t \to M_t$ is one-to-one of G into $\mathfrak{M}_K$.*

Denote by $TK_\omega(G)$ the "Tauberian part" of $K_\omega(G)$, that is, the closed subalgebra of $K_\omega(G)$ generated by the elements $\tilde{f}(t)$ which have compact support. Lemmas 1.2, 1.3, and 1.4 hold for the algebra $TK_\omega(G)$, and we denote its structure space by $\mathfrak{M}_{TK}$. Given the conditions of Lemma 1.4 we will identify G with its image in $\mathfrak{M}_{TK}$ or $\mathfrak{M}_K$. We will be interested in algebras $K_\omega(G)$ and $TK_\omega(G)$ primarily when they are regular. Whether there actually exists a non-regular $K_\omega(G)$ is an interesting open question to which we will refer again in some remarks at the end of this section.

THEOREM 1.5. *Let ω be continuous. If $TK_\omega(G)$ is not radical then $G = \mathfrak{M}_{TK}$. If $TK_\omega(G)$ is regular then the group topology in G is the same as the $\mathfrak{M}_{TK}$-topology.*

Proof. The proof of the first statement is very similar to Silov's proof of the analogous theorem for the compact case so we omit most of the details. If $M_0 \in M_{TK}$ consider $\tilde{e} \in TK_\omega(G)$ such that $\tilde{e}(t)$ has compact support and $\tilde{e}(M_0) = 1$. Let $m(\chi) = [\chi\tilde{e}](M_0)$. One shows that $m(\chi)$ is a homomorphism of $\hat{G}$ into the complexes of modulus 1. Since ω is continuous it follows that m is continuous. Thus by the duality theorem $m(\chi) = \chi(t_0)$ for some $t_0 \in G$. This says that $\tilde{f}(M_0) = f(t_0)$ for any element which is a linear combination of elements $\chi\tilde{e}$, hence, by definition of $TK_\omega(G)$, for any element $\tilde{g}\tilde{e}$ with $\tilde{g} \in TK_\omega(G)$. The desired result follows since $\tilde{e}(M_0) = 1$. The second statement in the theorem follows from standard theorems in topology. By definition of the Gelfond topology, the $\mathfrak{M}_{TK}$-topology is weaker than the group topology on G. Both are Hausdorff and locally compact, and if $TK_\omega(G)$ is regular then an $\mathfrak{M}_{TK}$-compact set K is G-compact (since $TK_\omega(G)$ has a unit modulo the kernel of K and all elements tend to zero at ∞ on G). Thus the topologies are the same.

The last part of the above proof also yields the following.

COROLLARY 1.6. *If $K_\omega(G)$ is regular then G is closed in $\mathfrak{M}_K$ and its topology is inherited from $\mathfrak{M}_K$.*

We can now formulate a necessary and sufficient condition for any regular $TK_\omega(G)$ to be semi-simple. Recall that $G = E_n \times G_c$ so that G clearly contains a discrete subgroup D for which G/D is compact (D is essentially the group I_n, where I is the group of integers) and a compact neighborhood C of the identity such that the natural map of C into G/D is one-to-one. $TK_\omega(G)$, or, more, generally, any algebra R of continuous K-valued functions on G, will be said to satisfy *Condition* (A) if:

(1) $TK_\omega(G)$ (or R) contains elements $\tilde{f}(t)$ with $f(t)$ not identically zero such that $\tilde{f}(t)$ has support contained in C, and

(2) every $\tilde{f} \in TK_\omega(G)$ (or R) with support in C is a uniform limit on C of functions of the form $\Sigma c_i \chi_i(t)\omega(\chi_i)$ where the χ_i are elements of $\hat{G}$ which are constant on D, i.e., each χ_i is a character of G/D.

Condition (A) implies that any $\tilde{f} \in TK_\omega(G)$ supported by C determines uniquely a function $\tilde{f}(\bar{t})$ on G/D such that $\tilde{f}(\bar{t})$ is an element of $K_{\bar{\omega}}(G/D)$ where $\bar{\omega}$ is the homomorphism of the character group of G/D into K which is induced by ω. Thus $TK_\omega(G)$ is locally rather firmly tied to the compact case.

The following lemma is stated in a form in which it will be applicable both in the present discussion and later in § 2.

LEMMA 1.7. *Let R be a semi-simple regular Banach algebra of continuous functions $\tilde{f}$ from G to K vanishing at ∞ with $\|\tilde{f}\| =$ $\sup|\tilde{f}(x)|$; $x \in G$. Suppose $\mathfrak{M}(R) = G$ and that R is closed under translation and multiplication by $\hat{G}$ in the sense of Lemma 1.2. Then*

(a) *for any $\tilde{f} \in R$, $\tilde{f}(t)$ vanishes on any open set in G on which $f(t) = \tilde{f}(M_t)$ vanishes, and*

(b) *R satisfies Condition* (A).

