CENTROID SURFACES

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1. Introduction. Let $M_1, \cdots, M_{n-1}$ denote $(n-1)$ bounded closed sets in E_n. Busemann [1] has established the expression

$$(1.1)\quad |M_1| \cdots |M_{n-1}| = \frac{(n-1)!}{2}\int_{\Omega_n}\left(\int_{M_1(u)} \cdots \int_{M_{n-1}(u)} T(z, p_1, \cdots, p_{n-1}) dV_{p_1}^{n-1} \cdots dV_{p_{n-1}}^{n-1}\right) d\omega_u^n$$

where $|M_i|$ is the n-dimensional Lebesgue measure or volume of M_i. On the righthand side $M_i(u)$ is the cross-section of M_i with the hyperplane through z normal to the unit vector u, the point p_i varies in $M_i(u)$ and the differential $dV_{p_i}^{n-1}$ is the $(n-1)$-dimensional volume element of $M_i(u)$ at p_i. The final integration is extended over the surface Ω_n of the solid-unit sphere U_n and $d\omega_u^n$ is the area element of Ω_n at point u. By $T(z, p_1, \cdots, p_r)$ we will denote the r-dimensional volume of the simplex (possibly degenerate) with vertices $z, p_1, \cdots, p_r$.

Let

$$(1.2)\qquad \pi_r = \frac{\pi^{r/2}}{\Gamma(r/2+1)}\,.$$

For $n \geqq 3$, Busemann also shows by Steiner's symmetrization that

$$(1.3)\quad |M_1| \cdots |M_{n-1}| \geqq \frac{1}{n}\frac{\pi_n^{n-2}}{\pi_{n-1}^n}\int_{\Omega_n} |M_1(u)|^{n/(n-1)} \cdots |M_{n-1}(u)|^{n/(n-1)} d\omega_u^n$$

for nondegenerate convex bodies M_i where the equality sign holds only when the M_i are homothetic solid ellipsoids with center z. Here $|M_i(u)|$, of course, denotes the $(n-1)$-dimensional volume of $M_i(u)$. In this regard we will also, as a matter of convenience, not index lower dimensional mixed discriminates and mixed volumes since the dimension will be evident from the number of components.

The primary purpose of this note is to reinterpret (1.1) as an integration of the type (1.3) retaining the equality sign. This is given in § 3 by (3.20). In addition other integral expressions and inequalities are derived which are geometrically of the same type as those considered above.

2. Fenchel's momental ellipsoid. Let M be a bounded closed set with positive volume. The centroid s of M is defined by its rectangular coordinates

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$$s_i = \frac{1}{|M|}\int_M x_i dV_x^n . \tag{2.1}$$

If L_ν is a ν-flat through the origin z, then the second moment of M with respect to $L_\nu (0 \leqq \nu \leqq n-1)$ is defined by

$$I(M, L_\nu) = \int_M r^2 \sin^2 \varphi dV_x^n \tag{2.2}$$

where the distance zx is r and φ is the angle between the ray zx and L_ν (for $\nu = 0$, we define $\varphi = \pi/2$). By the same type of integration technique in [1, pp. 5-6], the reader may verify that

$$I(U_n, L_\nu) = \frac{n-\nu}{n+2}\pi_n \tag{2.3}$$

where U_n has center z; a calculation which will be used later.

The matrix A_M given by

$$A_M = \left[\frac{1}{|M|}\int_M x_i x_j dV_x^n\right] \tag{2.4}$$

is positive definite since

$$y^T A_M y = \frac{1}{|M|}\int_M (\Sigma x_i y_i)^2 dV_x^n ,$$

where y is a column vector and y^T is its transpose. The ellipsoid with surface $x^T A_M x = 1$ will be called Fenchel's momental ellipsoid and its polar reciprocal with respect to Ω_n given by $x^T A_M^{-1} x = 1$ will be called simply Fenchel's ellipsoid. This name is chosen since *W.* Fenchel first observed the affine character of this polar reciprocal (unpublished):

(2.5) Let M be transformed into $\bar{M}$ by a central affinity with matrix B. If F and $\bar{F}$ are the Fenchel ellipsoids of M and $\bar{M}$ respectively, then this central affinity also carries F into $\bar{F}$.

To see this, it may be observed from (2.4) that $A_{\bar{M}} = BA_M B^T$ or $A_{\bar{M}}^{-1} = (B^{-1})^T A_M^{-1} B^{-1}$ which completes the proof.

