THE BERGMAN KERNEL FUNCTION FOR TUBES OVER CONVEX CONES

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In this article we determine the Bergman kernel function of the tube domain over an arbitrary convex cone not containing any entire straight line. For homogeneous self-dual cones this problem was solved by O. S. Rothaus ([3], Theorem 2.6). It turns out that his method can also be used in our considerably more general case. In fact, the proofs of our Theorems 1 and 2 follow closely the corresponding proofs of Rothaus; it is only in Lemma 2 that the proof of Rothaus has to be replaced by an essentially different convexity argument.

Let V be an n-dimensional real vector space. A set $D \subset V$ is called a cone if $x \in D$ and $\lambda > 0$ imply $\lambda x \in D$. Let V^* be the dual space of V. The dual cone D^* of D is defined as the set of all $\alpha \in V^*$ such that $\langle \alpha, x \rangle > 0$ for all $x \in \bar{D}$, $x \neq 0$. We call the cone D regular if it is

(i) open,

(ii) convex,

(iii) nonempty, and

(iv) contains no entire straight line, i.e. $x \in D$ implies $-x \notin D$. It is easy to see that if D is regular then D^* is regular too, and $D^{**} = D$.

We assume that a Euclidean norm $x \to |x|$ is defined on V. The dual norm on V^* will likewise be denoted by $\alpha \to |\alpha|$.

LEMMA 1. *If D is a regular cone and $K \subset D$ is a compact set then there exists a number $\rho > 0$ such that $\langle \alpha, x \rangle \geqq \rho |\alpha|$ for all $x \in K$, $\alpha \in \bar{D}^*$.*

Proof. The proof is the same as that of [2] Lemma 1. By homogeneity it suffices to prove the assertion for $|\alpha| = 1$. Let $S = \{\alpha \in V^* \,|\, |\alpha| = 1\}$ be the unit sphere in V^*. Now $\langle \alpha, x \rangle$ is a positive continuous function on the compact set $(S \cap \bar{D}^*) \times K$ and thus has a positive minimum ρ, finishing the proof.

We define the positive real-valued function M on D^* by

$$M(\alpha) = \int_D e^{-\langle \alpha, x \rangle} dx$$

for all $\alpha \in D^*$. By Lemma 1 the integral converges uniformly on compact sets. As it can immediately be seen, M is a homogeneous function of degree $-n$.

LEMMA 2. *Let D be a regular cone and let $\beta \in \partial D^*$ (the boundary of D^* in V^*). Then*

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$$\lim_{\alpha\to\beta} M(\alpha) = \infty \ .$$

Proof. If $\beta = 0$ the assertion is trivial. Let $\beta \neq 0$. For $\alpha \in D^*$ and $t > 0$ define $H_\alpha(t) = \{x \in D \mid \langle\alpha, x\rangle = t\}$ and let

$$V_\alpha(t) = \int_{H_\alpha(t)} dv_\alpha$$

be the volume of $H_\alpha(t)$ (dv_α denotes the volume element of the hyperplane $\{x \mid \langle\alpha, x\rangle = t\}$). Clearly we have $V_\alpha(t) = t^{n-1}V_\alpha(1)$ for all $t > 0$. Also

$$M(\alpha) = \int_D e^{-\langle\alpha,x\rangle}dx = \int_0^\infty dt \int_{H_\alpha(t)} e^{-\langle\alpha,x\rangle}dv_\alpha$$
$$= \int_0^\infty V_\alpha(t)e^{-t}dt = V_\alpha(1)\Gamma(n) \ .$$

Therefore the Lemma will be proved if we show that $\lim_{\alpha\to\beta} V_\alpha(1) = \infty$.

Let $U \subset \bar{D}^*$ be a compact neighborhood of β relative to $\bar{D}^*$. Then the set L of all $x \in D$ such that $\langle\alpha, x\rangle < 1$ for all $\alpha \in U$ has an interior. (In fact, if A is a bound for $|\alpha|$ on U, it is easy to see that L contains all $x \in D$ such that $|x| < A^{-1}$). Let K be an open sphere contained in L; let $c \in D$ be its center and $r > 0$ its radius.