Proof. The proof of (a) is exactly the proof of the corresponding lemma (4.7.1) in [6] so we omit the details.

Denote a general element of G by (s, t) where $s = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in E_n$, $t \in G_c$. For real $\alpha > 0$ define $S(\alpha) = \{(s, t) \mid |\alpha_i| \leqq \alpha, t \in G_c\}$. For the discrete subgroup D we can take the direct product of the usual discrete subgroup I_n of E_n and the identity subgroup of G_c. G/D is then the product of an n-torus and G_c. We may further assume that the compact neighborhood $S(\alpha)$ of 0 with the usual identifications, operations and topology is isomorphic and homeomorphic to G/D. If C is a compact subset of G containing $S(\alpha)$ then $\tilde{f} \in R$ is said to be D-periodic on C if for any $x \in C$, $d \in D$ for which $x + d \in C$ we have $\tilde{f}(x + d) = \tilde{f}(x)$. Clearly any D-periodic element on C determines uniquely both a continuous K-valued function on G/D and a similar complex valued function. R contains D-periodic functions on any compact set in G since regularity and part (a) of the theorem provide elements whose support is in $S(\alpha)$ and these can be extended to all of C by a finite number of translations by elements of D. (The possibility of multiplying a unit modulo the kernel of C by characters also yields D-periodic functions, but for reasons of later applicability we prefer not to make use of this hypothesis until later in the proof.) Suppose $\tilde{f}$ is D-periodic on $S(3\alpha)$ and that $h \in S(\alpha)$. Then the element $\tilde{f}_h$ is D-periodic on $S(2\alpha)$. Let I be the kernel of the subset $S(\alpha)$ of $\mathfrak{M} = G$ and let $\bar{R} = R/I$. Denote the image in $\bar{R}$ of a general $\tilde{f} \in R$ by $\bar{f}$. The norm of $\bar{f}$ in $\bar{R}$ is $\|\bar{f}\| = \|f\|_{S(\alpha)} = \inf \|g\|$; $g(x) = f(x)$ all $x \in S(\alpha)$. Let $\bar{R}_p$ be the closed subalgebra of $\bar{R}$ generated by all $\bar{f}_h$ with $\tilde{f}$ and h as above. Clearly $\bar{R}_p$ can be represented as an algebra of continuous complex functions on the compact abelian group G/D. Consider one of the generators $\bar{g} = \bar{f}_{h_1}$ and an element h in the interior of $S(\alpha)$. By adjusting h_1 by an element of D without changing the image $\bar{f}_{h_1}$ we can arrange to have $h_1 + h \in S(\alpha)$. Then $\tilde{g}_h = [\tilde{f}_{h_1}]_h$ is D-periodic on $S(2\alpha)$ and its image $\bar{g}_h$ is in $\bar{R}_p$. It is an easy exercise to show that if $\bar{t}$ denotes the image in G/D of $t \in G$ then $\bar{g}_h(\bar{t}) = \bar{g}(\bar{t} - \bar{h})$ so $\bar{g}_h$ is a translate of $\bar{g}$ in $\bar{R}_p$. The translation operator $T_{\bar{h}}$ is then defined on a dense subset of $\bar{R}_p$. We show that $T_{\bar{h}}$ is bounded. Let $\bar{f}$ be a general element of this dense set, i.e., $\tilde{f} = \Sigma_i[\tilde{f}_i]_{h_i}$ with $\tilde{f}_i$ D-periodic on $S(3\alpha)$, $h_i \in S(\alpha)$. Consider $\bar{f}_{\bar{h}}$, the image of $\tilde{f}_h$ as above. We must show that $\|\tilde{f}_h\|_{S(\alpha)} \leqq k\|\tilde{f}\|_{S(\alpha)}$ where k is independent of $\tilde{f}$. Let $S = S(\alpha) + h$. Clearly S is in the interior of $S(2\alpha)$. Choose a closed set T such that $S(\alpha) \cup S \subset T \subset$ interior $S(2\alpha)$. It is obvious that $\|\tilde{f}\|_{S(\alpha)} = \|\tilde{f}_h\|_S$. We show that $\|\tilde{f}_h\|_{S(\alpha)} \leqq \|\tilde{f}_h\|_T \leqq k\|\tilde{f}_h\|_S$. The first inequality is clear since $T \supset S(\alpha)$. Pick $\tilde{e} \in R$ such that $e(x) = 1$ on T, $e(x) = 0$ outside $S(2\alpha)$. Then $f_h e(x) = f_h(x)$ on T and

