THE EXTENSION OF LINEAR FUNCTIONALS DEFINED ON H^∞

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Dedicated to Marston Morse

1. Introduction. We consider the Banach space L^∞ of (classes of) bounded measurable complex functions on the unit circle Γ_1. It has a subspace $H = H^\infty$ consisting of those functions h which are the boundary value functions (existing almost everywhere by Fatou's theorem) of bounded analytic functions $\hat{h}$ in the interior S_1 of Γ_1. By the Hahn-Banach theorem, any bounded linear functional ϕ defined on H^∞ can be extended over L^∞ with no increase in norm. It is the primary purpose of this paper to prove that this extension is unique, provided that ϕ is defined (over H) by an integral with kernel in L^1. Without this hypothesis, uniqueness may fail,[1] as we shall see in 10.1.

The simplest case of the theorem occurs when $\phi(h)$ is defined to be $\hat{h}(0)(h \in H)$; then, for the unique norm preserving extension ϕ_e of ϕ, $\phi_e(f)$ is the average value of f on Γ_1.

If we assume that the extended functional is also defined through a kernel in L^1, then the uniqueness follows easily. The key to the proof (with no such assumption) is contained in lemma 3.1, asserting that any real nonnegative function f in L^∞ may be approximated from below in the L^1 norm by a nonnegative function f' whose harmonic extension through S_1 has a bounded conjugate function. At first sight this might seem surprising; for f' must in general be discontinuous, and the conjugate of a harmonic function whose boundary value function has a simple discontinuity is unbounded.

The proof, in the case of a nonnegative kernel, follows quickly from the lemma. For a general $K \in L^1$ which is extremal over H, i.e. such that $\sup \int hKd\theta = \int |K|\, d\theta$ (requiring $h \in H$ and $\|h\| \leqq 1$), an extremal function $h_0 \in H$(i.e. $\|h_0\| = 1$, $\int h_0 Kd\theta = \int |K|\, d\theta$) is made use of; with its help, the proof of uniqueness is reduced to the previous case. If ϕ is defined through an arbitrary kernel $J \in L^1$, we use a lemma of Rogosinski and Shapiro to replace J by an extremal kernel K.

In the second half of the paper we consider the analogous problem on a compact bordered Riemann surface $\bar{S}$, with border Γ. In general, uniqueness of the extension fails; a simple example is given in 10.2. The extension of the given functional ϕ is unique at least as far as a

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[1] An example of this was first given us by R. R. Phelps.

certain subspace of finite codimension ρ in L^∞, where ρ is the first Betti number of $\bar{S}$. If ϕ is defined in H by a real nonnegative kernel K, then the subspace mentioned is the annihilator G of a space P defined through considering the periods of conjugates of harmonic functions (see 6.16 and 6.17). The set of all norm preserving extensions through L^∞ forms a closed convex cell of dimension at most ρ; in particular, the extension is unique if $\rho = 0$. For $\rho > 0$, the extension may or may not be unique, depending on the nature of the kernel K. Some examples of this are given in § 10. The principal facts are summarized in Theorem 9.1.

2. Preliminaries. A function $f \in L^\infty$ has a unique harmonic extension $\hat{f}$ over S_1 such that the boundary value function of $\hat{f}$ equals f a.e. (almost everywhere). If f is continuous, the boundary values are taken on continuously. If $f = f_1 + if_2$, where f_1 and f_2 are real functions, then $\hat{f} = \hat{f}_1 + i\hat{f}_2$; now $f \in H$ if and only if $\hat{f}_2$ is a harmonic conjugate of $\hat{f}_1$; then $\hat{f}$ is the analytic function with boundary values f.

Furthermore, H is the set of all $h \in L^\infty$ such that

$$\int_{\Gamma_1} h(\theta)e^{in\theta}d\theta = 0 , \qquad n = 1, 2, \cdots .$$

It follows that, on considering L^∞ as the conjugate space of L^1 (setting $f \cdot K = \int_{\Gamma_1} fKd\theta$ for $f \in L^\infty$, $K \in L^1$), H is closed in the weak* topology. Since the bounded analytic functions in S_1 form an algebra, passing to boundary values shows that H is an algebra.

Denote by $\|\ \|$, $\|\ \|_1$, the norms in L^∞ and L^1 respectively. For a subspace E of L^∞, $\|\phi\|_E$ denotes the norm of the functional ϕ considered in E only.

The first (well known) lemma applies to $L^\infty(X)$ for any measure space X.

2.1. LEMMA. *Suppose ϕ is a bounded linear functional on L^∞ such that*

$$\phi(1) = \|\phi\| . \tag{2.2}$$

Then ϕ is real and nonnegative; that is, $\phi(f)$ is real if f is real a.e., and $\phi(f) \geqq 0$ if $f \geqq 0$ a.e. Moreover,

$$\phi(Re\, f) = Re\, \phi(f) , \qquad f \in L^\infty . \tag{2.3}$$

Proof. The case $\phi = 0$ being trivial, we suppose $\|\phi\| > 0$; we may normalize so that $\phi(1) = \|\phi\| = 1$. Take any $f \in L^\infty$. Now

$$Re\, \phi(f) = Re\, \phi(f + A) - A \leqq \|f + A\| - A$$

for any real constant A; letting $A \to \infty$ gives

$$Re\,\phi(f) \leqq \operatorname{ess\,sup}\{Re\,f(\theta)\}\ .$$

Replacing f by $-f$ gives

$$Re\,\phi(f) \geqq \operatorname{ess\,inf}\{Re\,f(\theta)\}\ .$$

Hence if f is purely imaginary, $Re\,\phi(f) = 0$, and $\phi(f)$ is purely imaginary; it follows that if f is real then $\phi(f)$ is real. The last inequality shows that ϕ is nonnegative. Finally, if $f = f_1 + if_2$ then $\phi(f) = \phi(f_1) + i\phi(f_2)$, and (2.3) follows.

To state that a function has bounded real part is to state that its values lie in a vertical strip. We approximate to such functions by bounded functions by means of the following lemma.

