DIRECT DECOMPOSITIONS WITH FINITE DIMENSIONAL FACTORS

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The principal results. A fundamental theorem of Ore [10] states that if an element in a finite dimensional modular lattice is represented in two ways as a direct join of indecomposable elements, then the factors of the two decompositions are projective in pairs. The Krull-Schmidt theorem is an immediate consequence of this result. Subsequently many authors have considered direct decompositions in modular lattices. In particular, Kurosh [8, 9] and Baer [1, 2] obtained conditions which imply the existence of projective refinements of two direct decompositions of an element in an upper continuous modular lattice. When applied to the decompositions of a group G, the conditions of Kurosh and Baer are reflected in certain chain conditions on the center of G. In a somewhat different direction, Zassenhaus [11] has shown that the representation of an operator group as a direct product of arbitrarily many indecomposable groups each with a principal series is unique up to isomorphism.

This paper studies the direct decompositions of an element in an upper continuous modular lattice under the assumption that the element has at least one decomposition with finite dimensional factors. It is then shown that every other decomposition of the element refines to one with finite dimensional factors, and that a strong exchange isomorphism exists between two decompositions with indecomposable factors. This latter result sharpens the uniqueness result of Zassenhaus.

Before explicitly stating the principal results, let us note the following definitions. A lattice L is *upper continuous* if L is complete and

$$a \cap \bigcup_{k \in K} x_k = \bigcup_{k \in K} a \cap x_k$$

for every element $a \in L$ and every *chain* of elements x_k $(k \in K)$ in L.

If a and a_i $(i \in I)$ are elements of a complete lattice L with a null element 0, then a is said to be a *direct join* of the elements a_i $(i \in I)$, in symbols

$$a = \dot{\bigcup_{i \in I}} a_i ,$$

if $a = \bigcup_{i \in I} a_i$, and for each index $h \in I$ we have $a_h \cap \bigcup_{i \neq h} a_i = 0$. The direct join of finitely many elements $a_1, \cdots, a_n$ is also denoted by $a_1 \dot{\cup} \cdots \dot{\cup} a_n$. An element b is called a *direct factor* of a if $a = b \dot{\cup} x$ for some element x. An element a is *indecomposable* if $a \neq 0$ and $a =$

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$x \dot{\cup} y$ implies $x = 0$ or $y = 0$. Finally, an element a is said to be *finite dimensional* if every chain of elements less than a is finite.

THEOREM 1. *If a is an element of an upper continuous modular lattice and*

$$a = x \dot{\cup} y = \dot{\bigcup_{i \in I}} t_i$$

where x is finite dimensional and indecomposable, then there exists an index $h \in I$ such that $t_h = r \dot{\cup} s$ and

$$a = r \dot{\cup} y = x \dot{\cup} s \dot{\cup} \dot{\bigcup_{i \neq h}} t_i \, .$$

THEOREM 2. *If a is an element of an upper continuous modular lattice and a is a direct join of finite dimensional elements, then every direct factor of a is also a direct join of finite dimensional elements.*

THEOREM 3. *If a is an element of an upper continuous modular lattice and*

$$a = \dot{\bigcup_{i \in I}} a_i = \dot{\bigcup_{j \in J}} b_j$$

where each a_i $(i \in I)$ and each b_j $(j \in J)$ is finite dimensional and indecomposable, then there is a one-to-one mapping φ of I onto J such that

$$a = a_i \dot{\cup} \dot{\bigcup_{j \neq \varphi(i)}} b_j$$

for each index $i \in I$.

These theorems may be applied directly to the lattice of admissible normal subgroups of an operator group to yield the following extension of the result of Zassenhaus mentioned above. *If an operator group G is a direct product $G \cong \prod_{i \in I} A_i$ where each of the factors A_i $(i \in I)$ has a principal series, then any two direct decompositions of G have centrally isomorphic refinements.*

Even with the strong continuity assumption it seems impossible to relax the assumption of finite dimensionality particularly in Theorems 1 and 3. The free abelian group of rank 2 shows that in general Theorems 1 and 3 fail for lattices satisfying only the ascending chain condition. The example in the following paragraph shows that continuity and the descending chain condition also are not sufficient for these results. It

[1] Cf. Jónsson and Tarski [6].

is curious that Theorems 1 and 3 hold for groups whose normal subgroup lattices satisfy only the descending chain condition[1] and yet fail for general continuous modular lattices satisfying the descending chain condition.