$$\|\tilde{f}_h\tilde{e}\| = \sup|\tilde{f}_h\tilde{e}(x)|\,(x \in S(2\alpha)) \leqq \|\tilde{e}\| \sup|\tilde{f}_h(x)|\,(x \in S)$$

by D-periodicity of $\tilde{f}_h$ on $S(2\alpha)$. By part (a) together with continuity of elements of R we see that $\|\tilde{f}_h\|_S \geqq \sup|\tilde{f}_h(x)|(x \in S)$ so we have $\|\tilde{f}_h\tilde{e}\| \leqq \|\tilde{f}_h\|_S \cdot \|\tilde{e}\|$. But $\|\tilde{f}_h\|_T \leqq \|\tilde{f}_h\tilde{e}\|$ so $\|\tilde{f}_h\|_T \leqq \|\tilde{e}\| \cdot \|\tilde{f}_h\|_S$. Hence T_h is bounded, hence extendible to $\bar{R}_p$ where it clearly defines the ordinary translate $\bar{f}_{\bar{h}}$ of an arbitrary $\bar{f} \in \bar{R}$. If h is on the boundary of $S(\alpha)$ we write $h = h_1 + h_2$, $h_i \in$ interior of $S(\alpha)$ and proceed as above. Since all $f \in R$ are *unjformly* continuous K-valued functions it follows that all elements of $\bar{R}_p$ are *continuous under translation*, that is, for any $\bar{f}$ and $\varepsilon > 0$, $\|\bar{f} - \bar{f}_{\bar{h}}\| < \varepsilon$ for all $\bar{h}$ in some neighborhood of $\bar{0}$. Thus $\bar{R}_p$ is a homogeneous space of functions in the sense of Silov satisfying the conditions of [6, 2.7]. We can therefore conclude that linear combinations of character of G/D are dense in $\bar{R}_p$.

If $\tilde{e} \in R$ is chosen so that $e(t) = 1$ on $S(3\alpha)$ and if χ_i are characters of G constant on D, then if $\tilde{g} = \Sigma c_i[\chi_i\tilde{e}]$ $\tilde{g}$ is in $\bar{R}_p$ and is the corresponding linear combination of characters in that algebra. $\tilde{g}(x) = \Sigma c_i \chi_i(x)\omega(\chi_i)$ for each $x \in S(\alpha)$ so Condition (A) follows from the fact, noted above, that $\|\bar{f}\| \geqq \sup|\tilde{f}(x)|(x \in S(\alpha))$.

THEOREM 1.8. *Let ω be continuous. If $TK_\omega(G)$ is regular then it is semi-simple if and only if it satisfies Condition* (A). *If $K_\omega(G)$ is regular then it satisfies Condition* (A) *if and only if it is semi-simple and $\mathfrak{M}_K = G$.*

Proof. Suppose $TK_\omega(G)$ is regular. Necessity of the condition is contained in Lemma 1.7 in view of the results of Theorem 1.5 and Lemma 1.2. Sufficiency follows readily from the fact that any $K_\omega(G)$ with G compact abelian is semi-simple [6, Theorem 4.6]. Suppose $\tilde{f} \in TK_\omega(G)$ and $f(t) = 0$ for all $t \in G$. Pick $\tilde{e} \in TK_\omega(G)$ with support contained in C and with $e(t_0) \neq 0$ (by Condition (A)). Then $\tilde{e}\tilde{f}$ is supported by C so $\tilde{e}\tilde{f}(\bar{t}) \in K_\omega(G/D)$ and $ef(\bar{t}) = 0$ for all $\bar{t}$. Thus $\tilde{e}\tilde{f}(\bar{t}) = 0$ for all $\bar{t}$ so that $\tilde{e}\tilde{f}(t_0) = 0$. $\tilde{e}(t_0)$ has an inverse in K since it is contained in no maximal ideal of K so we must have $\tilde{f}(t_0) = 0$. Thus, for each $s \in G$, $f_s(t_0) = 0$ which implies that $\tilde{f} = 0$. The statement for $K_\omega(G)$ follows by the same argument if we observe that we have actually proved that Condition (A) is equivalent to the vanishing of the kernel of G. For $TK_\omega(G)$ this is semi-simplicity since $G = \mathfrak{M}_{TK}$, For $K_\omega(G)$ the vanishing of this kernel is equivalent to semi-simplicity plus the condition that $G = \mathfrak{M}_K$, since we know by Corollary 1.6 that G is closed in $\mathfrak{M}_K$.

