AN INEQUALITY FOR LOGARITHMIC CAPACITIES

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1. Introduction. In his work on capacities, G. Choquet proved that for many capacities the inequality of strong subadditivity holds [1]. It is the purpose of this note to show that a similar inequality holds for logarithmic capacities. More precisely we shall prove the

THEOREM. *Let A and B be compact sets in the complex z-plane E. By $C(S)$ we denote the logarithmic capacity* [2] *of a given compact set $S, S \subset E$, where we agree to put $C(S) = 0$ whenever $S = \phi$. Then*

$$C(A \cup B) \cdot C(A \cap B) \leqq C(A) \cdot C(B) .$$

2. Proof of the theorem. Let $S, S \subset E$, be a compact set whose boundary consists of a finite number of analytic arcs. By S^* we denote that component of $E - S$ which is unbounded. Then Green's function of S^* is defined by the properties: it is harmonic in S^*, vanishes at the finite boundary points of S^* and has a logarithmic singularity at infinity. We will denote this function by $g_S(z, \infty)$.

First we shall deal with the case when the respective boundaries of A, B and $A \cap B$ consist of a finite number of non-degenerate analytic arcs. We remark that the difference $g_{A \cap B}(z, \infty) - g_A(z, \infty)$ is harmonic in $A^*, A^* \subset (A \cap B)^*$, and at infinity. It is furthermore non-negative on the boundary of A^* and hence non-negative in A^* by the maximum principle. Similarly $g_{A \cup B}(z, \infty) \geqq g_B(z, \infty)$ holds in $B^*, B^* \subset (A \cap B)^*$.

The function

$$h(z) = g_{A \cup B}(z, \infty) + g_{A \cap B}(z, \infty) - g_A(z, \infty) - g_B(z, \infty)$$

is harmonic in $(A \cup B)^*$ and at infinity. From $(A \cup B)^* = A^* \cap B^*$ it follows that the boundary points of $(A \cup B)^*$ belong either to the boundary of A^* or to the boundary of B^*. Therefore $g_{A \cup B}(z, \infty)$ and either $g_A(z, \infty)$ or $g_B(z, \infty)$ vanish at these boundary points. With the aid of the remark made above we get the result that $h(z)$ is non-negative in $(A \cup B)^*$.

Therefore

$$g_A(z, \infty) + g_B(z, \infty) \leqq g_{A \cup B}(z, \infty) + g_{A \cap B}(z, \infty)$$

holds in $(A \cup B)^*$. From this general inequality and using the fact that

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$$\lim_{z \to \infty} \{g_S(z, \infty) - \log |z|\}$$

is the constant $\gamma(S)$ of Robin [2] we deduce

$$\gamma(A) + \gamma(B) \leqq \gamma(A \cup B) + \gamma(A \cap B) .$$

But

$$C(S) = \exp \{-\gamma(S)\}$$

by definition. Hence our theorem is proven for the special case.

The general case follows by the usual approximation techniques [2].

References

1. G. Choquet, *Theory of capacities,* Ann. Inst. Fourier, Grenoble **5** (1953–54), 131–295.
2. R. Nevanlinna, *Eindeutige analytische Funktionen,* Grundlehren 46, 2. Aufl., Springer, 1953.

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