The example is as follows. Let p be an odd prime, and let G be an additive abelian group isomorphic with

$$Z(p) \times Z(p) \times Z(p^\infty) \times Z(p^\infty) \times Z(p^\infty) \times Z(p^\infty) ,$$

where $Z(p)$ denotes the cyclic group of order p and $Z(p^\infty)$ denotes the generalized cyclic p-group. Let $Q, R, S, T, U,$ and V be subgroups of G with $Q \cong R \cong Z(p)$, $S \cong T \cong U \cong V \cong Z(p^\infty)$, and

$$G = Q \dot{\cup} R \dot{\cup} S \dot{\cup} T \dot{\cup} U \dot{\cup} V .$$

Let q and r generate Q and R respectively, and let $S, T, U,$ and V be generated respectively by sets $\{s_n\}, \{t_n\}, \{u_n\}$, and $\{v_n\}$, where

$$ps_1 = 0,\ ps_{n+1} = s_n\, (n = 1, 2, \cdots) ,$$

with analogous relations holding for the t_n's, u_n's and v_n's. Set $A = Q \cup S \cup T$ and $B = R \cup U \cup V$. Let C be the subgroup generated by the set $\{q + r, s_1, s_2 + u_1, s_3 + u_2, \cdots, v_1, v_2 + t_1, v_3 + t_2, \cdots\}$, and let D be the subgroup generated by $\{q + 2r, u_1, u_2 + s_1, u_3 + s_2, \cdots, t_1, t_2 + v_1, t_3 + v_2, \cdots\}$. It then follows that

$$G = A \dot{\cup} B = C \dot{\cup} D ,$$

and $G = A \cup B = A \cup C = A \cup D = B \cup C = B \cup D = C \cup D$. Furthermore, $A \cap C, A \cap D, B \cap C, B \cap D \neq 0$. Now let L be the set of all subgroups $X \leqq S \cup T \cup U \cup V$, together with all subgroups of the form $A \cup X, B \cup X, C \cup X$, and $D \cup X$ with $X \leqq S \cup T \cup U \cup V$, and the group G. It is easily checked that under set inclusion the elements of L form a complete sublattice of the lattice of all subgroups of G. Hence L is an upper continuous modular lattice satisfying the descending chain condition. Moreover, the subgroups $A, B, C,$ and D are indecomposable in L, and each is projective only with itself. Thus Theorems 1 and 3 fail for the direct joins $G = A \dot{\cup} B = C \dot{\cup} D$.

Proofs of the theorems. The usual notation and terminology is used throughout. Lattice join, meet, inclusion, and proper inclusion are denoted respectively by $\cup, \cap, \leqq$, and $<$. If a and b are elements of a lattice and $b \leqq a$, then the quotient sublattice $\{x \mid b \leqq x \leqq a\}$ is denoted by a/b. The symbol $\cong$ denotes the isomorphism of two lattices. The null element of a lattice is always denoted by 0.

We begin with the following lemmas. The first is generally known.

LEMMA 1. *If L is an upper continuous lattice, S is a subset of L, and a is any element of L, then*

$$a \cap \bigcup S = \bigcup_{F \in \mathscr{F}} a \cap \bigcup F ,$$

where $\mathscr{F}$ is the collection of all finite subsets of S.

The lemma is trivial when S is finite. Suppose that S is an infinite subset of L, and suppose that the lemma is true for every subset S' of cardinality less than the cardinality of S. Then there is a chain S_i $(i \in I)$ of subsets of S such that each S_i has cardinality less than that of S and such that S is the set-sum of the subsets $S_i (i \in I)$. If $\mathscr{F}_i$ is the collection of all finite subsets of S_i, applying upper continuity and the inductive assumption we therefore have

$$\begin{aligned} a \cap \bigcup S &= a \cap [\bigcup_{i \in I} \bigcup S_i] = \bigcup_{i \in I} [a \cap \bigcup S_i] \\ &= \bigcup_{i \in I} \bigcup_{F \in \mathscr{F}_i} [a \cap \bigcup F] = \bigcup_{F \in \mathscr{F}} a \cap \bigcup F , \end{aligned}$$

and hence the lemma follows by induction.