THEOREM 1.9. *Let ω be continuous. If $TK_\omega(G)$ $(K_\omega(G))$ is regular* *and semi-simple then it is an algebra of type C.*[1]

Proof. Using part (a) of Lemma 1.7 one easily proves that the set $\{\tilde{f} \mid \tilde{f}(t_0) = 0\}$ is a closed primary ideal. It is immediate, then, that the norm in $K_\omega(G)$ is smaller than the type C norm. But the opposite inequality always holds.

Before turning to some structure theorems based on the above construction we mention several questions concerning the algebras $TK_\omega(G)$ and $K_\omega(G)$. The first one concerns the connectivity assumption on G. The results in this section hold in slightly more generality. The definitions and most of the early results require only that G be σ-compact. Condition (A), Lemma 1.7, and Theorem 1.8 require only that G be generated by a compact neighborhood of the identity (so that $G = E_n \times G_c \times G_a$, G_a discrete [7, section 29]). Full use of connectivity is used only in the next section in the proof of Theorem 2.3. Whether connectivity could be dropped in favor of, say, σ-compactness is an open question. Further open questions concern some of the separation conditions we have employed. Does there exist a radical $K_\omega(G)$? Does there exist a non-regular $K_\omega(G)$? Does a $K_\omega(G)$ exist for which $TK_\omega(G) \neq K_\omega(G)$? These questions are closely related to the question of regularity of $K_\omega(G)$ in the compact case, and a complete answer to this question is not known. Silov has sufficient conditions for regularity of $K_\omega(G)$ for compact G [6, section 5.8], but no necessary conditions. In case $G = E_n$ and K is finite dimensional there is some evidence which suggests that $TK_\omega(G)$ is regular and equal to $K_\omega(G)$. This is true, for instance, for dimension $\leqq 3$, but the proof requires a classification of primary algebras of these dimensions. This approach is not promising in the general finite dimensional case, however, since a classification of all finite dimensional primary algebras is not known. (Such a classification would involve a classification of finite dimensional nilpotent algebras, a more familiar unsolved problem.) In case $G = E_1$ it is not hard to exhibit sufficient conditions for regularity of $TK_\omega(G)$ or $K_\omega(G)$ by reducing to the compact case where Silov's conditions can be applied. We state one such result without proof. If $G = E_1$ we may identity $\hat{G}$ with E_1, the circle group C with $E_1/I(p)$ where $I(p)$ is the subgroup of integral multiples of p, p a positive integer, and $\hat{C}$ with the group of integers. The homomorphism ω of E_1 into K induces, for each p, a homomorphism ω_p of C into K: $\omega_p(n) = \omega(n/p)$. *If $K_{\omega_p}(C)$ is regular for each $p = 1, 2, 3, \cdots$ then $K_\omega(E_1)$ and $TK_\omega(E_1)$ are regular.*

[1] A commutative regular B-algebra R is of type C if its norm is equivalent to the norm $|||f||| = \sup \|f\|_M$, where M ranges over the structure space of maximal regular ideals and $\|f\|_M$ is the norm of the image of f in the difference algebra $R/J(M)$ (see section 2).

2. In § 1 we have seen that under certain conditions algebras $TK_\omega(G)$ or $K_\omega(G)$ are semi-simple commutative Banach algebras of type C closed under multiplication by $\hat{G}$ and under translation. In this section we consider the converse problem.

We follow Silov in calling a Banach algebra R *homogeneous over* G if R satisfies the following conditions: R is a semi-simple regular commutative Banach algebra whose space of maximal regular ideals is a locally compact abelian group G, R is closed under translation, the norm in R is translation invariant, and the elements of R are continuous under translation in the norm of R (it is sufficient to assume that R contains a set of generators continuous under translation). Further, in case G is not compact, we assume that R is Tauberian in the sense that the elements with compact support are dense in R.

For $t_0 \in G$ let the corresponding maximal regular ideal be M_{t_0}. M_{t_0} contains a unique minimal closed primary ideal $J(t_0)$ which can be characterized as the closure of the set of all $f \in R$ such that $f(t) = 0$ in a neighborhood of t_0. (If R were not Tauberian the above f would have to be assumed in addition to have compact support.) Also, since R is Tauberian, it is easy to see that an element e with compact support for which $e(t) = 1$ for all t in a neighborhood of t_0 is a unit modulo $J(t_0)$.

Later in this section we will make use of the extensions to algebras without unit element of the theorems on regular commutative Banach algebras contained in [6, section 3]. As far as we know, some of these generalizations are not available in the literature (in particular, the results of sections 3.5-3.9 on algebras of type C). However, they are all routine, and under the Tauberian condition the facts mentioned above make Silov's proofs applicable almost without change.