We consider a half closed vertical strip and half the unit disk;

$$T^+ = \left\{\lambda\colon 0 \leqq Re\,\lambda < \frac{\pi}{4}\right\}, \qquad S_1^+ = \{z\colon Re\,z \geqq 0, |z| < 1\}\ .$$

2.4. LEMMA. *The map $\lambda \to \tan\lambda$ carries T^+ conformally onto S_1^+. Moreover,*

(2.5) $$Re\arctan(\alpha\tan\lambda) \leqq Re\,\lambda \ \textit{ if } \lambda \in T^+, \alpha \textit{ real},\ 0 \leqq \alpha \leqq 1\ .$$

Proof. The first statement is standard. To prove the second, it is sufficient to show that Re arctan $\alpha\beta$, regarded as a function of the real variable α, is monotone increasing provided that $Re\,\beta > 0$. Since

$$c = [(1 + \alpha^2\beta^2)(1 + \alpha^2\bar{\beta}^2)]^{-1} > 0\ ,$$

we have

$$\begin{aligned}\frac{\partial}{\partial\alpha}Re\arctan\alpha\beta &= Re\frac{\partial}{\partial\alpha}\arctan\alpha\beta = Re\frac{\beta}{1+\alpha^2\beta^2}\\ &= cRe[\beta(1+\alpha^2\bar{\beta}^2)] = cRe[\beta + \alpha^2|\beta|^2\bar{\beta}] > 0\ ,\end{aligned}$$

as required.

3. Approximation from below.

We can now prove

3.1. LEMMA. *Given the real nonnegative function f in L^∞, there exists a sequence $\{h_n\}$ of functions in H such that*

(3.2) $$\textit{for all } n,\ 0 \leqq Re\,h_n \leqq f \quad \textit{a.e.}\ ,$$

(3.3) $$Re\,h_n \to f \ \textit{ in } L^1\ .$$

Proof. We may assume $f(\theta) < \pi/4$ a.e. The harmonic extension $\hat{f}$ is the real part of an analytic function g in S_1 where $g(0)$ is real. Since $0 \leqq Re\, g(z) < \pi/4$, g maps S_1 into T^+. Set

$$g_n(z) = \arctan [(1 - 1/n) \tan g(z)] . \tag{3.4}$$

Clearly g_n is a bounded analytic function in S_1. Let h_n be the boundary value function of g_n. By (2.5), $Re\, g_n(z) \leqq Re\, g(z)$, and (3.2) follows. Moreover,

$$\begin{aligned} \| f - Reh_n \|_1 &= \int_{\Gamma_1} (f - Reh_n) d\theta \\ &= 2\pi[Reg(0) - Reg_n(0)] = 2\pi[g(0) - g_n(0)] \end{aligned}$$

by the mean value theorem for harmonic functions. Since arctan is continuous in the real interval [0, 1], this tends to 0, and (3.3) is proved.

4. The theorem for nonnegative kernels.

The theorem in this case is:

4.1. THEOREM. *Suppose ϕ is a bounded linear functional on L^∞ such that*

$$\| \phi \| = \| \phi \|_H \tag{4.2}$$

and

$$\phi(f) = \int_{\Gamma_1} f(\theta) K(\theta) d\theta \tag{4.3}$$

for all f in H, where K is a real nonnegative function in L^1. Then (4.3) holds for all $f \in L^\infty$.

Proof. By (4.3), $|\phi(h)| \leqq \|h\| \cdot \|K\|_1$, $h \in H$; hence (since $1 \in H$)

$$\| \phi \| = \| \phi \|_H \leqq \| K \|_1 = \phi(1) \leqq \| \phi \| .$$

Therefore $\phi(1) = \| \phi \|$, and Lemma 2.1 applies.

Suppose $f \in L^\infty$ is real and nonnegative. Choose the sequence $\{h_n\}$ by Lemma 3.1. By Lemma 2.1, $\phi(f) \geqq \phi(\mathrm{Re}\, h_n) = \mathrm{Re}\, \phi(h_n)$, and hence

$$\int fK - \phi(f) \leqq \int fK - Re \int h_n K = \int (f - Re\, h_n) K .$$

Now $K \in L^1$, the functions $f - Re\, h_n$ are uniformly essentially bounded, and $f - Re\, h_n \to 0$ in L^1; it is elementary to show therefore that $\int (f - Re\, h_n) K \to 0$. Thus $\phi(f) \geqq \int fK$ for real nonnegative $f \in L^\infty$.

If $f \in L^\infty$ is real, pick a real number α so that $\alpha < f(\theta)$ a.e. Then

$$\phi(f) = \phi(f - \alpha) + \alpha\phi(1) \geqq \int (f - \alpha)K + \alpha \int K = \int fK .$$

Replacing f by $-f$ gives the opposite inequality; hence (4.3) holds for real f. Since ϕ is linear, (4.3) holds also for complex f.

5. The general theorem for the unit circle. Given any linear functional ψ on L^∞ and any measurable subset Z of Γ_1 with characteristic function χ_Z, define

$$\psi_Z(f) = \psi(\chi_Z f) . \tag{5.1}$$

5.2. LEMMA. *Let ψ be a bounded linear functional on L^∞, and let X and Y be complementary measurable subsets of Γ_1. Then*

$$\| \psi \| = \| \psi_X \| + \| \psi_Y \| . \tag{5.3}$$

Proof. Since $\psi = \psi_X + \psi_Y$, $\| \psi \| \leqq \|\psi_X \| + \| \psi_Y \|$. Conversely, given $\varepsilon > 0$, choose g_1 and g_2 in L^∞ so that $\psi_X(g_1)$ and $\psi_Y(g_2)$ are real, $\| g_1 \| = \| g_2 \| = 1$, and

$$\psi_X(g_1) \geqq \| \psi_X \| - \varepsilon , \qquad \psi_Y(g_2) \geqq \| \psi_Y \| - \varepsilon .$$

Set $g = \chi_X g_1 + \chi_Y g_2$. Then $\| g \| \leqq 1$, and

$$\psi(g) = \psi_X(g_1) + \psi_Y(g_2) \geqq \| \psi_X \| + \| \psi_Y \| - 2\varepsilon ,$$

giving the opposite inequality.

5.4. THEOREM. *Suppose that ψ is a linear functional on L^∞, K is in L^1,*

$$\| \psi \| = \| \psi \|_H = \| K \|_1 , \tag{5.5}$$

and

$$\psi(f) = \int_{\Gamma_1} f(\theta) K(\theta) d\theta \tag{5.6}$$

for all $f \in H$. Then (5.6) *holds for all $f \in L^\infty$.*

Proof. Since the unit ball B in $L^\infty = (L^1)^*$ is weak* compact (see for instance [2], p. 37) and H is weak* closed, $H \cap B$ is weak* compact. Because of (5.6), ψ is weak* continuous in H; hence $\psi(H \cap B)$ is compact, and there is an $h_0 \in H$ such that

$$\| h_0 \| \leqq 1 , \qquad \psi(h_0) = \| \psi \|_H . \tag{5.7}$$

The equality, with (5.6) and (5.5), gives

$$\int_{\Gamma_1} h_0(\theta)K(\theta)d\theta = \int_{\Gamma_1} |K(\theta)|\,d\theta\,, \tag{5.8}$$

from which we conclude that

(5.9) $h_0(\theta)K(\theta)$ is real and nonnegative a.e., and

(5.10) for almost all θ, either $|h_0(\theta)| = 1$ or $K(\theta) = 0$.