For $\alpha \in U$ let K_α be the $(n-1)$-dimensional sphere of radius r and center $c_\alpha = \langle\alpha, c\rangle^{-1}c$ contained in the hyperplane $\{x \mid \langle\alpha, x\rangle = 1\}$. By convexity and by $\langle\alpha, c\rangle^{-1} > 1$ we have $K_\alpha \subset H_\alpha(1)$. Since $|c_\alpha| = \langle\alpha, c\rangle^{-1}|c|$ and since the continuous function $\langle\alpha, c\rangle^{-1}$ is bounded on the compact set U, there exists a number R such that

$$(1) \qquad |c_\alpha| \leqq R$$

for all $\alpha \in U$.

Now let $\Omega > R$ be an arbitrarily large number. There exists an element $a \in \bar{D}$, $|a| = 1$ such that $\langle\beta, a\rangle = 0$, for otherwise we would have $\beta \in D^*$. Hence there exists an element $x \in D$, $|x| = 1$ such that $\langle\beta, x\rangle < (R + \Omega)^{-1}$. It follows then that there exists a neighborhood $U(\Omega) \subset U$ of β relative to D^* such that $\langle\alpha, x\rangle < (R + \Omega)^{-1}$ for all $\alpha \in U(\Omega)$. Let $x_\alpha = \langle\alpha, x\rangle^{-1}x$. Clearly we have $x_\alpha \in H_\alpha(1)$ and

$$(2) \qquad |x_\alpha| > R + \Omega$$

for all $\alpha \in U(\Omega)$. Now $H_\alpha(1)$ is convex, and thus contains the convex hull B_α of K_α and x_α; hence, be (1) and (2),

$$V_\alpha(1) \geqq \int_{B_\alpha} dv_\alpha > \frac{C}{n-2}\Omega$$

for all $\alpha \in U(\Omega)$, C denoting the volume of the $(n-2)$-dimensional sphere of radius r. This completes the proof.

Let $V_C = V \oplus iV$ be the complexification of V. The tube over D in V_C is the domain $T_D = \{x + iy \mid x \in D, y \in V\}$. For $z = x + iy \in V_C$ and $\alpha \in V^*$ we write $\langle \alpha, z \rangle = \langle \alpha, x \rangle + i\langle \alpha, y \rangle$. We denote by $\mathscr{L}^2(T_D)$ the Hilbert space of holomorphic functions on T_D, square integrable with respect to $dxdy$, and by $L^2_M(D^*)$ the Hilbert space of functions on D^* square integrable with respect to $M(\alpha)d\alpha$.

THEOREM 1. *The mapping* $\varphi \rightarrow f$ *defined by*

$$(3) \qquad f(z) = \pi^{-n/2} \int_{D^*} \varphi(\alpha) e^{-\langle \alpha, z \rangle} d\alpha$$

is an isomorphism of $L^2_M(D^*)$ *onto* $\mathscr{L}^2(T_D)$.

Proof. Let $\varphi \in L^2_M(D^*)$. Then

$$\int_{D^*} | \varphi(\alpha) e^{-\langle \alpha, z \rangle} | d\alpha = \int_{D^*} | \varphi(\alpha) | e^{-\langle \alpha, x \rangle} d\alpha$$
$$\leqq \left(\int_{D^*} | \varphi(\alpha) |^2 M(\alpha) d\alpha \right)^{1/2} \left(\int_{D^*} e^{-2\langle \alpha, x \rangle} M(\alpha)^{-1} d\alpha \right)^{1/2}$$

by the Schwarz inequality. The first integral is just $\| \varphi \|^2$, the second is also convergent by Lemma 2 and by the homogeneity of M; by Lemma 1 it is even bounded on compact subsets of D. Thus (3) converges absolutely and uniformly on compact subsets of T_D, and hence represents a holomorphic function. Furthermore, reversing the order of integration (which is possible since the integrand is positive and measurable), and then applying the Plancherel theorem we have

$$(4) \qquad \| \varphi \|^2 = \int_{D^*} | \varphi(\alpha) |^2 M(\alpha) d\alpha = \int_{D^*} | \varphi(\alpha) |^2 d\alpha \int_D e^{-\langle \alpha, x \rangle} dx$$
$$= 2^n \int_{D^*} | \varphi(\alpha) |^2 d\alpha \int_D e^{-2\langle \alpha, x \rangle} dx = 2^n \int dx \int_{D^*} | \varphi(\alpha) e^{-\langle \alpha, x \rangle} |^2 d\alpha$$
$$= \int_D dx \int_V | f(x + iy) |^2 dy = \| f \|^2 ,$$

which shows that $f \in \mathscr{L}^2(T_D)$ and also that the mapping is an isomorphism.