If $|F|$ is the volume of the Fenchel ellipsoid F of M, then

$$|F|^2 = \pi_n^2 \det(A_M) . \tag{2.6}$$

The result (2.5) enables one to prove readily that

$$\pi_n^{-2}|F|^2 = \det(A_M) \geqq (n+2)^{-n}\pi_n^{-2}|M|^2 \tag{2.7}$$

with equality only if, except for a set of measure zero, M is a solid ellipsoid with center z. For if we transform M into $\bar{M}$ by a unimodular central affinity so that $\bar{F}$ is a sphere, then

$$\det(A_M) = \left[\frac{1}{n\,|M|}\int_{\bar{M}} r^2 dV_x^n\right]^n .$$

Comparison of $\int_{\bar{M}} r^2 dV_x^n$ with that for a sphere with center z and volume $|M|$ proves (2.7).

We will adopt the same notation for mixed discriminates as in [2, pp. 51–57] where the reader will find an exposition of their properties. Consider the r quadratic forms $q_i = x^T A_i x$, $i = 1, \cdots, r$, where $A_k = [a_{ij}^{(k)}]$ is a real symmetric matrix. For any real $\lambda_1, \cdots, \lambda_r$, set $q = \lambda_1 q_1 + \cdots + \lambda_r q_r = x^T A x$ where $A = \sum_{k=1}^{r} \lambda_k A_k$. The discriminant $D(q) = \det(A)$ can be written

$$D(q) = \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} \lambda_{i_1} \cdots \lambda_{i_n} D(q_{i_1}, \cdots, q_{i_n})$$

where $D(q_{i_1}, \cdots, q_{i_n})$ is independent of the order of the q_{i_k} and is called the mixed discriminant of $q_{i_1}, \cdots, q_{i_n}$. For n forms q_i we have

$$D(q_1, \cdots, q_n) = \frac{1}{n!} \sum_{(i_1 \cdots i_n)} \begin{vmatrix} a_{11}^{(i_1)} & \cdots & a_{1n}^{(i_n)} \\ \vdots & & \vdots \\ a_{n1}^{(i_1)} & \cdots & a_{nn}^{(i_n)} \end{vmatrix} \tag{2.8}$$

where $(i_1 \cdots i_n)$ is a permutation of $(1 \cdots n)$.

Now consider n closed and bounded sets M_i with positive volume and let $q_i = x^T A_{M_i} x$ be the quadratic form associated with the Fenchel momental ellipsoid of M_i. By (2.4) and (2.8) we have

$$\begin{aligned} D(q_1, \cdots, q_n) &= \frac{1}{n!\,|M_1| \cdots |M_n|} \\ &\sum_{(i_1 \cdots i_n)} \int_{M_{i_1}} \cdots \int_{M_{i_n}} x_1^{(i_1)} \cdots x_n^{(i_n)} \begin{vmatrix} x_1^{(i_1)} & \cdots & x_1^{(i_n)} \\ \vdots & & \vdots \\ x_n^{(i_1)} & \cdots & x_n^{(i_n)} \end{vmatrix} dV_{x^{(i_1)}}^n \cdots dV_{x^{(i_n)}}^n \\ &= \frac{1}{n!\,|M_1| \cdots |M_n|} \int_{M_1} \cdots \int_{M_n} \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & & \vdots \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix}^2 dV_{x^{(1)}}^n \cdots dV_{x^{(n)}}^n . \end{aligned} \tag{2.9}$$

Since $T(z, x^{(1)}, \cdots, x^{(n)}) = \pm (1/n!) \det(x_i^{(j)})$ we then have

$$D(q_1, \cdots, q_n) = \frac{n!}{|M_1| \cdots |M_n|} \int_{M_1} \cdots \int_{M_n} T^2(z, p_1, \cdots, p_n)\, dV_{p_1}^n \cdots dV_{p_n}^n . \tag{2.10}$$

The fundamental inequality for mixed discriminants (see [2, p. 53]) is: (2.11) If the forms $q_1, \cdots, q_{n-1}$ are positive definite and Q is any symmetric form, then

$$D^2(q_1, \cdots, q_{n-1}, Q) \geqq D(q_1, \cdots, q_{n-1}, q_{n-1}D(q_1, \cdots, q_{n-2}, Q, Q)$$

where the equality sign holds only if $Q = \lambda q_{n-1}$.

If we set

$$D_p(q, Q) = D(q_1, \cdots, q_{n-p}, \underbrace{Q, \cdots, Q}_{p}) , \tag{2.12}$$

then for n positive definite forms q_i, (2.11) generalizes to

$$D^r(q_1, \cdots, q_n) \geqq \prod_{k=0}^{r-1} D_r(q, q_{n-k}), \; r = 2, 3, \cdots, n \tag{2.13}$$

with equality only if $q_{n-k} = \lambda_{n-k} q_n$ for $k = 0, \cdots, r-1$.