If 0 is the identity element of G let $K = R/J(0)$. K is a commutative primary Banach algebra with identity and maximal ideal $Q = M_0/J(0)$. As before, denote the norm in K by $|\cdot|$.

LEMMA 2.1. *If R is a homogeneous algebra over the locally compct abelian group G then for all $s \in G$, $R/J(s)$ is isomorphic and isometric to $R/J(0) = K$, and R can be represented as an algebra of continuous K-valued functions on G vanishing at ∞.*

Proof. The isomorphism is $f + J(0) \rightarrow f_s + J(s)$. Clearly it is a homomorphism of K onto $R/J(s)$. It is an isomorphism since by definition $f \in J(0)$ implies $f_s \in J(s)$. By invariance of the norm in R under translation it is immediate that $\|f\|_0 = \|f_s\|_s$ where $\|g\|_t$ denotes the norm of the image of g in $R/J(t)$. For $f \in R$, $t \in G$ let $\tilde{f}(t)$ be the image of f under the mapping $R \to R/J(t) \to K$. The collection of functions $\tilde{f}(t)$ is the algebra isomorphic to R. Since $\|f\| \equiv \|\tilde{f}\| \geqq \sup |\tilde{f}(t)|$ $(t \in G)$, continuity of the functions $\tilde{f}$ follows from continuity of the elements of R under translation. Since R is Tauberian it is an easy exercise to show that each $\tilde{f}(t)$ vanishes at ∞.

LEMMA 2.2. *Let R be homogeneous over G and let R' be the set of all elements of R with compact support. Suppose R' is closed under multiplication by $\hat{G}$,* i.e., *for each $f \in R'$ and $\chi \in \hat{G}$ there exists an element $\chi f \in R'$ such that $\chi f(t) = \chi(t)f(t)$ for all t. Then*

(a) *R determines a homomorphism ω of $\hat{G}$ into the coset of 1 in $K = R/J(0)$ modulo $Q = M_0/J(0)$,*

(b) *for any $f \in R$ and any $\chi \in \hat{G}$ for which $g = \chi f$ exsists in R, $\tilde{g}(t) = \chi(t)\omega(\chi)\tilde{f}(t)$ for all $t \in G$, and*

(c) *if the mapping $\chi \to \chi f$ is continuous then ω is continuous.*

Proof. Pick $e \in R$ with compact support and with $e(t) = 1$ on a compact neighborhood C of 0. As we have remarked above, e is a unit modulo $J(0)$. For $\chi \in \hat{G}$ consider the element χe. If $\omega(\chi)$ denotes the image of χe in $R/J(0) = K$ the homomorphism is $\chi \to \omega(\chi)$. Clearly $\omega(\chi)(Q)=1$. Since, for $\chi_1, \chi_2 \in \hat{G}$, $[\chi_1\chi_2 e - \chi_1 e \cdot \chi_2 e](t)=0$ in a neighborhood of 0 and outside a compact set we have $\omega(\chi_1\chi_2) = \omega(\chi_1)\omega(\chi_2)$. A similar argument shows that ω is independent of the choice of C and the choice of e. Let $h = \chi e$, then $h(t) = \chi(t)e(e) = \chi(s)\chi(t-s)e(t-s)$ provided both t and $t-s$ are in C. If s is in the interior of C then let U be a neighborhood of 0 such that $U \subset C$, $U + s \subset C$ then the above holds for all $t \in U + s$. Thus $h - \chi(s)h_s \in J(s)$ so via the mapping $R \to R/J(s) \to K$ we have $h \to h + J(s) = \chi(s)h_s + J(s) \to \chi(s)h + J(0) = \chi(s)\omega(\chi)$, this is, $\tilde{h}(s) = \chi(s)\omega(\chi)$ for s in the interior of C. The equality extends to all of C by continuity. Now let $g = \chi f$ for any $f \in R$ for which the product is defined. Fix $t_0 \in G$, let C be a compact neighborhood of 0 containing t_0 in its interior, and consider the corresponding e and $h = \chi e$. It follows easily that $\tilde{g}(t_0) = \tilde{g}\tilde{e}(t_0) = \tilde{h}\tilde{f}(t_0) = \chi(t_0)\omega(\chi)\tilde{f}(t_0)$. Part (c) is obvious.

Two Banach algebras R_1 and R_2 with the same structure space $\mathfrak{M}$ will be called *locally isomorphic* in case for each $t \in \mathfrak{M}$ there exist homeomorphic neighborhoods U_1 and U_2 of t such that every element of R_1 restricted to U_1 is carried by the homeomorphism into an element of R_2 restricted to U_2, and conversely. Two algebras of K-valued functions on G will be called *locally K-isomorphic* in case the analogous condition holds for the K-valued functions.