Remains to show (and this is the more important part) that the isomorphism is onto.

First we prove that there exists a measurable function φ on V^* such that

$$f(z) = f(x + iy) = \lim \pi^{-n/2} \int_{V^*} \varphi(\alpha) e^{-\langle \alpha, z \rangle} d\alpha$$

for almost all $x \in D$. In fact, by Fubini's theorem $f(x + iy)$ as a function of y is in $L^2(V)$ for almost all x; so the Fourier transform

$$\psi(x, \alpha) = \lim \pi^{-n/2}\int_V f(x + iy)e^{-i\langle\alpha, y\rangle}dy$$

exists. The assertion is that $\psi(x, \alpha) = \varphi(\alpha)e^{-\langle\alpha, x\rangle}$ with some measurable φ. Let $N \subset D$ be a subset whose distance from ∂D is $d > 0$. Then, by a well-known property of $\mathscr{L}^2$-spaces, $|f(z)| = |f(x + iy)| \leqq C_d \|f\|$ for all $x \in N$, $f \in \mathscr{L}^2(T_D)$. Using this remark the proof of our assertion is the same as that of a similar assertion in [1], p. 128, and will not be reproduced here.

Next we show that $\varphi(\alpha) = 0$ for almost all $\alpha \notin D^*$. In fact, using the Plancherel theorem and reversing the order of integration we obtain

$$\|f\|^2 = 2^n \int_{V^*} d\alpha \int_D |\varphi(\alpha)|^2 e^{-2\langle\alpha, x\rangle}dx .$$

In particular, $\int_D |\varphi(\alpha)|^2 e^{-2\langle\alpha, x\rangle}dx$ exists for almost all α and is integrable. Now if $\alpha \notin D^*$, then $\langle\alpha, x\rangle < 0$ for some $x \in D$ and hence $\int_D e^{-2\langle\alpha, x\rangle}dx$ diverges. Therefore $\varphi(\alpha) = 0$ for almost all such α.

Finally we must show that $\varphi \in L^2_M(D^*)$. This however follows at once from the Plancherel theorem through the equalities (4).

THEOREM 2. *The Bergman kernel function of T_D is*

$$K(z, w) = \frac{1}{\pi^n}\int_{D^*} e^{-\langle\alpha, z+\bar{w}\rangle}M(\alpha)^{-1}d\alpha .$$

Proof. From Theorem 1 it is clear that, for fixed $w \in T_D$, $K(z, w)$ as a function of z is in $\mathscr{L}^2(T_D)$. Also for fixed $w \in T_D$ and $x \in D$, $K(z, w)$ is in $L^2(V)$ as a function of y.

Let $f \in \mathscr{L}^2(T_D)$, then f can be represented in the form (3). Using the Plancherel theorem and then reversing the order of integration (which can be done since the integrand is measurable and the repeated integral in reverse order exists absolutely), we obtain

$$\begin{aligned}\int_{T_D} f(z)\bar{K}(z, w)\,dxdy &= \int_D dx \int_V f(z)\bar{K}(z, w)dy \\ &= 2^n \int_D dx \int_{V^*} \varphi(\alpha)e^{-\langle\alpha, x\rangle}e^{-\langle\alpha, x+w\rangle}M(\alpha)^{-1}d\alpha \\ &= 2^n \int_{V^*} d\alpha\varphi(\alpha)e^{-\langle\alpha, w\rangle}M(\alpha)^{-1}\int_D e^{-2\langle\alpha, x\rangle}dx \\ &= \int_{V^*} \varphi(\alpha)e^{-\langle\alpha, w\rangle}d\alpha = f(w)\end{aligned}$$

for all $w \in T_D$. Owing to the fact that the Bergman kernel is uniquely determined by its reproducing property, the proof is finished.

References

1. S. Bochner and W. T. Martin, *Several Complex Variables*, Princeton University Press, 1948.
2. M. Koecher, *Positivitätsbereiche im R^n*, Amer. J. Math., **79** (1957), 575-596.
3. O. S. Rothaus, *Domains of Positivity*, Abh. Math. Semin. Hamburg **24** (1960), 189-235.

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