The proof of (2.13) and the condition for equality proceed by induction from the case $r = 2$. The proof is analogous to Alexandrov's generalization [2, p. 50] of a corresponding inequality for mixed volumes and consequently will be omitted here.

If we now set

$$W(M_1, \cdots, M_n, z) = \int_{M_1} \cdots \int_{M_n} T^2(z, p_1, \cdots, p_n) d V_{p_1}^n \cdots d V_{p_n}^n \tag{2.14}$$

and

$$W_p(M, M_k, z) = W(M_1, \cdots, M_{n-p}, \underbrace{M_k, \cdots M_k}_{p}, z) , \tag{2.15}$$

then by (2.13) and (2.10) we have

$$W^r(M_1, \cdots, M_n, z) \geqq \prod_{k=0}^{r-1} W_r(M, M_{n-k}, z), \; r = 2, \cdots, n \tag{2.16}$$

with the equality sign only if the Fenchel ellipsoids of M_{n-k} are homothetic for $k = 0, \cdots, r-1$. Applying (2.16) to the case $r = n$ and using (2.10) and (2.7), we have

$$[\,|M_1| \cdots |M_n|\,]^{(n+2)/n} \leqq n! \pi_n^2 (n+2)^n W(M_1, \cdots, M_n, z) \tag{2.17}$$

with equality only if (except for a set of measure zero) the M_i are homothetic ellipsoids with center z.

The reader will find other inequalities of the above type in [3, pp. 70–71].

3. Centroid surfaces. As before, M is a bounded closed set with positive volume. An oriented hyperplane $L(u)$ through z normal to the direction $u\,(u \neq 0)$ bounds a closed half-space lying on its positive side.

The intersection of this halfspace with M will be denoted by $C(u)$.

Consider the function

$$(3.1)\qquad H(u) = \frac{1}{|M|}\int_M |u\cdot x|\,dV_x^n,\ u\cdot x = \sum_{i=1}^n u_i x_i\ .$$

Since

(a) $H(0) = 0$,
(b) $H(\mu u) = \mu H(u)$ for $\mu > 0$,
(c) $H(u+v) \leqq H(u) + H(v)$,

$H(u)$ is the supporting function (s.f.) of a convex body K^* (see [4, p. 26]), which is nondegenerate and has center z. Let P_0 be the supporting plane (s.p.) to K^* in the direction $u^{(0)}$, the supporting function of $K^* \cap P_0$ is given by the directional derivative

$$(3.2)\qquad \begin{aligned} H'(u^{(0)}; u) &= \lim_{h\to 0+}\frac{H(u^{(0)}+hu) - H(u^{(0)})}{h}\\ &= \frac{1}{|M|}\int_{C(u^{(0)})} u\cdot x\,dV_x^n - \frac{1}{|M|}\int_{C(-u^{(0)})} u\cdot x\,dV_x^n\ . \end{aligned}$$

Since $H'(u^{(0)}; u)$ is a linear function of the u_i, P_0 touches K^* in a single point and thus every s.p. of K^* is regular and K^* is strictly convex. (See [4, pp. 25–26].) The derivatives $\partial H/\partial u_i$ are continuous, homogeneous of degree 0, and if y is the point of contact of the s.p. to K^* in the direction u, then

$$(3.3)\qquad y_i = \frac{\partial H}{\partial u_i} = \frac{1}{|M|}\int_{C(u)} x_i\,dV_x^n - \frac{1}{|M|}\int_{C(-u)} x_i\,dV_x^n\ .$$

We will call K^* the centroid body of M (with respect to z) and the surface of K^* will be called the centroid surface of M. One may observe that if M happens to have center z, then the centroid surface of M is precisely the set of all controids of $C(u)$ for $u \in \Omega_n$. In general, let $s^{(1)}$ and $s^{(2)}$ be the centroids of $C(u)$ and $C(-u)$ respectively, the y is the center of mass of the two points $s^{(1)}$ and $-s^{(2)}$ provided with mass $|C(u)|/|M|$ and $|C(-u)|/|M|$ respectively. If $|C(u)| = 0$, we will define the centroid of $C(u)$ to be the point z.

It is evident that if M is transformed into $\bar{M}$ by a central affinity, then this transformation also carries the centroid surface of M into the centroid surface of $\bar{M}$.