An element c in a complete lattice L is said to be *compact* if for every subset $S \subseteqq L$ with $c \leqq \bigcup S$ there is a finite subset $S' \subseteqq S$ such that $c \leqq \bigcup S'$. A lattice L is *compactly generated* if L is complete and every element of L is a join of compact elements.[2] The next lemma is an immediate consequnce of the definition of compactness.

LEMMA 2. *If $\{c_1, \cdots, c_n\}$ is a finite set of compact elements in a complete lattice, then $c_1 \cup \cdots \cup c_n$ is also compact.*

LEMMA 3. *Every finite dimensional element in an upper continuous lattice is compact.*

We shall first show that if q is completely join irreducible, then q is compact. Suppose $S \subseteqq L$ and $q \leqq \bigcup S$. Let $p = \bigcup \{x \mid x < q\}$. Then $p < q$ since q is completely join irreducible. Let $\mathscr{F}$ denote the collection of all finite subsets of S. If $\bigcup F \ngeqq q$ for every $F \in \mathscr{F}$, then $q \cap \bigcup F < q$ and hence $q \cap \bigcup F \leqq p$ for every $F \in \mathscr{F}$. And it follows by Lemma 1 that

$$q = q \cap \bigcup S = \bigcup_{F \in \mathscr{F}} q \cap \bigcup F \leqq p ,$$

a contradiction. Hence q is compact.

Now suppose that a is a finite dimensional element different from

[2] For a discussion of compactly generated lattices see [4].

0 and suppose that every element properly contained in a is compact. If a is join irreducible, then a is compact from above. If a is not join irreducible, then there are two elements $b, c < a$ such that $a = b \cup c$. Since b and c are compact, a is therefore compact, and the lemma follows by induction.

LEMMA 4. *If an element a of an upper continuous modular lattice is a join of finite dimensional elements, then the quotient sublattice $a/0$ is compactly generated, and each compact element is finite dimensional.*

For suppose $a = \bigcup C$ where each $c \in C$ is finite dimensional. If $x \leqq a$, then with $\mathscr{F}$ denoting the set of all finite subsets of C we have

$$x = x \cap \bigcup C = \bigcup_{F \in \mathscr{F}} x \cap \bigcup F .$$

Since the lattice is modular, $x \cap \bigcup F$ is finite dimensional and hence compact for each $F \in \mathscr{F}$. The lemma now follows.

LEMMA 5. *If $c, a_1, a_2, \cdots, a_n$ are elements of a compactly generated lattice, c is compact, and $c \leqq a_1 \cup \cdots \cup a_n$, then for each $m = 1, \cdots, n$ there is a compact element $d_m \leqq a_m$ such that $c \leqq d_1 \cup \cdots \cup d_n$.*

Since the lattice is compactly generated, for each $m = 1, \cdots, n$ there is a set C_m of compact elements such that $a_m = \bigcup C_m$. Then $c \leqq \bigcup C_1 \cup \cdots \cup \bigcup C_n$, and since c is compact there are finite subsets $C'_m \subseteqq C_m$ such that $c \leqq \bigcup C'_1 \cup \cdots \cup \bigcup C'_n$. By Lemma 2, $\bigcup C'_m$ is a compact element for each $m = 1, \cdots, n$.

LEMMA 6. *If a, x, y are elements of a modular lattice, $x \cup y = x \dot{\cup} y$, and $x \leqq a \leqq x \cup y$, then $a = x \dot{\cup} (a \cap y)$.*

For $x \cap (a \cap y) = x \cap y = 0$, and $x \cup (a \cap y) = a \cap (x \cup y) = a$.