We remark aside that h_0 is unique a.e. (unless $\psi = 0$); for a second such element h_1 would agree a.e. with h_0 wherever $K \neq 0$, by (5.9) and (5.10), hence $h_1 = h_0$ on a set of positive measure, and it is known that this implies that $h_1 = h_0$ a.e. (see for instance [6], vol. II, p. 203).

Set

$$\phi(f) = \psi(h_0 f)\,, \qquad f \in L^\infty\,. \tag{5.11}$$

Since H is a ring,

$$\phi(h) = \psi(h_0 h) = \int h h_0 K\,, \qquad h \in H\,.$$

Also $\|h_0\| \leqq 1$, $\|h_0 f\| \leqq \|f\|$, and hence $\|\phi\| \leqq \|\psi\|$. Now

$$\|\phi\| \leqq \|\psi\| = \|K\|_1 = \phi(1) \leqq \|\phi\|_H \leqq \|\phi\|\,,$$

giving $\|\phi\| = \|\phi\|_H$. By Theorem 4.1,

$$\psi(h_0 f) = \phi(f) = \int f h_0 K \text{ for all } f \in L^\infty\,.$$

If h_0 had a bounded inverse h_1, we could use this to give $\psi(f) = \psi(h_0 h_1 f) = \int fK$; but this need not be the case, as we shall see by an example in 10.4.

By (5.10), $h_0 \bar{h}_0 K = K$ a.e.; hence replacing f by $\bar{h}_0 f$ in the above equation gives

$$\psi(h_0 \bar{h}_0 f) = \int fK \text{ for all } f \in L^\infty\,. \tag{5.12}$$

Let X be the subset of Γ_1 where K vanishes, and let Y be the complementary subset. Then $h_0 \bar{h}_0 = 1$ a.e. on Y. In the notation of Lemma 5.2,

$$\psi_Y(f) = \psi(\chi_Y f) = \psi(\chi_Y h_0 \bar{h}_0 f) = \int \chi_Y f K = \int fK$$

for all $f \in L^\infty$, whence, with the aid of (5.5) and (5.3),

$$\|\psi_Y\| = \|K\|_1 = \|\psi\| = \|\psi_X\| + \|\psi_Y\|\,.$$

Therefore $\psi_X = 0$, and

$$\psi(f - h_0\bar{h}_0 f) = \psi(\chi_X(f - h_0\bar{h}_0 f)) = \psi_X(f - h_0\bar{h}_0 f) = 0$$

for all $f \in L^\infty$. Thus (5.12) gives (5.6) for all $f \in L^\infty$, as claimed.

We wish to give a theorem without the hypothesis on the norm of the kernel. For this, the given kernel must be replaced by an "extremal kernel". This is possible, by the following lemma of Rogosinski and Shapiro, [4], Theorem 8, p. 303. (A proof will be given in 8.9.)

5.13. LEMMA. *Suppose ψ is a linear functional defined on H by $\psi(h) = \int hJ\,(h \in H)$, where $J \in L^1$. Then there is a function $K \in L^1$, unique a.e., such that $\psi(h) = \int hK\ (h \in H)$ and $\|K\|_1 = \|\psi\|_H$.*

Combining the last two results gives

5.14. THEOREM. *Suppose ψ is a bounded linear functional on H such that*

$$\psi(h) = \int_{\Gamma_1} h(\theta)J(\theta)d\theta \text{ for all } h \in \mathrm{H}\,, \tag{5.15}$$

where $J \in L^1$. Then ψ has a unique norm preserving extension ψ_e over L^∞. Moreover, there is a $K \in L^1$, unique a.e., such that

$$\psi_e(f) = \int_{\Gamma_1} f(\theta)K(\theta)d\theta \text{ for all } f \in L^\infty\,, \tag{5.16}$$

we have $\|K\|_1 = \|\psi\|_H = \|\psi_e\|$. If $\|J\|_1 = \|\psi\|_H$, then $K = J$ a.e.

6. Bordered Riemann surfaces. We now turn our attention to the analogous problem for a compact bordered Riemann surface $\bar{S}$. The theorems are similar to those of the preceding sections and the methods of proof are essentially the same, but we must take account of the fact that not every harmonic function is the real part of an analytic function. In this section we recall some of the facts about harmonic and analytic functions on $\bar{S}$.

6.1. A bordered Riemann surface $\bar{S}$ is a connected surface carrying an oriented conformal structure which is everywhere locally isomorphic to a relatively open subset of the closed upper half plane, these isomorphisms being given by the local coordinate functions (see for instance [3]). Let S denote the interior of $\bar{S}$, and Γ, the border; S is a Riemann surface in the ordinary sense. We assume always that $\bar{S}$ is compact and Γ is not empty; then Γ is a real analytic manifold with a natural orientation, consisting of a finite number of disjoint Jordan curves.

6.2. For any real continuous function f on Γ there is a unique harmonic function $\hat{f}$ in S which extends f continuously over $\bar{S}$ (see also 6.7). For each point $p \in S$ there is a real and everywhere positive analytic differential ω_p on Γ such that, for all f as above,

$$\hat{f}(p) = \int_\Gamma f\omega_p \,. \tag{6.3}$$

Choose $p_0 \in S$; this point will be fixed throughout the paper. The differential ω_{p_0} defines a measure μ on Γ (called the harmonic measure for p_0). Expressing ω_p in terms of ω_{p_0}, (6.3) becomes

$$\hat{f}(p) = \int_\Gamma f(q)k(p, q)d\mu(q) \,. \tag{6.4}$$

For each $p \in S$, $k(p, q)$ is an everywhere positive real analytic function on Γ. Furthermore, k is continuous in $S \times \Gamma$, and for each fixed q, is harmonic in S.

6.5. Measure theoretic statements will always refer to the measure μ. We form as usual the spaces L^1 and L^∞ of integrable and of bounded complex valued measurable functions respectively, and use $\| \;\|_1$ and $\| \;\|$ for the corresponding norms. Let L^1_{Re} and L^∞_{Re} denote the spaces of real functions in L^1 and L^∞ respectively. To reduce the notation we shall not distinguish between measurable functions and their equivalence classes modulo null sets. With the pointwise operations of addition and multiplication, the Banach space L^∞ becomes a Banach algebra.

As in § 2, L^∞ is the conjugate space of L^1, under the definition $f \cdot K = \int_\Gamma fK d\mu$. Again, the closed unit ball B in L^∞ is weak* compact, as is $B_{Re} = B \cap L^\infty_{Re}$ in L^∞_{Re}. Since L^1 is separable, the weak* topology of B is metrizable ([1], p. 426), so that questions of convergence in B can be handled with sequences and the Bolzano-Weierstrass theorem applies.