We now wish to impose additional restrictions on M such that $H(u)$ has continuous second partial derivatives and the surface of K^* has positive Gauss curvature. The following two conditions are sufficient for this purpose:

(a) The set $M(u)$ has positive $(n-1)$-dimensional measure for all $u \in \Omega_n$.

(b) For any $u^{(0)} \in \Omega_n$ and any sequence $u^{(i)} \to u^{(0)}$, the $\lim_{u^{(i)} \to u^{(0)}} M(u^{(i)})$ coincides with $M(u^{(0)})$ except for a possible set of zero $(n-1)$-dimensional measure.

To simplify the calculation of the second partial derivatives at a point $u^{(0)}$, we introduce what Busemann [2, p. 57] calls "standard coordinates." With the same origin and orientation, the x_n axis is chosen such that $u_1^{(0)} = \cdots = u_{n-1}^{(0)} = 0$ and $u_n^{(0)} > 0$. It then follows from (3.3) that

$$\frac{\partial^2 H(u^{(0)})}{\partial u_k \partial u_n} = \frac{\partial^2 H(u^{(0)})}{\partial u_n \partial u_k} = 0 \, . \tag{3.4}$$

Although standard coordinates vary from point to point, the end result (3.9) is expressed geometrically and therefore independent of the coordinate system.

For $j < n$, let $u = (0, \cdots, 0, u_j, 0, \cdots, 0, u_n^{(0)})$ and set

$$N_1 = C(u) \cap C(u^{(0)}),\ N_1^* = C(-u) \cap C(-u^{(0)}),\ N_2 = C(u^{(0)}) - N_1,$$
$$N_2^* = C(-u) - N_1^*,\quad N_3 = C(u) - N_1, \qquad N_3^* = C(-u^{(0)}) - N_1^*.$$

Except for a set of zero n-dimensional measure, $N_2 = N_2^*$ and $N_3 = N_3^*$. By (3.3) we have for $i, j < n$

$$\frac{\left(\dfrac{\partial H}{\partial u_i}\right)_u - \left(\dfrac{\partial H}{\partial u_i}\right)_{u^{(0)}}}{u_j} = \frac{2}{u_j\,|M|}\left(\int_{N_3} x_i d V_x^n - \int_{N_2} x_i d V_x^n\right) . \tag{3.5}$$

We will calculate the limit of (3.5) as either $u_j \to 0+$ or $u_j \to 0-$. In either case for $x \in N_3$, $x_j u_j \geqq 0$, $x_n \leqq 0$ and for $x \in N_2$, $x_j u_j \geqq 0$, $x_n \geqq 0$. For $-\pi/2 < v_n < \pi/2$, let the hyperplane $x_n = (\tan v_n) x_j$ intersect M in $M^+(v_n)$ for $x_n \geqq 0$ and in $M^-(v_n)$ for $x_n \leqq 0$. Also the volume element $d V_x^{n-1}$ of this hyperplane is

$$d V_x^{n-1} = dx_1 \cdots dx_{n-1} \sec v_n \, . \tag{3.6}$$

We introduce new coordinates $v_1, \cdots, v_n$ by $x_i = v_i$ for $i = 1, \cdots, n-1$ and $x_n = v_j \tan v_n$ which uniquely define the v_i with $-\pi/2 < v_n < \pi/2$ for all x for which $x_j \neq 0$. The Jacobian J of this transformation is $J = v_j \sec^2 v_n$. Also define α, $0 \leqq \alpha < \pi/2$, by $u_n^{(0)} \tan \alpha = |u_j|$. Then $|J|/u_j = \pm v_j \sec^2 v_n / u_n^{(0)} \tan \alpha$ with the plus sign for $x \in N_3$ and the minus sign for $x \in N_2$. The difference quotient (3.5) is, consequently, given by

$$\frac{2}{u_n^{(0)}\,|M| \tan \alpha}\left[\int_0^\alpha \sec v_n \left(\int_{M^-(v_n)} v_i v_j d V_v^{n-1}\right) dv_n\right.$$

$$+ \int_0^{\alpha} \sec v_n \Big(\int_{M^+(v_n)} v_i v_j d V_v^{n-1} \Big) dv_n \Big]$$

and since the integrands are continuous functions of v_n by assumption (b) we have

$$(3.7) \qquad H_{ij}(u^{(0)}) = \frac{\partial^2 H(u^{(0)})}{\partial u_i \partial u_j} = \frac{2}{u_n^{(0)} \mid M \mid} \int_{M(u^{(0)})} x_i x_j d V_x^{n-1}, \ (i, j < n) .$$