Proof of Theorem 1. Suppose a, x, y, t_i $(i \in I)$ are elements of an upper continuous modular lattice, x is finite dimensional and indecomposable, and

$$a = x \dot{\cup} y = \dot{\bigcup_{i \in I}} t_i .$$

Since x is compact by Lemma 2, there is a finite subset of indices $\{i_1, \cdots, i_n\} \subseteqq I$ such that $x \leqq t_{i_1} \dot{\cup} \cdots \dot{\cup} t_{i_n}$. For each $m = 1, 2, \cdots, n$ let us set

$$\bar{t}_m = t_{i_1} \cup \cdots \cup t_{i_{m-1}} \cup t_{i_{m+1}} \cup \cdots \cup t_{i_n} ,$$

and define $x_m = (x \cup \bar{t}_m) \cap t_{i_m}$. Then it follows that[3]

$$x \leqq b = x_1 \dot{\cup} \cdots \dot{\cup} x_n \, .$$

Now $x_m \cap \bar{t}_m = (x \cup \bar{t}_m) \cap t_{i_m} \cap \bar{t}_m = t_{i_m} \cap \bar{t}_m = 0$, and

$$x_m \cup \bar{t}_m = [(x \cup \bar{t}_m) \cap t_{i_m}] \cup \bar{t}_m = (t_{i_m} \cup \bar{t}_m) \cap (x \cup \bar{t}_m) = x \cup \bar{t}_m \, .$$

Thus $x_m/0 = x_m/x_m \cap \bar{t}_m \cong x_m \cup \bar{t}_m/\bar{t}_m = x \cup \bar{t}_m/\bar{t}_m \cong x/x \cap \bar{t}_m$, and hence each x_m is finite dimensional, and its dimension does not exceed the dimension of x. It follows that $b = x_1 \dot{\cup} \cdots \dot{\cup} x_n$ is finite dimensional. Since $x \leqq b \leqq x \dot{\cup} y$, we infer from Lemma 6 that

$$b = x \dot{\cup} (b \cap y) \, .$$

Therefore, since x is indecomposable and the dimension of each x_m is at most the dimension of x, it follows from Ore's theorem[4] that (renumbering the x_m's if necessary)

$$b = x_1 \dot{\cup} (b \cap y) = x \dot{\cup} x_2 \dot{\cup} \cdots \dot{\cup} x_n \, .$$

Then $y \cup x_1 = y \cup (b \cap y) \cup x_1 = y \cup b = a$. From the fact that x_1 is finite dimensional and $x_1/x_1 \cap y \cong x_1 \cup y/y = y \cup b/y \cong b/y \cap b \cong x_1/0$, it follows that $x_1 \cap y = 0$. Thus

$$a = x_1 \dot{\cup} y \, .$$

Moreover, since $x_1 \leqq t_{i_1}$, it follows from Lemma 6 that

$$t_{i_1} = x_1 \dot{\cup} (y \cap t_{i_1}) \, .$$

Let us set

$$t^*_{i_1} = \dot{\bigcup_{i \neq i_1}} t_i \, .$$

Then since $x_2 \cup \cdots \cup x_n \leqq t_{i_2} \cup \cdots \cup t_{i_n} \leqq t^*_{i_1}$ we have

$$\begin{aligned} x \cup [(y \cap t_{i_1}) \dot{\cup} t^*_{i_1}] &= x \cup x_2 \cup \cdots \cup x_n \cup (y \cap t_{i_1}) \cup t^*_{i_1} \\ &= b \cup (y \cap t_{i_1}) \cup t^*_{i_1} = t_{i_1} \cup t^*_{i_1} = a \, , \end{aligned}$$

and since $x/x \cap [(y \cap t_{i_1}) \cup t^*_{i_1}] \cong a/(y \cap t_{i_1}) \cup t^*_{i_1} \cong x_1/0 \cong x/0$, it follows that $x \cap [(y \cap t_{i_1}) \dot{\cup} t^*_{i_1}] = 0$. Hence

$$a = x \dot{\cup} [(y \cap t_{i_1}) \dot{\cup} t^*_{i_1}] = x \dot{\cup} (y \cap t_{i_1}) \dot{\cup} \dot{\bigcup_{i \neq i_1}} t_i \, ,$$

[3] See for example [3, p. 95].

[4] Actually we use the somewhat stronger version of Ore's theorem given in [5, pp. 128–130].

and the proof of Theorem 1 is complete.

Proof of Theorem 2. Throughout the proof of Theorem 2 we will assume that a is an element of an upper continuous modular lattice and

$$a = \dot{\bigcup_{i \in I}} a_i$$

where each a_i $(i \in I)$ is finite dimensional and indecomposable.