THEOREM 2.3. *Let R be a homogeneous Banach algebra over a* *connected locally compact abelian group. If R is of type C then R is closed under multiplication by $\hat{G}$. R can be represented as a closed subalgebra of $TK_\omega(G)$ where $K = R/J(0)$ and ω is the homomorphism given in Lemma 2.2. $TK_\omega(G)$ is semi-simple and R and $TK_\omega(G)$ are locally K-isomorphic. If ω is continuous then R and $TK_\omega(G)$ are locally isomorphic.*

Proof. Several remarks on Lemma 1.7 and its proof will produce a large part of the proof of the present theorem. In the first place, we know by Lemma 2.1 that R satisfies all the conditions of Lemma 1.7 except closure under multiplication by $\hat{G}$. This hypothesis is expendable, however. It was needed in the lemma only because we lacked the machinery for an intrinsic construction of the homomorphism ω. The proof of 1.7 shows (without using the hypothesis in question) the existence in $\bar{R}_p$ of a generating set X of characters which distinguish between points of G/D. Since the set $S(\alpha)$ is the structure space of $\bar{R}$ and $\chi(t) \neq 0$ for all t, it follows from standard Banach algebra theorems that with each $\chi \in X$ $\bar{R}$ contains its complex conjugate χ^{-1}. But the only subgroup of $(G/D)^\frown$ which separates points of G/D is $(G/D)^\frown$ itself (by Stone-Weierstrass and orthonormality of $(G/D)^\frown$) so $\bar{R}$ contains all characters of G/D. Thus for any character χ which is identically 1 on D, R contains an element which is $\chi(t)$ on $S(\alpha)$. Furthermore, in the proof of 1.7 more general "rectangles"

$$S(\xi_1, \xi_2, \cdots, \xi_n) = \{(s, t) \in G \mid |\alpha_i| \leqq \xi_i, t \in G_c\} ,$$

with the obvious corresponding discrete subgroups D, could have been used in place of the sets $S(\alpha)$. Since $\hat{G} = E_n \times \hat{G}_c$ [2, 35A] it is clear that *any* $\chi \in \hat{G}$ is identically 1 on some such D. It follows that for any $\chi \in \hat{G}$ there exists a set $S(\xi_1, \xi_2, \cdots, \xi_n)$ such that R contains a sequence $\tilde{f}_k$, $k = 1, 2, \cdots$, with $f_k(t) = \chi(t)$ on $S(k\xi_1, k\xi_2, \cdots, k\xi_n)$. Since this latter collection of compact sets is a σ-covering of G we conclude that for any $\chi \in \hat{G}$ and compact set $C \subset G$ R contains an element which is $\chi(t)$ on C. Any element of R with compact support can therefore be multiplied by a character, so Lemma 2.2 applies and the homomorphism ω is defined. The second part of 2.2, together with the fact that R is of type C, implies that if $f_k \to f$, f_k with compact support, then $\{\chi f_k\}$ is Cauchy and $\chi f_k \to \chi f$. Thus R is closed under multiplication by $\hat{G}$. Conclusion (b) of 1.7 implies that R is a subalgebra of $TK_\omega(G)$. For it is clear that if $\{C_n\}$ is any σ-covering there exist discrete subgroups D_n such that the mapping $C_n \to G/D_n$ (compact) is one-to-one and Condition (A) holds for each pair C_n, D_n. If $f \in R$, $f = \lim f_n$, with the support of f_n contained in C_n, and if $\tilde{f}_n(t)$ is approximated to within

$1/n$ uniformly on C_n by a function $\tilde{f}^{(n)}$ of the form $\Sigma c_i \chi_i(t)\omega(\chi)$, then clearly f corresponds to the element $\{f^{(n)}\}$ of $K_\omega(G)$. Since R is Tauberian it is in $TK_\omega(G)$, and R is closed since its norm is the $K_\omega(G)$ norm. The local K-isomorphism and resulting semi-simplicity of $TK_\omega(G)$ follow from Lemma 2.2 and regularity of R, and the final conclusion follows from Theorem 1.5.