6.6. The space L^1, defined through the choice of p_0, is not intrinsically related to $\bar{S}$. This defect may be remedied as follows. Let C be the space of continuous functions on Γ. An analytic differential ω on Γ determines a linear functional ϕ_ω on C by the formula

$$\phi_\omega(f) = \int_\Gamma f\omega \,.$$

The norm is given by $\| \phi_\omega \| = \int_\Gamma | \omega |$. Let M be the closure, in the conjugate C^* of C, of the set of all ϕ_ω. Clearly, M is intrinsically related to $\bar{S}$. It is easy to identify L^1 with M, since any differential form on Γ may be written as a multiple of ω_{p_0} (see 6.2). The choice of

p_0 enables us to interpret elements of M as functions on Γ; this is valuable because of the corresponding identification of M^* with L^∞ and because the analytic properties of k may be used. Since the null sets on Γ do not depend on the choice of p_0, the space L^∞ is intrinsic a priori; it is easy to check that its identification with M^* is independent of the choice of p_0.

6.7 For each complex function $f \in L^\infty$, its harmonic extension $\hat{f}$ through S may be defined by (6.4); it is bounded by $||f||$, and has the boundary value functions f a.e. on Γ.

6.8. LEMMA. *Let f and the sequence $\{f_n\}$ be in L^∞. If $f_n \to f$ weak*, then $\hat{f}_n \to \hat{f}$ pointwise. If the $\hat{f}_n$ are uniformly bounded and $\hat{f}_n \to \hat{f}$ pointwise, then $f_n \to f$ weak*.*

The first statement follows from (6.4). Conversely, suppose the $\hat{f}_n$ are uniformly bounded, $\hat{f}_n \to \hat{f}$ in S, but $f_n \to f$ weak* is false. Since λB in the weak* topology is metrizable and compact for any real λ, we may find a subsequence $\{f_{m_i}\}$ converging weak* to g, where $||f - g|| > 0$. Then $\hat{f}_{m_i} \to \hat{g}$ in S, hence $\hat{g} = \hat{f}$ in S, and hence $g = f$ a.e. on Γ, a contradiction.

6.9. As in the case of the unit circle, H is the space of boundary values of bounded analytic functions in S. Again, for any $f \in L^\infty$, $f \in H$ if and only if $\hat{f}$ is analytic in S, or (setting $f = f_1 + if_2$ with f_1 and f_2 real) $\hat{f}_2$ is a conjugate of $\hat{f}_1$. As before, H is a subalgebra of L^∞.

6.10. LEMMA. *H is weak* closed in L^∞.*

Proof. By [2], p. 39, it is sufficient to show that $H \cap B$ is weak* closed. As noted in 6.5, convergence in B is sequential. Suppose that $\{h_n\}$ is a sequence in $H \cap B$ which converges to $f \in L^\infty$ weak*. By Lemma 6.8, $\hat{h}_n \to \hat{f}$ in S. Since the $\hat{h}_n$ are uniformly bounded, $\hat{f}$ is analytic, by a general convergence theorem; hence $f \in H$, as required.

6.11. Let $f \in L^\infty$. We form the harmonic extension $\hat{f}$, its differential $d\hat{f}$, and the conjugate harmonic differential $(d\hat{f})^*$. Suppose $\hat{f}$ had a conjugate function $\hat{f}^*$. Then if ν is a Jordan arc from p_0 to p_1 in S, with normal vector n, we would have

$$\hat{f}^*(p_1) - \hat{f}^*(p_0) = \int_\nu \frac{\partial \hat{f}^*(p)}{\partial s} ds = \int_\nu \frac{\partial \hat{f}(p)}{\partial n} ds \; ;$$

substituting in (6.4) and changing the order of integration gives

$$\hat{f}^*(p_1) - \hat{f}^*(p_0) = \int_\Gamma f(q)\Phi_\nu(q)d\mu(q) \text{ if } \hat{f}^* \text{ exists,} \tag{6.12}$$

where

$$\Phi_\nu(q) = \int_\nu \frac{\partial k(p, q)}{\partial n} ds , \tag{6.13}$$

the integral being defined with the help of the local conformal structure of S.

6.14. In the general case, $\hat{f}^*$ exists if and only if $\int_\Gamma f\Phi_\delta d\mu = 0$ for each closed curve δ in S, or equivalently, for each closed curve of a homology basis. Let $\delta_1, \cdots, \delta_\rho$ be such a basis. Here ρ is the first Betti mumber of S or $\bar{S}$, and

$$\rho = 1 - \chi = 2g + m - 1 ,$$

where χ is the Euler characteristic of $\bar{S}, g$ is its genus, and m is the number of components of Γ.

Now $\hat{f}^*$ exists if and only if the periods

$$\int_{\delta_i} (d\hat{f})^* = \int_\Gamma f(q)\alpha_i(q)d\mu(q) , \qquad \alpha_i(q) = \Phi_{\delta_i}(q) , \tag{6.15}$$

all vanish. The α_i are real analytic on Γ and are therefore in L^1.

6.16. Let P and P_{Re} be the complex and real spaces respectively spanned by the α_i; then $P_{Re} = P \cap L^1_{Re}$.

Let $c_1, \cdots, c_\rho$ be given real numbers. There exists an analytic differential ω on $\bar{S}$ which can be extended analytically over the double of $\bar{S}$ and which has the periods $ic_1, \cdots, ic_\rho$ (see [3], p. 172). Since the periods of $Re\ \omega$ vanish there is a harmonic function $\hat{f}$ in $\bar{S}$ with $d\hat{f} = Re\,\omega$; now $\hat{f}$ is bounded in $\bar{S}$, and $(d\hat{f})^* = Im\,\omega$ has the periods $c_1, \cdots, c_\rho$, It follows from (6.15) that $\alpha_1, \cdots, \alpha_\rho$ are linearly independent over the reals. Therefore P_{Re} has real dimension ρ, and P has complex dimension ρ.

6.17. Considering $P \subset L^1$, let $G \subset L^\infty$ be its annihilator space. Then G is weak* closed and has codimension ρ in L^∞. Hence every linear functional on L^∞ which vanishes on G has the form $f \to \int_\Gamma f\alpha$ for some $\alpha \in P$. Moreover $G_{Re} = G \cap L^\infty_{Re}$ is the annihilator of P_{Re} in L^∞_{Re}, and $G = G_{Re} + iG_{Re}$. The definitions of P and G show that for any $f \in L^\infty$, $\hat{f}$ has a harmonic conjugate (not necessarily bounded) if and only if $f \in G$. It follows that $H \subset G$.