Suppose $a = r \dot{\cup} s$. We shall first show that r and s are direct joins of elements which are joins of a countable number of compact elements.[5]

Consider the collection $\mathscr{P}$ of all subsets P of the lattice which satisfy the following conditions:

$$\text{(1)} \qquad \bigcup P = \dot{\bigcup_{t \in P}} t = \dot{\bigcup_{i \in K}} a_i$$

for some subset $K \subseteqq I$.

(2) $t = (t \cap r) \dot{\cup} (t \cap s)$ for each $t \in P$.

(3) $t \cap r$ and $t \cap s$ are both joins of a countable number of compact elements for each $t \in P$.

$\mathscr{P}$ is nonempty since the null set is in $\mathscr{P}$. Moreover, since by Lemma 1 a set is independent if every finite subset is independent, it follows that the set-sum of a chain of sets in $\mathscr{P}$ also belongs to $\mathscr{P}$. By the Maximal Principle $\mathscr{P}$ contains a maximal element Q.

Set

$$q = \dot{\bigcup} Q = \dot{\bigcup_{i \in M}} a_i \,, \quad u = \dot{\bigcup_{t \in Q}} (t \cap r) \,, \quad v = \dot{\bigcup_{t \in Q}} (t \cap s) \,.$$

Then it follows from condition (2) that $q = u \dot{\cup} v$, and from condition (1) that $a = q \dot{\cup} b = u \dot{\cup} v \dot{\cup} b$ where $b = \bigcup_{i \in I-M} a_i$. Furthermore, if we set $r' = r \cap (b \cup v)$ and $s' = s \cap (b \cup v)$, then it follows from Lemma 6 that $r = r' \dot{\cup} u$ and $s = s' \dot{\cup} v$. Hence

$$a = r' \dot{\cup} s' \dot{\cup} q \,.$$

Suppose $q \neq a$. Then for some $i_0 \in I$ we must have $a_{i_0} \nleqq q$. Since a_{i_0} is compact and $a/0$ is compactly generated by Lemma 3, it follows by Lemma 5 that compact elements $c_1 \leqq r'$ and $d_1 \leqq s'$ exist such that

$$a_{i_0} \leqq c_1 \cup d_1 \cup q \,.$$

$c_1 \cup d_1$ is also compact, and hence there is a finite subset $M_1 \subseteqq I$ such

[5] The proof of this part was suggested by the main theorem of Kaplansky [7].

that

$$c_1 \cup d_1 \leqq \bigcup_{i \in M_1} a_i \, .$$

Again $\bigcup_{i \in M_1} a_i$ is compact, and hence there are compact elements $c_2 \leqq r'$ and $d_2 \leqq s'$ such that

$$\bigcup_{i \in M_1} a_i \leqq c_2 \cup d_2 \cup q \, .$$

Continuing in this way we get a sequence $\{i_0\}$, $M_1, M_2, \cdots, M_n, \cdots$ of finite subsets of I and two sequences of compact elements $c_1, c_2, \cdots, c_n, \cdots \leqq r'$ and $d_1, d_2, \cdots, d_n, \cdots \leqq s'$ such that

$$c_n \cup d_n \leqq \bigcup_{i \in M_n} a_i \leqq c_{n+1} \cup d_{n+1} \cup q$$

for each $n = 1, 2, \cdots$.

Let

$$r^* = \bigcup_{n < \infty} c_n \, , \quad s^* = \bigcup_{n < \infty} d_n \, .$$

Then $r^* \leqq r'$ and $s^* \leqq s'$, and if M^* is the set-sum of the sets M, $\{i_0\}$, $M_1, M_2, \cdots$, it is clear that

$$t^* = r^* \cup s^* = r^* \dot{\cup} s^* \, ,$$

and

$$t^* \cup q = t^* \dot{\cup} q = \bigcup_{i \in M^*} a_i \, .$$

Hence the set-sum of Q and $\{t^*\}$ is a member of $\mathscr{P}$ properly containing Q. Since this is contrary to the maximality of Q, we must have $q = a$. It follows that $r = u$ and $s = v$, and thus r and s are direct joins of elements which are joins of a countable number of compact elements.

