THE ENVELOPES OF HOLOMORPHY OF TUBE DOMAINS IN INFINITE DIMENSIONAL BANACH SPACES

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1. Introduction. Let B be a Banach space with the strong topology generated by the norm. An open and connected set is called a *domain*. Let f be a complex valued functional defined in a domain D of a complex Banach space B_c. Let L be a finite dimensional translated complex linear subspace of $B_c : L = \{z \mid z = z_0 + \tau_1 a_1 + \cdots \tau_n a_n\}$ where $z_0, a_1, \cdots, a_n$ are fixed elements $\tau_1, \cdots, \tau_n$ complex parameters. (In the following we will call L an "affine subspace"). f is called "G-holomorphic" (=Gâteaux-holomorphic) if and only if the restriction of f to the intersection $D \cap L$ of D with any finite dimensional affine subspace L of B_c is holomorphic (in the ordinary sense). (Compare Hille-Phillips [7], Soeder [9], Bremermann [5].)

A functional that is G-holomorphic and locally bounded is called "F-holomorphic" (Fréchet-holomorphic). For finite dimension the notions (ordinary) "holomorphic function" and "G- and F-holomorphic functional" coincide. (The theory of holomorphic functionals in finite dimensional Banach spaces is equivalent to the theory of n complex variables.) For infinite dimension, in general, there exist already linear (and hence G-holomorphic) functionals that are not locally bounded (and hence not F-holomorphic).

In Bremermann [5] it has been shown that the phenomenon of "simultaneous holomorphic continuation," well known for n complex variables, persists for infinite dimension even for the very general G-holomorphic functionals: There exist domains such that all G-holomorphic functionals can be continued into a larger domain.

A domain for which a G-holomorphic functional exists that cannot be continued is called (in analogy to finite dimension) a "domain of G-holomorphy." In Bremermann [5] it has been shown that a domain of G-holomorphy is "pseudo-convex" (in a sense which is a natural extension from finite dimension).

We will apply these notions in the following to infinite dimensional tube domains and moreover we will show that it is possible to define and to determine the envelope of holomorphy of tube domains.

Finite dimensional tube domains and their envelopes of holomorphy have been studied by Bochner [1], Bochner-Martin [2], Hitotumatu [8], and Bremermann [3], [4]. It has been shown that a tube domain is pseudo-convex if and only if it is convex, and that the envelope of

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holomorphy of any tube domain is its convex envelope. The former property has been extended to infinite dimension in [5]. We extend here the latter. *To the author's knowledge this is the first time that the envelope of holomorphy of a class of infinite dimensional domains has been determined.* At the same time the proof given in the following is simpler than some previous proofs for finite dimension.

2. Tube domains, envelopes of holomorphy. Let B_c be a complex Banach space that is split into a real and imaginary part, such that every $z \in B_c$ is written

$$z = x + iy, \text{ where } x \in B_r,\ y \in B_r\ ,$$

where B_r is a real Banach space. Then a *domain* T_X *is called a tube domain with basis* X *if and only if it is of the form* $T_X = \{z \mid x \in X, y$ arbitrary$\}$, *where* X *is a domain in* B_r.

Obviously, T_X is convex if and only if X is convex, and X is convex if and only if the intersection of X with every finite dimensional affine subspace L_r of B_r is convex. ($L_r = \{x \mid x = x_0 + t_1 a_1 + \cdots, t_n a_n\}$, where $x_0, a_1, \cdots, a_n$ are fixed elements in B_r, and $t_1, \cdots, t_n$ real parameters).

It is somewhat difficult to define the envelope of holomorphy for arbitrary domains. Already for finite dimension it may not be schlicht. (Comp. [3], [6]). However, for finite dimension the following is true. Let D be a given domain. Suppose we have a domain $E(D)$ with the following properties:

(I) Every function holomorphic in D can be continued as a (single-valued) holomorphic function to $E(D)$.

(II) To every finite boundary point z_0 of $E(D)$ there exists a function that is holomorphic throughout $E(D)$ and is singular at z_0. If $E(D)$ has these properties, then $E(D)$ is the envelope of holomorphy of D.

Analogously, if we have an infinite dimensional domain D and a domain $E(D)$ with the properties (I) and (II) (with respect to G-holomorphic functionals), then we call $E(D)$ the *envelope of G-holomorphy of D.*

3. Proof of the main theorem. Let T_X be a tube domain that is not convex. Then, there exists an affine subspace

$$L_r = \{x \mid x = x_0 + t_1 a_1 + \cdots t_n a_n\}$$

($x_0, a_1, \cdots, a_n \in B_r, t_1, \cdots, t_n$ real parameters) such that $X \cap L_r$ is not convex.

Now it would be possible that $X \cap L_r$ is not connected and each connected component is convex (for instance if L_r is one-dimensional).