7. The theorem with an extremal kernel. Given the linear functional ϕ on H, defined by means of an extremal kernel K, we show here that any norm preserving extension over L^∞ is given by means of a modified kernel, and we study the set of norm preserving extensions. As before, the proof is based on an approximation lemma. Recall the definition of G_{Re} in 6.17.

7.1. LEMMA. *Given the nonnegative function $f \in G_{Re}$, there exists a sequence $\{h_n\}$ of functions in H such that $0 \leqq Reh_n \leqq f$ a.e. in Γ and $Reh_n \to f$ in L^1.*

Proof. As in § 3, assume $f(q) < \pi/4$ a.e. in Γ. Since $f \in G_{Re}$, we may find g, g_n and h_n as before. Then $0 \leqq Re\, h_n \leqq f$, and that $Re\, h_n \to f$ in L^1 follows from

$$\| f - Re\, h_n \|_1 = \int_\Gamma (f - Re\, h_n) d\mu = Re\, g(p_0) - Re\, g_n(p_0) \to 0 \,.$$

7.2. THEOREM. *Suppose ϕ is a linear functional on L^∞ such that $\| \phi \| = \| \phi \|_H$ and*

$$\phi(f) = \int_\Gamma fK d\mu \tag{7.3}$$

for all $f \in H$, where $K \in L^1$ is real and nonnegative. Then (7.3) *holds for all $f \in G$. Moreover, there exists a function $\alpha \in P_{Re}$ such that $K + \alpha$ is nonnegative and*

$$\phi(f) = \int_\Gamma f(K + \alpha) d\mu \text{ for all } f \in L^\infty \,. \tag{7.4}$$

Proof. The first statement follows as in the proof of Theorem 4.1, with Lemma 7.1 replacing Lemma 3.1.

To derive the second statement, set

$$\psi(f) = \phi(f) - \int_\Gamma fK d\mu \,, \qquad f \in L^\infty \,;$$

then $\psi(g) = 0$ for all $g \in G$, and (see 6.17) there is an $\alpha \in P$ such that

$$\psi(f) = \int_\Gamma f\alpha d\mu \qquad f \in L^\infty \,.$$

These two relations give (7.4). Since $\phi(1) = \| \phi \|$ (compare § 4), ϕ is real and nonnegative (Lemma 2.1); hence $K + \alpha$ is real and nonnegative.

7.5. THEOREM. *Let ϕ be a linear functional defined on H by* (7.3), *where $K \in L^1$ is real and nonnegative. Then the set of norm preserving* *extensions of* ϕ *over* L^∞ *is in one to one correspondence with the set* W *of functions* $\alpha \in P_{Re}$ *such that* $K + \alpha$ *is nonnegative. The set* W *is a compact convex cell of real dimension at most* ρ, *the first Betti number of* $\bar{S}$.

Proof. By the last theorem, every norm preserving extension corresponds to some $\alpha \in W$; clearly this correspondence is one to one. It is onto W. For suppose $\alpha \in W$. Then if $\phi_e(f) = \int_\Gamma f(K + \alpha)d\mu$, $\phi_e = \phi$ over H, since $H \subset G$. Since $1 \in G$,

$$\| K + \alpha \|_1 = \int (K + \alpha) d\mu = \int K d\mu = \| K \|_1 \ ;$$

therefore $\| \phi_e \| = \| \phi \|$, and ϕ_e is a norm preserving extension of ϕ corresponding to α.

Clearly W is a closed convex subset of P_{Re}, and hence is of dimension $\leqq \rho$. Since $\| K + \alpha \|_1 = \| K \|_1$, W is bounded and hence compact.

7.6. THEOREM. *Suppose that* ψ *is a linear functional defined on* H *by*

$$\psi(h) = \int_\Gamma hK d\mu \ , \tag{7.7}$$

where $K \in L^1$ *and* $\| K \|_1 = \| \psi \|_H$. *Then there is a function* $h_0 \in H$ *such that every norm preserving extension* ψ_e *of* ψ *over* L^∞ *has the form*

$$\psi_e(f) = \int_\Gamma f(K + \alpha \bar{h}_0) d\mu, \ \textit{where} \ K + \alpha \bar{h}_0 \in L^1 \ , \tag{7.8}$$

for some $\alpha \in P_{Re}$. *(See also* (7.14), (7.15) *and* (7.12).)

Proof. We follow the proof of Theorem 5.4 (using 6.5 and 6.10), finding the extremal function h_0 for ψ such that

$$\| h_0 \| = 1 \ , \qquad \psi(h_0) = \| \psi \|_H \ , \qquad \int h_0 K d\mu = \int | K | d\mu \ . \tag{7.9}$$

Again we conclude that for almost all $q \in \Gamma$,

$$h_0(q)K(q) \geqq 0 \text{ and either } | h_0(q) | = 1 \text{ or } K(q) = 0 \ . \tag{7.10}$$

Given ψ_e, set

$$\phi(f) = \psi_e(h_0 f) \ , \qquad f \in L^\infty \ .$$

Then

$$\phi(h) = \psi_e(h_0 h) = \int_\Gamma h h_0 K d\mu \,, \qquad h \in H \,,$$

since $h_0 h \in H$. Also, as in § 5, $\| \phi \| = \| \psi \|_H = \| \phi \|_H$. Therefore for some $\alpha \in P_{Re}$,

$$\psi_e(h_0 f) = \phi(f) = \int_\Gamma f(h_0 K + \alpha) d\mu \,, \qquad f \in L^\infty \,, \tag{7.11}$$

by Theorem 7.2, and

$$h_0 K + \alpha \geqq 0 \text{ a.e.}, \ \| h_0 K + a \|_1 = \| \phi \| = \| \psi \|_H \,. \tag{7.12}$$

Since $h_0 \not\equiv 0, h_0 = 0$ at most in a set a measure 0 (see [6], vol. II, p. 203); hence there is a measurable function g on Γ such that $g h_0 = 1$ a.e. on Γ. Set

$$E_n = \{ q \in \Gamma : | g(q) | \leqq n \} \,,$$

let χ_n be the characteristic function of E_n, and set $\psi_n(f) = \psi_e(f \chi_n)$. By Lemma 5.2, which extends to the present case,

$$\| \psi \|_H = \| \psi_e \| = \| \psi_n \| + \| \psi_e - \psi_n \| \,. \tag{7.13}$$