Now let $H^{(i)}(u)$ be the supporting function (3.1) for the set M_i, $i = 1, \cdots, n-1$. Set $H = \lambda_1 H^{(1)} + \cdots + \lambda_n H^{(n-1)}$, then $D_{n-1}(H)$ is defined as the sum of all principal $(n-1)$ rowed minors of the matrix H_{ij} (with components evaluated for a unit vector) and is a homogeneous polynomial of degree $(n-1)$ in the λ_i. (See [4, p. 59] or [2, pp. 45–46].) The quantity $D(H^{(1)}, \cdots, H^{(n-1)})$ denotes the factor of $\lambda_1 \cdots \lambda_{n-1}$ in $D_{n-1}(H)$ divided by $(n-1)!$. If we calculate (3.7) for each of the $H^{(i)}$ using the same standard coordinates we have, because of (3.4),

$$(3.8) \qquad D(H^{(1)}, \cdots, H^{(n-1)}) = \frac{1}{(n-1)!} \sum_{(i_1 \cdots i_{n-1})} \begin{vmatrix} H_{11}^{(i_1)} & \cdots & H_{n(n-1)}^{(i_{n-1})} \\ \vdots & & \vdots \\ H_{(n-1)1}^{(i_1)} & \cdots & H_{(n-1)(n-1)}^{(i_{n-1})} \end{vmatrix} .$$

In the same way as we derived (2.10), we find for any $u \in \Omega_n$,

$$(3.9) \qquad D(H^{(1)}, \cdots, H^{(n-1)}) = \frac{(n-1)! 2^{n-1}}{\mid M_1 \mid \cdots \mid M_{n-1} \mid} \int_{M_1(u)} \cdots \int_{M_{n-1}(u)} T^2(z, p_1, \cdots, p_{n-1}) d V_{p_1}^{n-1} \cdots d V_{p_{n-1}}^{n-1} .$$

By comparison with (2.10) we observe that

$$(3.10) \qquad D(H^{(1)}, \cdots, H^{(n-1)}) = \frac{2^{n-1} \mid M_1(u) \mid \cdots \mid M_{n-1}(u) \mid}{\mid M_1 \mid \cdots \mid M_{n-1} \mid} D(q_1, \cdots, q_{n-1})$$

where q_i is the quadratic from associated with the Fenchel momental ellipsoid of $M_i(u)$ in the $(n-1)$-dimensional space $L(u)$.

From (3.9), we may give an integral interpretation of an elementary symmetric function $\{R_1 \cdots R_m\}$ of the principal radii of curvature of the centroid surface of M. With H given by (3.1) we have for $m = 1, \cdots, n-1$ (see [4, p. 63]),

$$(3.11) \qquad \{R_1 \cdots R_m\} = \binom{n-1}{m} D(\underbrace{\mid u \mid, \cdots, \mid u \mid}_{n-m-1}, \underbrace{H, \cdots, H}_{m}) = D_m(H) .$$

Set $M = M_1 = \cdots = M_m$ and $U_n = M_{m+1} = \cdots = M_{n-1}$. Since

$$(3.12) \qquad \frac{1}{\mid U_n \mid} \int_{U_n} \mid u \cdot x \mid d V_x = \frac{2\pi_{n-1}}{(n+1)\pi_n} \mid u \mid ,$$

we obtain

$$\left[\frac{2\pi_{n-1}}{(n+1)\pi_n}\right]^{n-m-1} D_m(H) = \binom{n-1}{m} \frac{(n-1)!2^{n-1}}{|M|^m \pi_n^{n-m-1}}$$

$$\cdot \underbrace{\int_{M(u)}\int_{M(u)}}_{m} \underbrace{\int_{U_n(u)}\int_{U_n(u)}}_{n-m-1} T^2(z, p_1, \cdots, p_{n-1}) dV_{p_1}^{n-1} \cdots dV_{p_{n-1}}^{n-1} .$$

By integrating successively over the $U_n(u)$ and using (2.3) applied to the appropriate dimensions we obtain

$$(3.13) \quad \{R_1 \cdots R_m\} = \frac{m!2^m}{|M|^m} \int_{M(u)} \cdots \int_{M(u)} T^2(z, p_1, \cdots, p_m) dV_{p_1}^{n-1} \cdots dV_{p_m}^{n-1}$$

for $m = 1, \cdots, n-1$.

We may also give an interpretation of each individual principal radius of curvature. First we show:

(3.14) The Dupin indicatrix of the centroid surface of $M(wrtz)$ at the point of contact y of the tangent plane in the direction u is homothetic to the Fenchel ellipsoid ($wrt\ z$) of $M(u)$ in the space $L(u)$.