We now prove the following: if b is a direct factor of a and c is a compact element with $c \leqq b$, then there exists a finite dimensional direct factor w of b such that $c \leqq w$. Suppose $a = b \dot{\cup} e$. Since c is compact, there is a finite subset $\{i_1, \cdots, i_n\} \subseteqq I$ such that $f = a_{i_1} \cup \cdots \cup a_{i_n} \geqq c$. Applying Theorem 1 to the element a_{i_1} and the decompositions

$$a = a_{i_1} \dot{\cup} \dot{\bigcup_{i \neq i_1}} a_i = b \dot{\cup} e \, ,$$

it follows that $b = b_1' \dot{\cup} b_1''$, $e = e_1' \dot{\cup} e_1''$ (where either $b_1'' = 0$ or $e_1'' = 0$), and

$$a = b_1'' \dot{\cup} e_1'' \dot{\cup} \dot{\bigcup_{i \neq i_1}} a_i = a_{i_1} \dot{\cup} b_1' \dot{\cup} e_1' \, .$$

Now consider the direct decompositions

$$a = a_{i_2} \dot{\cup} \dot{\bigcup_{i \neq i_2}} a_i = a_{i_1} \dot{\cup} b_1' \dot{\cup} e_1' .$$

If we apply Theorem 1 to the element a_{i_2} and these decompositions, then since $a_{i_1} \cap \bigcup_{i \neq i_2} a_i = a_{i_1} > 0$, it follows that $b_1' = b_2' \dot{\cup} b_2''$, $e_1' = e_2' \dot{\cup} e_2''$, and

$$a = b_2'' \dot{\cup} e_2'' \dot{\cup} \dot{\bigcup_{i \neq i_2}} a_i = a_{i_2} \dot{\cup} a_{i_1} \dot{\cup} b_2' \dot{\cup} e_2' .$$

Repeating this replacement for each a_{i_k} we conclude that for every $k = 1, \cdots, n$ there exist elements $b_k', b_k'' \leqq b$ and $e_k', e_k'' \leqq e$ such that

$$a = b_k'' \dot{\cup} e_k'' \dot{\cup} \dot{\bigcup_{i \neq i_k}} a_i = a_{i_k} \dot{\cup} \cdots \dot{\cup} a_{i_1} \dot{\cup} b_k' \dot{\cup} e_k' .$$

In particular

$$a = f \dot{\cup} b_n' \dot{\cup} e_n' .$$

Let $w = b \cap (e_n' \cup f)$. Then w is finite dimensional, and $w \geqq b \cap f \geqq c$. Moreover, Lemma 6 implies that $b = b_n' \dot{\cup} w$, and the assertion follows.

In view of what has been proved above, to complete the proof of Theorem 2 it suffices to show that if b is a direct factor of a, and b is a join of a countable number of compact elements, then b is a direct join of finite dimensional elements. To this end, suppose

$$b = \bigcup_{n < \infty} c_n$$

where c_n is compact for each $n = 1, 2, \cdots$. Then it follows from the preceding paragraph that elements w_1 and v_1 exist such that w_1 is finite dimensional, $w_1 \geqq c_1$, and

$$b = w_1 \dot{\cup} v_1 .$$

Since c_2 is compact and $a/0$ is compactly generated, by Lemma 5 there is a compact element $d_1 \leqq v_1$ such that $c_2 \leqq w_1 \cup d_1$. Now v_1 is a direct factor of a, and again applying the result of the preceding paragraph we obtain elements w_2 and v_2 such that w_2 is finite dimensional, $w_2 \geqq d_1$, and $v_1 = w_2 \dot{\cup} v_2$. Thus $c_2 \leqq w_1 \cup w_2$, and

$$b = w_1 \dot{\cup} w_2 \dot{\cup} v_2 .$$

Continuing in this way we get a sequence of finite dimensional elements $w_1, w_2, \cdots, w_n, \cdots \leqq b$ such that

$$w_1 \cup \cdots \cup w_n = w_1 \dot{\cup} \cdots \dot{\cup} w_n \geqq c_n$$

for each $n = 1, 2, \cdots$. Thus the set $\{w_n | n = 1, 2, \cdots\}$ is independent since every finite subset is independent, and hence