Since $g f \chi_n \in L^\infty$, (7.11) gives

$$\begin{aligned} \psi_n(f) = \psi_e(g h_0 f \chi_n) &= \int_\Gamma g f \chi_n (h_0 K + \alpha) d\mu \\ &= \int_{E_n} f(K + \alpha g) d\mu \,, \qquad f \in L^\infty \,. \end{aligned}$$

Hence

$$\int_{E_n} | K + \alpha g | d\mu = \| \psi_n \| \leqq \| \psi_e \| \,,$$

and letting $n \to \infty$ gives

$$K + \alpha g \in L^1 \,, \qquad K + \alpha g \|_1 \leqq \| \psi_e \| \,.$$

Therefore, with the help of (7.9) and (7.11),

$$\begin{aligned} \| \psi_e \| = \| \psi \|_H = \psi(h_0) &= \int_\Gamma (h_0 K + \alpha) d\mu \\ &= \int_\Gamma h_0 (K + \alpha g) d\mu \leqq \int_\Gamma | K + \alpha g | d\mu \leqq \| \psi_e \| \,, \end{aligned}$$

and

$$\lim_{n \to \infty} \| \psi_n \| = \int_\Gamma | K + \alpha g | d\mu = \| \psi_e \| \,.$$

Now (7.13) gives $\lim \| \psi_e - \psi_n \| = 0$; consequently

$$\psi_e(f) = \lim_{n\to\infty} \psi_n(f) = \int_\Gamma f(K + \alpha g)d\mu\ , \qquad f \in L^\infty\ .$$

From the equalities $\| h_0 \| = 1$ and $\int h_0(K + \alpha g)d\mu = \int | K + \alpha g \,| \,d\mu$ we conclude that for almost all $q \in \Gamma$, either $| h_0(q) | = 1$ or $K(q) + \alpha(q)g(q) = 0$. Comparing this with (7.10) shows that

(7.14) $\qquad$ a.e. in Γ, either $| h_0(q) | = 1$ or $K(q) = \alpha(q) = 0$.

Since $gh_0 = 1$ a.e., we have $\alpha g = \alpha \bar{h}_0$ a.e., and the theorem is proved.

Note that $Kh_0\bar{h}_0 = K$ a.e., by (7.10); hence (7.8) may be written in the form

(7.15) $$\psi_e(f) = \int_\Gamma f\bar{h}_0(h_0K + \alpha)d\mu \text{ for all } f \in L^\infty\ .$$

8. **Existence of extremal kernels.** In order to obtain a result analogous to Theorem 5.14 for bordered Riemann surfaces we must extend the lemma of Rogosinski and Shapiro 5.13 to the more general setting. Our first object is to prove that, if A is the space of complex valued, real-analytic functions on Γ, then $H \cap A$ is weak* dense in H. Note that $H \cap A$ is the space of restrictions to Γ of functions holomorphic on $\bar{S}$.

8.1. LEMMA. *Let E be any real or complex Banach space and let B be the closed unit ball in its conjugate space E^*. Suppose G is a weak* closed linear subspace of E^* of finite codimension. Suppose $0 < \lambda \leqq 1$ and that A is a linear subspace of E^* such that $A \cap B$ is weak* dense in λB. Then $A \cap G \cap B$ is weak* dense in $G \cap \lambda B$.*

Proof. Since G is weak* closed, G is the annihilator of a certain closed linear manifold $P \subset E$. Since G has finite codimension in E^*, say m, P has dimension m. Elements of A are linear functionals on P by restriction, and indeed a total set of linear functionals, for if not there would be a $p \in P$ such that $p \neq 0$ and $a(p) = 0$ for all $a \in A$ and $A \cap B$ would not be weak* dense in λB. Hence we can find an m-dimensional linear subspace $F \subset A$ which is total on P. Then F is a linear complement of G in E^*.

Now let g_0 be any element of $G \cap \lambda B$. We must show that g_0 is the weak* limit of some directed system in $A \cap G \cap B$. By hypothesis there is a directed system $\{a_\alpha\}$ in $A \cap B$ such that $a_\alpha \to g_0$ weak*. Write $a_\alpha = f_\alpha + g_\alpha$ where $f_\alpha \in F$, $g_\alpha \in G$. Then $g_\alpha \in A \cap G$. For any $p \in P$, $a_\alpha(p) \to g_0(p) = 0$; hence $f_\alpha(p) \to 0$. Since F is finite dimensional and the elements of P determine a total set of linear functionals on F, this implies that $f_\alpha \to 0$ in the norm. Hence $g_\alpha \to g_0$ weak*. Now put

$\xi_\alpha = (1 + \|f_\alpha\|)^{-1}$. Then $\xi_\alpha \to 1$ so $\xi_\alpha g_\alpha \to g_0$ weak*. Moreover, $\|\xi_\alpha g_\alpha\| = \xi_\alpha \|a_\alpha - f_\alpha\| \leqq \xi_\alpha(1 + \|f_\alpha\|) = 1$, so $\xi_\alpha g_\alpha \in A \cap G \cap B$. This completes the proof.

For the next lemma, let A be the space defined at the beginning of § 8, and set $A_{Re} = A \cap L^\infty_{Re}$.

8.2. LEMMA. *Let $g \in G_{Re}$ (defined in 6.17). Then there exists a sequence $\{h_n\}$ in $H \cap A$ such that*

$$\|Re\, h_n\| \leqq \|g\|\ , \tag{8.3}$$

$$Re\, h_n \to g \text{ weak* in } L^\infty\ . \tag{8.4}$$

Proof. L^∞_{Re} is the real conjugate space of L^1_{Re} and G_{Re} is the annihilator of the finite dimensional subspace P_{Re} of L^1_{Re}. It is well known that $A_{Re} \cap B_{Re}$ is weak* dense in B_{Re}, where B_{Re} is the unit ball of L^∞_{Re}. By Lemma 8.1, $A_{Re} \cap G_{Re} \cap B_{Re}$ is weak* dense in $G_{Re} \cap B_{Re}$.

To prove the lemma we may clearly suppose that $\|g\| = 1$. As noted in 6.5, the weak* topology of B_{Re} is metrizable, so there is a sequence $\{f_n\}$ in $A_{Re} \cap G_{Re} \cap B_{Re}$ such that $f_n \to g$ weak*. Since $f_n \in G_{Re}$, $\hat{f}_n$ has a conjugate function, and $\hat{f}_n = Re\, \hat{h}_n$ where $\hat{h}_n$ is a holomorphic function on S. On the other hand, since $\hat{f}_n$ has real analytic boundary values, $\hat{h}_n$ is analytic on $\bar{S}$. Thus $h_n \in H \cap A$ and $Re\, h_n = f_n$. Now (8.4) has already been established and (8.3) follows because $f_n \in B_{Re}$ so that $\|f_n\| \leqq 1 = \|g\|$.