A central affinity sends homothetic figures in parallel hyperplanes into homothetic figures. Due to the affine nature of Fenchel ellipsoids and centroid surfaces, we need only show that if the Fenchel ellipsoid of $M(u)$ is a sphere, then the Dupin indicatrix at y is a sphere. However, this follows at once from (2.4) and the representation (3.7) in standard coordinates since the principal radii of curvature R_i must satisfy

$$\begin{vmatrix} H_{11}-R & \cdots & H_{1n} \\ \vdots & & \vdots \\ H_{n1} & \cdots & H_{nn} - R \end{vmatrix} = 0$$

where H_{ij} are evaluated for a unit vector. (See [4, p. 61].)

Now, let the line through z, parallel to the ith principal direction of the centroid surface at y, be normal to the $(n-2)$ space L_{n-2} through z in $L(u)$. Then R_i is given by

$$(3.14) \qquad R_i = \frac{2}{|M|} I(M(u), L_{n-2})$$

where $I(M(u), L_{n-2})$ is the second moment, in $L(u)$, of $M(u)$ with respect to L_{n-2}.

Returning to the $(n-1)$ bodies $M_1, \cdots, M_{n-1}$ for which we obtained (3.9), let $H^{(n)}(u)$ be the supporting function (3.1) corresponding to any bounded closed set M_n with positive volume. Then (see [2, p. 46]),

$$(3.15) \qquad V(K_1^*, \cdots, K_n^*) = n^{-1} \int_{\Omega_n} H^{(n)} D(H^{(1)}, \cdots, H^{(n-1)}) d\omega_u^n, \ |u| = 1 ,$$

where $V(K_1^*, \cdots, K_n^*)$ is the mixed volume of $K_1^*, \cdots, K_n^*$. Using (3.9), (3.15), (3.1) and the integration technique of Busemann in [1] where it is shown that

$$dV_{p_1}^n \cdots dV_{p_{n-1}}^n = (n-1)!\,T(z, p_1, \cdots, p_{n-1})dV_{p_1}^{n-1} \cdots dV_{p_{n-1}}^{n-1}d\omega_u^n ,$$

we obtain

$$(3.16)\qquad V(K_1^*, \cdots, K_n^*) = \frac{2^n}{|M_1| \cdots |M_n|}\int_{M_1} \cdots \int_{M_n} T(z, p_1, \cdots, p_n)dV_{p_1}^n \cdots dV_{p_n}^n .$$

Since both sides of (3.16) vary continuously with the M_i, we may extend this result to any n bounded and closed sets M_i with $|M_i| > 0$. Briefly, we may assume $z \in M_i$ and let $\varepsilon_j > 0$ be a sequence such that $\varepsilon_j \to 0$. A covering of open spheres of radius ε_j with centers in M_i may be reduced to a finite covering since M_i is compact. Conditions (a) and (b) are then satisfied for the closure of such a finite covering and the extention of (3.16) follows.

There is an alternate proof of (3.16) which proceeds directly from (3.1). We did not resort to this at the outset since the intervening results are of interest in themselves. Briefly, the alternate proof is as follows: We approximate the $H^{(i)}(u)$ of (3.1) by

$$E^{(i,k)}(u) = \frac{1}{|M_i|}\sum_{j=1}^{k} |u \cdot x^{(j)}| \Delta V_j^n$$

such that $E^{(i,k)} \to H^{(i)}$ as $k \to +\infty$. Now $|u \cdot x|$ is the supporting function of the segment $\bar{x}$ with end-points x and $-x$. Also, by induction, one shows that

$$V(\bar{x}^{(1)}, \cdots, \bar{x}^{(n)}) = 2^n T(z, x^{(1)}, \cdots, x^{(n)}) .$$

The function $E^{(i,k)}$ is the supporting function of the linear combination

$$E_{(i,k)} = \frac{1}{|M_i|}\sum_{j=1}^{k} \bar{x}^{(j)} \Delta V_j^n .$$

For $\lambda_j > 0$ the linear combination $E_k = \lambda_1 E_{(1,k)} + \cdots + \lambda_n E_{(n,k)}$ may also be expressed as a linear combination of the nk segments $\bar{x}^{(j_i)}$. Expressing the volume of E_k as a polynomial in the λ_i in two ways we have by comparing the coefficient of $\lambda_1 \cdots \lambda_n$

$$V(E_{(1,k)}, \cdots, E_{(n,k)}) = \frac{2^n}{|M_1| \cdots |M_n|}\sum_{j_1=1}^{k} \cdots \sum_{j_n=1}^{k} T(z, x^{(j_1)}, \cdots, x^{(j_n)})\Delta V_{j_1}^n \cdots \Delta V_{j_n}^n$$

and (3.16) follows in the limit as $k \to +\infty$.