$$b = \dot{\bigcup_{n<\infty}} w_n .$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Let a be an element of an upper continuous modular lattice and suppose that

$$(1) \qquad a = \dot{\bigcup_{i\in I}} a_i = \dot{\bigcup_{j\in J}} b_i$$

where each a_i $(i \in I)$ and each b_j $(j \in J)$ is finite dimensional and indecomposable. We shall show the following: *there exists a well-ordering* $(\prec)$ *on the index set I and a one-to-one mapping* φ *of I onto J such that for each index* $h \in I$ *we have*

$$a = \dot{\bigcup_{i\leqq h}} b_{\varphi(i)} \mathbin{\dot\cup} \dot{\bigcup_{i>h}} a_i = a_h \mathbin{\dot\cup} \dot{\bigcup_{j\neq\varphi(h)}} b_j .$$

Let $\mathscr{P}$ be the collection of all ordered triples $(H, \prec, \psi)$, where $H \subseteqq I$, $(\prec)$ is a well-ordering of H, ψ is a one-to-one mapping of H into J, and such that the following conditions are satisfied:

(i) for each index $h \in H$ we have

$$\begin{aligned} a &= \dot{\bigcup_{i\in H, i\leqq h}} b_{\psi(i)} \mathbin{\dot\cup} \dot{\bigcup_{i\in H, i\succ h}} a_i \mathbin{\dot\cup} \dot{\bigcup_{i\in I-H}} a_i \\ &= a_h \mathbin{\dot\cup} \dot{\bigcup_{j\neq\psi(h)}} b_j ; \end{aligned}$$

(ii) $\bigcup_{i\in H} a_i \leqq \bigcup_{i\in H} b_{\psi(i)}$.

Partially order $\mathscr{P}$ by defining $(H', \prec', \psi') \geqq (H, \prec, \psi)$ if and only if $H = H'$ or there is an element $h' \in H'$ such that

$$H = \{i \in H' | i \prec' h'\} ,$$

$(\prec')$ on H coincides with $(\prec)$, and ψ', restricted to H coincides with ψ. Note that $\mathscr{P}$ is nonempty since it contains the triple $(\phi, \prec^0, \psi^0)$ where ϕ, $\prec^0$, and ψ^0 are respectively the empty set, relation, and mapping.

Suppose that $(H^\sigma, \prec^\sigma, \psi^\sigma)$ $(\sigma \in \Sigma)$ is a chain of elements in $\mathscr{P}$. Let $\bar{H}$ be the set-sum of the subsets $H^\sigma(\sigma \in \Sigma)$. Define a well-ordering $(\prec)$ on $\bar{H}$ by $i \prec i'$ if and only if $i, i' \in H^\sigma$ and $i \prec^\sigma i'$ for some $\sigma \in \Sigma$. And define the mapping $\bar{\psi}$ on $\bar{H}$ into J by $\bar{\psi}(i) = \psi^\sigma(i)$ where $i \in H^\sigma$. Then it is easily verified that $(\bar{H}, \prec, \bar{\psi}) \in \mathscr{P}$ and that $(\bar{H}, \prec, \bar{\psi})$ is an upper bound of the chain $(H^\sigma, \prec^\sigma, \psi^\sigma)$ $(\sigma \in \Sigma)$. Thus by the Maximal Principle $\mathscr{P}$ contains a maximal element $(M, \prec, \varphi)$.

Now the set-sum of $\{b_{\varphi(i)} | i \in M\}$ and $\{a_i | i \in I - M\}$ is independent since by (i) every finite subset is independent. Therefore it follows from (ii) that

$$a = \dot{\bigcup_{i \in M}} b_{\varphi(i)} \mathbin{\dot\cup} \dot{\bigcup_{i \in I-M}} a_i \,. \tag{2}$$

Suppose that $M \neq I$. This implies that $\varphi(M) \neq J$. Pick an index $j_0 \in J - \varphi(M)$. Then applying Theorem 1 to the element b_{j_0} and the direct decompositions (1) and (2), it follows that an index $i_0 \in I - M$ exists such that

$$a = b_{j_0} \mathbin{\dot\cup} \dot{\bigcup_{i \in M}} b_{\varphi(i)} \mathbin{\dot\cup} \dot{\bigcup_{i \notin M, i \neq i_0}} a_i = a_{i_0} \mathbin{\dot\cup} \dot{\bigcup_{j \neq j_0}} b_j \,. \tag{3}$$