We can immediately deduce the following fact which may be useful in other connections.

8.5. THEOREM. *Let $\bar{H} = \{\bar{h} : h \in H\}$. Then $H + \bar{H}$ is weak* dense in G.*

Proof. Let M be the weak* closure of $H + \bar{H}$. Since $H \subset G$ and $\bar{H} \subset G$ while G is weak* closed, $M \subset G$. On the other hand, by the last lemma, every $g \in G_{Re}$ is the weak* limit of a sequence $\{1/2(h_n + \bar{h}_n)\}$ in $H + \bar{H}$, so $g \in M$. Thus $M \supset G_{Re}$. By linearity $M \supset G_{Re} + iG_{Re} = G$.

8.6. LEMMA. *$H \cap A$ is weak* dense in H. Indeed, $H \cap A \cap B$ is sequentially weak* dense in $H \cap B$.*

Proof. Given $h \in H \cap B$ we must find a sequence in $H \cap A \cap B$ which converges weak* to h. If h is constant, this is trivial, so we suppose not. Then arctan $\hat{h}$ maps S holomorphically into the open strip $T = \{\lambda : -\pi/4 < Re\, \lambda < \pi/4\}$. Now Re arctan $\hat{h}$ has a boundary value function $g \in G_{Re}$; by Lemma 8.2 we can choose a sequence $\{h_n\}$ in $H \cap A$ so that

$$\begin{cases} \| Re\, h_n \| \leqq \pi/4\ , \qquad Re\, h_n \to g \text{ weak}^*\ , \\ Im\, \hat{h}_n(p_0) = Im \arctan \hat{h}(p_0). \end{cases} \tag{8.7}$$

Since $h_n \in H \cap A$ and tan is holomorphic in $\bar{T}$, tan $h_n \in H \cap A \cap B$.

Given $p \in S$, let ν be an arc from p_0 to p; now (6.12) and (8.7) give

$$\begin{aligned} Im\, \hat{h}_n(p) - Im\, \hat{h}_n(p_0) &= \int_\Gamma Re\, h_n \Phi_\nu d\mu \\ &\to \int_\Gamma g\Phi_\nu d\mu = \hat{g}^*(p) - \hat{g}^*(p_0)\ ; \end{aligned}$$

using (8.7) again gives $Im\, \hat{h}_n(p) \to Im$ arctan $\hat{h}(p)$. Also, by Lemma 6.8, $Re\, \hat{h}_n(p) \to Re \arctan \hat{h}(p)$; thus $\hat{h}_n(p) \to \arctan \hat{h}(p)$. Therefore $\tan \hat{h}_n(p) \to \hat{h}(p)$ in S, and by Lemma 6.8, $\tan h_n \to h$ weak*, completing the proof.

We shall need the following extension of a theorem of F. and M. Riesz.

8.8. Lemma. *Let C be the space of continuous complex-valued functions on Γ. A Borel measure on Γ which is orthogonal to $H \cap C$ is absolutely continuous with respect to any harmonic measure.*

For a proof see Wermer [5] Lemma 3. Because of the context in which he is working, Wermer's statement includes the hypothesis that Γ consists of a single Jordan arc, but this hypothesis is not used in the proof.

We are finally in a position to extend the lemma of Rogosinski and Shapiro.

8.9. Lemma. *Suppose ψ is a linear functional defined on H by*

$$\psi(h) = \int_\Gamma hJd\mu \qquad \text{for all } h \in H\ ,$$

where $J \in L^1$. Then there exists a function K in L^1 such that $\| K \|_1 = \| \psi \|_H$ and

$$\psi(h) = \int_\Gamma hKd\mu \qquad \text{for all } h \in H\ . \tag{8.10}$$

Proof. Following [4], we consider ψ on $H \cap C$ and take a norm preserving extension over C. The extension can be represented by a Borel measure ν. Now $\| \nu \| = \| \psi \|_{H \cap C}$ and $\psi(h) = \int hd\nu$ for $h \in H \cap C$. Hence $\int h(d\nu - Jd\mu) = 0$ for $h \in H \cap C$. By Lemma 8.8, we may write $d\nu - Jd\mu = Md\mu$ where $M \in L^1$. Put $K = J + M \in L^1$ and we have $d\nu = Kd\mu$ and $\| \nu \| = \| K \|_1$.

The linear functional $h \to \int h(K-J)d\mu$ vanishes for $h \in H \cap C$. Since $H \cap A \subset H \cap C$ is weak* dense in H (Lemma 8.6), $\int h(K-J)d\mu = 0$ for all $h \in H$, giving 8.10. Moreover, $\|\psi\|_H \leqq \|K\|_1 = \|\nu\| = \|\psi\|_{H \cap C} \leqq \|\psi\|_H$. This gives $\|K\|_1 = \|\psi\|_H$ and completes the proof.

9. The final theorem. Combining Lemma 8.9 with the theorems of § 7 gives

9.1. THEOREM. *Let $\bar{S}$ be a compact bordered Riemann surface with non-empty boundary Γ and first Betti number ρ. Let H be the space of boundary value functions of bounded analytic functions defined on S. Suppose ψ is a bounded linear functional defined on H by*

$$\psi(h) = \int_\Gamma h(q)J(q)d\mu(q) \qquad \textit{for all } h \in H, \tag{9.2}$$

where μ is the harmonic measure on Γ associated with some point $p_0 \in R$, and J is in $L^1(\mu)$. Then every norm preserving extension ψ_e of ψ over L^∞ has the form

$$\psi_e(f) = \int_\Gamma f(q)K(q)d\mu(q) \qquad \textit{for all } f \in L^\infty, \tag{9.3}$$

where $K \in L^1(\mu)$. The set of all such extensions forms a compact convex cell of dimension at most ρ; the extension is unique if $\rho = 0$.

The first part of the theorem follows directly from Lemma 8.9 and Theorem 7.6.

By 7.6 again, the set of norm preserving extensions corresponds to the set W of those functions $\alpha \in P_{Re}$ such that insertion in (7.8) or in (7.15) gives such an extension. As in the proof of Theorem 7.5, W is a compact subset of P_{Re}; this space has dimension ρ.

To prove convexity, suppose

$$\alpha_1, \alpha_2 \in W, \qquad 0 < t < 1, \qquad \alpha_3 = (1-t)\alpha_1 + t\alpha_2;$$

let ψ_i be determined by (7.15), using α_i. That ψ_3 is an extension of ψ follows at once. Moreover, by (7.12), $h_0K + \alpha_i \geqq 0$ a.e. ($i = 1, 2$); hence $h_0K + \alpha_3 \geqq 0$ a.e., and (7.15) and (7.14) give

$$\|\psi_3\| = \int_\Gamma |\bar{h}_0(h_0K + \alpha_3)| d\mu = \int_\Gamma (h_0K + \alpha_3)d\mu;$$

the similar formulas hold for $\|\psi_1\|$ and $\|\psi_2\|$. Since $\|\psi_1\| = \|\psi_2\| = \|\psi\|_H$, we have $\|\psi_3\| = \|\psi\|_H$, which completes the proof.