The formula (3.16) may be substituted into inequalities of mixed volumes to yield inequalities of the integrals. Since the number of times a component appears on each side of a mixed volume inequality is always the same, the coefficient on the righthand side of (3.16) cancels leaving, as in (2.16), inequalities among the integrals only. However, in this case when the uniqueness theorem (4.1) applies, the condition for equality may be passed through the K_i^* to the M_i.

In [1, p. 11], Busemann shows that if M is a nondegenerate convex body, then

$$(3.17) \qquad \int_M \cdots \int_M T(z, p_1, \cdots, p_n) d V_{p_1} \cdots d V_{p_n} \geqq \frac{2}{(n+1)!} \frac{\pi_{n+1}^{n-1}}{\pi_n^{n+1}} |M|^{n+1}$$

with equality only if M is an ellipsoid with center z. We define the expanded centroid body K of M to be the dilation of K^* about z by the factor $(n+1)\pi_n/2\pi_{n-1}$. By (3.12), we see that this is the factor which dilates the centroid body of an ellipsoid with center z into coincidence with the ellipsoid.

From (3.16) we obtain a reinterpretation of (3.17) by observing the identity $n!\pi_n\pi_{n-1} = 2^n\pi^{n-1}$:

(3.18) If K is the expanded centroid body of a nondegenerate convex body M, then $|K| \geqq |M|$ with equality only if M is an ellipsoid with center z.

The convexity of M is not an essential feature in (3.18) and the Steiner symmetrization used to prove (3.17) may be extended to include nonconvex sets.

Using the expanded centroid bodies K_i of M_i, we may write (3.16) as

$$(3.19) \qquad |M_1| \cdots |M_n| \; V(K_1, \cdots, K_n) = \frac{(n+1)!\pi_n^{n+1}}{2\pi_{n+1}^{n-1}} \int_{M_1} \cdots \int_{M_n} T(z, p_1, \cdots, p_n) d V_{p_1}^n \cdots d V_{p_n}^n$$

and if we define K_i to be the point z if $|M_i| = 0$ then (3.19) holds for any bounded closed sets M_i.

Substituting (3.19) into (1.1) we have

(3.20) THEOREM. *If $K_i(u)$ is the expanded centroid body of $M_i(u)$ in the $(n-1)$-dimensional space $L(u)$, then*

$$|M_1| \cdots |M_{n-1}| = \frac{1}{n} \frac{\pi_n^{n-2}}{\pi_{n-1}^n} \int_{\Omega_n} |M_1(u)| \cdots |M_{n-1}(u)| \; V(K_1(u), \cdots, K_{n-1}(u)) d\omega_u^n .$$

The inequality $V^{n-1}(K_1(u), \cdots, K_{n-1}(u)) \geqq |K_1(u)| \cdots |K_{n-1}(u)|$ (see [2, p. 50]) and (3.18) reproduces (1.3).

There are two special cases of (3.20) of particular geometric interest. First, set $M = M_1 = \cdots = M_{n-1}$, then

$$(3.21) \qquad |M|^{n-1} = \frac{1}{n}\frac{\pi_n^{n-2}}{\pi_{n-1}^n}\int_{\Omega_n} |M(u)|^{n-1}|K(u)|\,d\omega_u^n .$$

Next, for $n \geqq 3$, set $M = M_1 = \cdots = M_{n-2}$, $M_{n-1} = U_n$, then

$$(3.22) \qquad |M|^{n-2} = \frac{1}{n(n-1)}\frac{\pi_n^{n-3}}{\pi_{n-1}^{n-1}}\int_{\Omega_n} |M(u)|^{n-2} S(K(u))d\omega_u^n$$

where $S(K(u))$ is the surface area of $K(u)$ in the space $L(u)$.

4. **Uniqueness theorems.** In order for K^* to determine M, additional restrictions on M are necessary as may be seen by consideration of a set M bounded by two concentric spheres.

(4.1) THEOREM. *Suppose M_i $(i = 1, 2)$ can be represented in polar coordinates by $0 \leqq r \leqq \rho_i(u)$, $u \in \Omega_n$ where $\rho_i(u)$ is an even,* i.e., *$\rho_i(u) = \rho_i(-u)$, continuous function on Ω_n. If the centroid surface of M_1(wrt z) is identical to the centroid surface of M_2(wrt z), then M_1 and M_2 are identical.*

(4.2) THEOREM. *Suppose M_i $(i = 1, 2)$ have the same representation as in* (4.1). *If $|M_1(u)| = |M_2(u)|$ for all $u \in \Omega_n$, then M_1 and M_2 are identical.*

The latter theorem is a result, for $n = 3$, of *P.* Funk [6].