If X is not convex, then there exist two points x_1 and x_2 that cannot be connected by a straight line segment in X. However, X is connected, and even arcwise connected. Hence we can connect x_1 and x_2 by an arc in X, and even by a "polygon" that is by finitely many straight line segments. The polygonal arc spans a finite dimensional affine subspace L_r and the connected component of $L_r \cap X$ that contains x_1 and x_2 is not convex since x_1 and x_2 cannot be connected by a straight line.

Thus $L_r \cap X$ has a connected component that is not convex. Hence there exists a point x_3 on the boundary of $L_r \cap X$ and a line segment s containing x_3 such that s is locally a supporting line segment of the complement of $L_r \cap X$. In particular, x_3 and s can be chosen such that in a neighborhood of x_3 the line segment s has with the boundary $\partial(X \cap L_r)$ only the point x_3 in common.

Let the equation of the line containing s be

$$s = \{x \mid x = x_3 + bt\} ,$$

where b is a fixed element in B_r, t a real parameter. Let b be normalized such that $\|b\| = 1$. This real line lies in the analytic plane:

$$A = \{z \mid z = x_3 + b\tau\} ,$$

where τ is a complex parameter.

Let S_ρ be a disc on A with center at x_3, radius ρ:

$$S_\rho = \{z \mid z = x_3 + b\tau,\ |\tau| < \rho\} .$$

If ρ is small enough, then S_ρ will lie entirely in T_X, except for the points

$$\{z \mid z = x_3 + ibt,\ |t| < \rho,\ t \text{ real}\} .$$

We now apply the following lemma (which is an immediate consequence of the "fundamental Lemma" 3.1 (and 3.2) of [5] and Theorem 6.3 of [6]).

To formulate the lemma we need the distance function $d_D(z)$ which is defined as follows: Given a domain D, then

$$d_D(z) = \sup r \ni \{z' \mid \|z - z'\| < r\} \subset D ,$$

in other words $d_D(z)$ is the distance of the points z from the boundary of D, measured in the norm of B_c.

LEMMA. *Let $h(z)$ be the solution of the boundary value problem*

$$h(\tau) = \log d_{T_X}(x_3 + b\tau) \text{ for } |\tau| = \rho ,$$
$$h(\tau) \text{ harmonic for } |\tau| < \rho .$$

Then any function that is G-holomorphic in T_X can be continued G-holomorphically into the point set:

$$C = \{z \mid z' = x_3 + \tau b, \mid \tau \mid < \rho, \| z - z' \| < e^{h(\tau)}\} .$$

(We note that even though $\log d_{T_X}(x)$ becomes infinite at the two points $z = x_3 \pm i\rho b$, the solution of the boundary value problem exists and is finite for all $\mid \tau \mid < \rho$).

The pointset C is a neighborhood of the point $z = x_3$. In particular it contains the points $\| z - x_3 \| < e^{h(0)}$, and $e^{h(0)} \neq 0$. This continuation procedure can be repeated at any point $z = x_3 + iy$, where y is arbitrary. We always get the same neighborhood, independently of y, because the function $d_{T_X}(x_3 + iy)$ and hence h does not depend upon y. Hence any function G-holomorphic in T_X can not only be continued into a larger domain but into a larger tube domain $T_{X'}$, that means $X \subset X'$, $X \neq X'$.

We have to observe however one difficulty: If the intersection $X \cap \{X \mid \| x - x_3 \| < e^{h(0)}\}$ consists of more than one component, then continuation into $T_{X'}$ with $X' = X \cup \{x \mid \| x - x_3 \| < e^{h(0)}\}$ could possibly be such that the continued function would no longer be single-valued in $T_{X'}$. In order to keep the continuation single-valued we remove from X' all components of $X \cap \{x \mid \| x - x_3 \| < e^{h(0)}\}$ except the one that intersects S_ρ. In this way the continuation remains single-valued.

Thus we have the result: If T_X is a tube domain such that X is not convex, then any function that is G-holomorphic can be continued G-holomorphically (and single-valued) into a larger tube domain with basis X'. Then we can apply the same result to $T_{X'}$, and obviously the process can be iterated as long as the enlarged tube is not yet convex. Thus we have proved:

Given a tube domain T_X, then any function that is G-holomorphic in T_X can be continued G-holomorphically into the convex envelope of T_X.

(The convex envelope of T_X equals $T_{C(X)}$, where $C(X)$ is the convex envelope of X.)

On the other hand there exists to every boundary point z_0 of $T_{C(X)}$ a supporting affine subspace of B_c and a linear functional $l(z)$ that becomes zero exactly on the affine subspace. (This is an immediate consequence of the Hahn-Banach theorem.) The functional $1/l(z)$ is then G-holomorphic in $T_{C(X)}$ and becomes singular at z_0. Hence we have shown:

To every boundary point z_0 of a *convex* tube domain there exists a functional that is G-holomorphic in the domain and singular at z_0. The two statements combined give:

THEOREM. *Let T_X be a tube domain in a complex Banach space (of arbitrary dimension). Then the envelope of G-holomorphy of T_X is the convex envelope of T_X, which equals $T_{C(X)}$, where $C(X)$ is the convex envelope of X.*

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