THEOREM 2.4. *Let R be a homogeneous Banach algebra of type C over the connected locally compact abelian group G with R closed under multiplication by $\hat{G}$. Suppose that for some σ-covering $\{C_n\}$ of G there exists a bounded sequence $\{e_n\}$ of elements of R with compact support such that $e_n(t) = 1$ on C_n. Then $R = TK_\omega(G) = K_\omega(G)$.*

Proof. By Theorem 2.3 we need only show that $R \supset K_\omega(G)$. Let $k = \sup \| e_n \|$ and suppose that $e_n(t)$ vanishes outside $C_{n'}$. Let $\{f^{(n)}\}$ be any ω-Cauchy sequence of linear combinations of characters defining an element of $K_\omega(G)$. Consider the sequence $\{f^{(n')}e_n\}$ in R. Choose $\varepsilon > 0$, then since $\{f^{(n)}\}$ is ω-Cauchy it follows from Lemma 2.2 that there exists a compact set C_ε such that for sufficiently large n $| f^{(n')}\tilde{e}_n(t) | < \varepsilon k$ for $t \notin C_\varepsilon$. It is also clear that if m and n are sufficiently large $(m > n)$ then $| f^{(n')}\tilde{e}_n(t) - f^{(m')}\tilde{e}_m(t) | < \varepsilon$ for $t \in C_n$. Thus, for sufficiently large m and n

$$\| f^{(n')}e_n - f^{(m')}e_m \| < \max(\varepsilon, 2k\varepsilon) ,$$

so $\{f^{(n')}e_n\}$ is Cauchy. Its limit is the element we seek.

3. In this section we exhibit three examples of algebras of the type discussed above.

(1) Let $G = E_1$ and $R = D_m(E_1)$ be the algebra of all complex functions f on E_1 which have m continuous derivatives all of which tend to zero together with f at ∞. $\| f \| = \sup \sum_{i=0}^{m} 1/i! \, | f^{(i)}(t) | \; (-\infty < t < \infty)$. It is easy to verify that $\mathfrak{M}(D_m) = E_1$ and that $J(t) = \{f \in D_m \,|\, f^{(i)}(t) = 0, i = 1, 2, \cdots, m\}$. D_m is locally isomorphic to $D_m[a, b]$, which is thoroughly discussed by Silov and to $D_m(C)$, C the circle group [6]. $D_m/J(t_0)$ is easily seen to be an $(m + 1)$-dimensional "truncated" polynomial algebra generated by images of functions which are $(t - t_0)^k$, $k = 0, 1, \cdots, m$ in a neighborhood of t_0. D_m is of type C; indeed, the norm of f modulo $J(t_0)$ is exactly $\sum 1/i! \, | f^{(i)}(t_0) |$. It is also clearly closed under multiplication by $\hat{G}$. Since each $f^{(i)}(t) \to 0$ at ∞ it is uniformly continuous on E_1. Consequently, for each $f \in D_m$ $\| f - f_s \| \to 0$ as $s \to 0$. D_m is regular and Tauberian by easy proofs. Finally, it is clear that there exist $e_n \in D_m$ with $\| e_m \|$ *constant* and $e_n(t) = 1$ on $[-n, n]$, $e_n(t) = 0$ outside $[-n - 1, n + 1]$. This is true for any σ-covering of E_1 provided

that the distance between C_n and the complement of the support of e_n is bounded away from zero. Thus $D_m = K_\omega(G)$. Here $K = \{\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_m x^m \mid \alpha_i \text{ complex}, x^{m+1} = 0\}$ and for $f \in D_m$ $\tilde{f}(t) = f(t) + f'(t)x + \cdots + (1/m!)f^{(m)}(t)x^m$. ω is given, then, by $e^{i\lambda t} \to 1 + i\lambda x + \cdots + (1/m!)(i\lambda)^m x^m$ and is clearly continuous.

(2) Let G be any direct product of copies of E_1 and the circle group C. One can define a wide variety of algebras on G analogous to $D_m(E_1)$. For the circle and torus examples have been discussed by Silov [6]. We illustrate by considering the algebra D_m^θ $(-\pi/2 \leqq \theta \leqq \pi/2)$ of all continuous functions on the cylinder $E_1 \times C$ which have m continuous directional derivatives in the direction making an angle θ from the generating circle C, all vanishing at ∞. D_m^θ can easily be seen to be homogeneous of type C over $E_1 \times C$ and to have a bounded set of units modulo a σ-covering of $E_1 \times C$. Thus $D_m^\theta = K_\omega(E_1 \times C)$. It is easily seen that K is the same $(m+1)$-dimensional algebra which occurred in (1) and that ω is given by

$$\omega[e^{i\lambda t}e^{int}] = 1 + \sum_{k=1}^{m} (1/k!)[(i\lambda)^k \cos\theta + (in)^k \sin\theta]x^k .$$

All D_m^θ, m fixed, are locally isomorphic. If we call a curve in $E_1 \times C$ which intersects each generating circle in a constant angle α an α-curve then it is clear that given non-zero $\alpha \neq \beta$ there is a homeomorphism of G onto itself sending each α-curve into a β-curve and each β-curve into an α-curve, but that no homeomorphism can send a $\pi/2$-curve into a 0-curve. From this it is easy to see that all D_m^θ, $\theta \neq 0$ are isomorphic to each other, but that D_m^0 is *not* isomorphic to D_m^θ, $\theta \neq 0$.