We first prove (4.1). From (3.1) and the assumption on the representation of M_i we have

$$H^{(i)}(u) = \frac{1}{(n+1)|M_i|}\int_{\Omega_n} |u\cdot\tau|\,\rho_i^{n+1}(\tau)d\omega_\tau^n,\ |\tau| = 1 .$$

Consequently, (4.1) follows from the uniqueness of the solution of an integral equation of the first kind. Namely:

(4.3) THEOREM. *Let $h(\tau)$ be an even, continuous function on Ω_n. If for unit vectors u and τ*

$$\int_{\Omega_n} |u\cdot\tau|\,h(\tau)d\omega_\tau^n = 0$$

for all $u \in \Omega_n$, then $h(\tau)$ vanishes identically.

The result (4.3) is well known for $n = 2, 3$ and the recent extension of surface harmonics to n-dimensions, in particular the Funk-Hecke theorem, enables one to prove (4.3) for all n. There are two steps in the following proof (which applies for $n \geqq 3$). First, from the completeness [5, p. 241] it suffices to show that

$$\int_{\Omega_n} S_m(\tau)h(\tau)d\omega_\tau^n = 0$$

for all the linearly independent surface harmonics $S_m(\tau)$ of degree m and for $m = 0, 1, 2, \cdots$. Since $h(\tau)$ is an even function we need only to consider, now, even m. Next, from the Funk-Hecke theorem [5, pp. 247–248] we have

$$\int_{\Omega_n} | u \cdot \tau | S_m(u)d\omega_u^n = \lambda_m S_m(\tau)$$

where

$$\lambda_m = \frac{(4\pi)^\nu m! \Gamma(\nu)}{(m + 2\nu - 1)!} \int_{-1}^{1} | x | C_m^\nu(x)(1 - x^2)^{\nu - 1/2} dx \tag{4.4}$$

and $\nu = (n - 2)/2 \geqq 1/2$. Thus, we need only to verify that $\lambda_m \neq 0$ for $m = 0, 2, 4, \cdots$. For $m = 0$, $C_0^\nu(x) = 1$ and $\lambda_0 \neq 0$. For $m > 0$,

$$C_m^\nu(x) = a_{m,\nu}(1 - x^2)^{-\nu+1/2} \frac{d^m}{dx^m}[(1 - x^2)^{m+\nu-1/2}]$$

where $a_{m,\nu} \neq 0$. See [5, p. 236] for the explicit expression of the coefficient $a_{m,\nu}$. Thus the integral in (4.4) is

$$I_{m,\nu} = 2a_{m,\nu} \int_0^1 x \frac{d^m}{dx^m}[(1 - x^2)^{m+\nu-1/2}]dx$$

and using integration by parts

$$I_{m,\nu} = 2a_{m,\nu}(-1)^{\frac{m-2}{2}}(m - 2)! \binom{m + \nu - 1/2}{\frac{m}{2} - 1} \neq 0$$

for $m = 2, 4, 6, \cdots$ which completes the proof.

The result (4.2) is clearly a consequence of the following spherical integration theorem.

(4.5) Let $f(\tau)$ be a continuous even function defined on Ω_n. If

$$\int_{\Omega_n(u)} f(\tau)d\omega_\tau^{n-1} = 0$$

for all $u \in \Omega_n$, then $f(\tau)$ vanishes identically.

A proof of (4.5) for $n = 3$ can be found in [4, pp. 136–138]. However, a proof for all $n \geqq 3$ is easily obtained from (4.1). To see this, set $g(\tau) = f(\tau) - [\min f(\tau)] + 1 > 0$. Let $\rho(\tau) = [g(\tau)]^{1/(n+1)}$ and let M be the set whose polar coordinates satisfy $0 \leqq r \leqq \rho(\tau)$, $\tau \in \Omega_n$. Using (**3.13**) for $m = 1$, the sum of the principal radii of curvature of the centroid surface of M *wrt* z may be expressed, in this case, by

$$R_1 + \cdots + R_{n-1} = \frac{2}{(n+1)\,|\,M\,|}\int_{\Omega_n(u)} g(\tau)d\omega_\tau^{n-1}$$

and, by hypothesis, this is a positive constant for $u \in \Omega_n$. However, this implies (see [4, pp. 117–118]) that the centroid surface is a sphere and by (4.1), M is a solid sphere and $g(\tau)$ is a constant which completes the proof.

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