The element a_{i_0} is compact, and hence there is a finite subset $J_0 \subseteqq J$ such that

$$a_{i_0} \leqq \bigcup_{j \in J_0} b_j \,.$$

Let M_0 be the set-sum of M and $\{i_0\}$, and let $\{j_1, \cdots, j_m\}$ denote the subset of J_0 consisting of those indices different from j_0 which are *not* contained in $\varphi(M)$. Then repeated application of Theorem 1 yields the following: there exist m distinct indices $i_1, \cdots, i_m \in I - M_0$ such that for each $n = 1, \cdots, m$ we have

$$\begin{aligned} a &= b_{j_n} \mathbin{\dot\cup} b_{j_{n-1}} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} b_{j_0} \dot{\bigcup_{i \in M}} b_{\varphi(i)} \mathbin{\dot\cup} \dot{\bigcup_{i \notin M, i \neq i_0, \cdots, i_n}} a_i \\ &= a_{i_n} \mathbin{\dot\cup} \dot{\bigcup_{j \neq j_n}} b_j \,. \end{aligned} \tag{4}$$

Again the element $a_{i_1} \cup \cdots \cup a_{i_m}$ is compact, and therefore a finite subset $J_1 \subseteqq J$ exists such that

$$a_{i_1} \cup \cdots \cup a_{i_m} \leqq \bigcup_{j \in J_1} b_j \,.$$

Let M_1 denote the set-sum of M and $\{i_0, \cdots, i_m\}$, and let $\{j_{m+1}, \cdots, j_p\}$ denote the subset of indices contained in J_1 but *not* contained in either $\varphi(M)$ or $\{j_0, \cdots, j_m\}$. Applying Theorem 1 repeatedly to the elements $b_{j_{m+1}}, \cdots, b_{j_p}$ it follows that indices $i_{m+1}, \cdots, i_p \in I - M_1$ exist such that

$$\begin{aligned} a &= b_{j_n} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} b_{j_{m+1}} \mathbin{\dot\cup} b_{j_m} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} b_{j_0} \mathbin{\dot\cup} \dot{\bigcup_{i \in M}} b_{\varphi(i)} \mathbin{\dot\cup} \dot{\bigcup_{i \notin M_1, i \neq i_{m+1}, \cdots, i_n}} a_i \\ &= a_{i_n} \mathbin{\dot\cup} \dot{\bigcup_{j \neq j_n}} b_j \end{aligned}$$

for each $n = m + 1, \cdots, p$. We may continue this procedure obtaining two sequences of indices $i_0, i_1, \cdots, i_n, \cdots$ in I and $j_0, j_1, \cdots, j_n, \cdots$ in

J (both of which may be finite with an equal number of terms) such that equations (4) hold for every $n = 0, 1, 2, \cdots$ and such that

$$\bigcup_{n \geqq 0} a_{i_n} \leqq \bigcup_{n \geqq 0} b_{j_n} \cup \bigcup_{i \in M} b_{\varphi(i)} .$$

Let M^* be the set-sum of M and $\{i_0, i_1, \cdots, i_n, \cdots\}$. Define the well-ordering $(\prec^*)$ on M^* as follows: if $i, i' \in M$, then $i \prec^* i'$ if $i \prec i'$ in M; and

$$i \prec^* i_0 \prec^* i_1 \prec^* \cdots \prec^* i_n \prec^* \cdots$$

for every $i \in M$. Define the mapping φ^* on M^* into J by $\varphi^*(i) = \varphi(i)$ for each $i \in M$, and $\varphi^*(i_n) = j_n$ for each $n = 0, 1, 2, \cdots$. Then it is clear that $(M^*, \prec^*, \varphi^*) \in \mathscr{P}$ and $(M^*, \prec^*, \varphi^*) > (M, \prec, \varphi)$. Since this contradicts the maximality of $(M, \prec, \varphi)$ we must have $M = I$. From (2) it follows that $\varphi(M) = \varphi(I) = J$. Hence the proof is complete.

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