10. Examples. We give here some particular cases where non-uniqueness or uniqueness holds, and an example concerning the proof of Theorem 5.4.

10.1. We give first the example promised in the introduction.

Suppose ψ is a nonzero bounded linear functional on L^∞ which vanishes on H and takes real values on L^∞_{Re}. By the Hahn decomposition theorem ψ may be represented as the difference $\psi^+ - \psi^-$ of two nonnegative linear functionals; here, ψ^+ is defined for nonnegative f by

$$\psi^+(f) = \sup\{\psi(g) : 0 \leqq g \leqq f\}\ ,$$

and is extended over the rest of L^∞ by linearity. Since ψ^+ and ψ^- are real and nonnegative, $\|\psi^+\| = \psi^+(1)$ and $\|\psi^-\| = \psi^-(1)$; since $1 \in H$,

$$\|\psi^+\|_H = \|\psi^+\|\ , \qquad \|\psi^-\|_H = \|\psi^-\|\ .$$

Furthermore, $\psi^+ = \psi^-$ over H, since ψ vanishes in H; thus ψ^+ and ψ^- are distinct norm preserving extensions over L^∞ of the same linear functional over H.

To complete the example we must construct ψ with the given properties. Let ν be an arc of the unit circle, and let ν' be its complement. Let f_0 equal 1 on ν and equal -1 on ν'. Let $Re\, H$ be the set of real parts of functions in H; we shall show that f_0 is at a distance 1 from $Re\, H$. If this were not so, we could find $f \in Re\, H$ with $\|f - f_0\| = 1 - \varepsilon$, $\varepsilon > 0$. Now $f \geqq \varepsilon$ on ν and $f \leqq -\varepsilon$ on ν'; the expression, in terms of f, of the conjugate $\hat{f}^*$ of the harmonic extension $\hat{f}$ of f over the unit disk shows that $\hat{f}^*$ is unbounded near the ends of ν, so $f \notin Re\, H$, a contradiction. By the Hahn-Banach extension theorem there is a bounded real linear functional on L^∞_{Re} which vanishes on $Re\, H$ but not at f_0. This extends by complex linearity over L^∞ to give the required functional ψ.

10.2. To illustrate the theorem for Riemann surfaces, let $\bar{S}$ be the annulus $\{z : a \leqq |z| \leqq b\}$, where $0 < a < b$. Here Γ consists of two oppositely oriented circles, and the first Betti number ρ of $\bar{S}$ is 1. A function $f \in L^\infty_{Re}$ is in G_{Re} if and only if $\int_\Gamma f d\theta = 0$; using Lebesgue measure μ on Γ, this shows that P is spanned by the single function α on Γ, defined by

$$\alpha(z) = \begin{cases} 1/b \text{ if } |z| = b\ , \\ -1/a \text{ if } |z| = a\ . \end{cases}$$

Let K_0 be the constant function 1, defining the linear functional ϕ

on H. Then for any real t,

$$K_t(z) = K_0(z) + t\alpha(z) \geqq 0 \text{ on } \Gamma \text{ if } -b \leqq t \leqq a\ ,$$

but this fails for other values of t. By Theorem 7.5, the set of norm preserving extensions of ϕ corresponds to this set of values of t, and is thus one dimensional.

On the other hand, if for instance $K_1(z) = |Re\, z|$ on Γ, then K_1 is nonnegative but $K_1 + t\alpha$ takes on negative values if t is real and $\neq 0$; by Theorem 7.5, the linear functional on H determined by K_1 has a unique norm preserving extension over L^∞.

10.3. More generally, let $\bar{S}$ be any compact bordered Riemann surface with first Betti number $\rho > 0$. Suppose ϕ is the linear functional defined on H by a real kernel K_0 such that $K_0 \geqq a > 0$ on Γ. Then for all α in P_{Re} such that $\|\alpha\| \leqq a$, $K_0 + \alpha$ is nonnegative, so that (Theorem 7.5) the set of extensions has real dimension ρ.

On the other hand, we shall show that (with measure μ as in 6.2) there is an $\varepsilon > 0$ such that if Q is a measurable subset of Γ with $\mu(Q) < \varepsilon$ and K is the characteristic function of Q, then the linear functional over H defined by K has a unique norm preserving extension over L^∞.

To this end, note that every nonzero α in P_{Re} is continuous and takes on both positive and negative values (since $1 \in H$ and hence $\int 1 \cdot \alpha d\mu = 0$). Let $\alpha_1, \cdots, \alpha_\rho$ be a basis for P_{Re}, and let D denote the set of n-tuples $v = (v_1, \cdots, v_\rho)$ with $\Sigma v_i^2 = 1$. For each $v \in D$, set

$$U_v = \{q \in \Gamma : \Sigma v_i \alpha_i(q) < 0\}\ , \qquad \mu_v = \mu(U_v)\ .$$

Clearly the function μ_v on D is lower semicontinuous; since D is compact, μ_v takes on its lower bound $\varepsilon > 0$. Now if $Q \subset \Gamma$, $\mu(Q) < \varepsilon$, then every $\alpha \neq 0$ in P takes on negative values outside of Q; hence (see Theorem 7.5) the characteristic function K of Q has the required property.

10.4. We show that there is a kernel on the unit circle Γ_1 whose extremal function has a zero on Γ_1 and hence does not have a bounded inverse; this was promised in the proof of Theorem 5.4. Through conformal mapping, we can replace the unit disk by the half disk $S^+ = \{z : Re\, z \geqq 0, |z| \leqq 1\}$, its boundary Γ replacing Γ_1. (We are now in the setting of Theorem 7.6.)

Define K on Γ by

$$K(z) = \begin{cases} \bar{z} & \text{if } |z| = 1\ ,\ Re\, z > 0\ , \\ 0 & \text{if } Re\, z = 0\ ,\ |Im\, z| \leqq 1\ . \end{cases}$$

Set $h_0(z) = z$ in S^+. If ψ is the linear functional on L^∞ defined by K, using Lebesgue measure μ on Γ,

$$\psi(h_0) = \int_\Gamma zK d\mu = \pi = \int_\Gamma K d\mu = \| K \|_1 \, ;$$

hence h_0 is the extremal function for K. But $0 \in \Gamma$ and $h_0(0) = 0$.

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