In the next example we introduce the C-completion R^c of a non-type C Banach algebra R, that is, the completion of R relative to the type C norm. The general situation is somewhat as follows: Silov has shown that if R^c is semi-simple then it is an algebra of type C, and he has examined the connections between R and R^c for regular commutative Banach algebras ([6] contains an account assuming an identity, and the results generalize easily to algebras without identity.). If R is a homogeneous algebra over a compact abelian group R^c is automatically a $K_\omega(G)$ and is therefore semi-simple. No such clear cut answers appear to be available in the non-compact case, but given various additional bits of information about R it is possible to obtain information about R^c from the results in § 2. The algebra of the next example is one for which such additional bits are available.

(3) Let G be a σ-compact abelian group, R the Banach algebra of Fourier transforms f of elements $\hat{f}$ of $L_1(\hat{G})$. If $f \in R$ with $f(t) = \int \hat{f}(\chi)\overline{\chi(t)}d\chi$ then for $\|f\|$ we use the L_1-norm of $\hat{f}$. Multiplication in R

is pointwise and R is isomorphic and isometric to $L_1(\hat{G})$ with convolution as multiplication. Several properties of R are immediate or well-known.

(a) G is the structure space of R and R is semi-simple, regular and Tauberian [2]. If $\hat{f} \in L_1(\hat{G})$ and $h \in G$ then the function $\chi(h)\hat{f}(\chi)$ is also in $L_1(\hat{G})$. But this function corresponds to the function $f_h(t) = f(t-h)$ in R so

(b) R is closed under translation. Clearly $\|f\| = \|f_h\|$. It is easy to verify that $\|f - f_h\|$ tends to 0 at $h = 0$ so

(c) the elements of R are continuous under translation. If $f \in R$ and $\chi_0 \in \hat{G}$ then $\chi_0(t)f(t)$ is the Fourier transform of the translate $\hat{f}_{\chi_0} \in L_1(\hat{G})$, so

(d) R is closed under multiplication by $\hat{G}$. Moreover, by a well-known theorem on the Haar integral, if $\hat{f}$ and $\tilde{e}$ are in $L_1(\hat{G})$ then $\hat{f} * \hat{e}$ can be L_1-approximated by linear combinations of translates of $\hat{e}$. In R this means that

(e) Re is generated by $\hat{G}e$. Finally

(f) $\|\chi f\| = \|f\|$ for all $f \in R$, $\chi \in \hat{G}$ by an easy proof. From properties (a)-(d) it can easily be seen that R^c satisfies all the conditions of Lemmas 2.1 and 2.2 with the possible exception of semi-simplicity. The fact that for any unit e modulo a compact set of G Re is generated by $\hat{G}e$ enables one to show directly that $R^c \subset TK_\omega(G)$; the type C condition on R and the connectivity condition on G were used in Theorem 2.3 essentially to establish property (e). Property (f) (or, more generally, $\|\chi^{\pm n}f\| = o(n)$ for all χ, f) implies that $TK_\omega(G) = C_0(G)$. For if e is such that $\chi e \to \omega(\chi)$ in K then $|\omega(\chi)| \leqq \|\chi e\|$. Thus $|\omega(\chi^{\pm n})| = |\omega(\chi)^{\pm n}|$ is $o(n)$ and a theorem of Gelfond-Hille [1, p. 715] shows that this implies in a primary algebra that $\omega(\chi) = 1$. Example 1 of § 1 completes the proof. Thus R^c is semi-simple, hence homogeneous of type C so by Theorem 2.3 R^c is locally isomorphic to $C_0(G)$. By theorems of Silov [6; 3.5, 3.9] extended to algebras without identity $R/J(0)$ is isomorphic to the corresponding difference algebra in $C_0(G)$, but this is the complex field. Thus $J(0)$ and hence each $J(t)$ is maximal. This provides a proof of the well known theorem (first proved by Beurling and Segal for the real line and then by Kaplansky in general) which says that in the group algebra of a locally compact abelian group closed primary ideals are maximal. Finally, if G is connected then $R^c = C_0(R)$. For R contains elements with compact support for which $f(t) = 1$ on a compact set and $0 \leqq f(t) \leqq 1$ for all t. Since $J(t) = M_t$ this says that the type C norm of f is 1, so Theorem 2.4 applies to R^c.

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