ABELIAN SUBGROUPS OF p-GROUPS

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Let G be a finite p-group where p is an odd prime. We say that G has *property* A_n if every abelian normal subgroup of G can be generated by n elements. Further, if G_n denotes the nth element in the descending central series of G, we say that G has property $A_n(G_n)$ if every abelian subgroup of G_n which is normal in G can be generated by n elements. If G has property A_1, then G is cyclic. N. Blackburn [1] found all of the groups which have property A_2. It follows from the work of Blackburn that if G has property A_2 then the derived group of G is abelian and every subgroup of G has property A_2. We shall show that if G has property A_3 then every subgroup of G has property A_3. There exist groups which have property A_3 in which the derived series is arbitrarily long [2] so no analogue of Blackburn's result on the derived group is possible. We next consider groups G which have property $A_n(G_n)$ and show that G_n can be generated by n elements. This leads to the existence of a bound on the derived length of G which depends only on n and the exponent of G_n.

We shall use the following notation: p is an odd prime; $G=G_1\supset G_2\supset\cdots$ is the descending central series of G; $Z(G)=Z_1(G)\subset Z_2(G)\subset\cdots$ is the ascending central series of G; $G^{(k)}$ is the kth derived group of G; (H, K) is the subgroup of G generated by all elements $(h, k)=h^{-1}k^{-1}hk$ for $h\in H$, $k\in K$; $N\lhd G$ means N is normal in G; $N\subset G$ means N is properly contained in G; $C_G(N)$ is the centralizer of N in G; H^G is the normal subgroup of G generated by H; $\mho(G)$ is the subgroup generated by pth powers of elements of G. $\Omega(G)$ is the subgroup generated by all elements of order p in G; $\phi(G)$ is the Frattini subgroup of G; $|G|$ is the order of G.

If $A\lhd G$ and $A\subset C_G(A)$, then there is a subgroup B of $C_G(A)$ such that $B\lhd G$ and $[B\colon A]=p$. It follows that if a normal subgroup A of G is properly contained in an abelian subgroup C of G, then A is properly contained in some abelian normal subgroup B of G.

LEMMA 1. *Suppose $A\lhd G$ and $A\subset C$ where C is an elementary abelian subgroup of G. Then G contains an elementary abelian normal subgroup B such that A is a subgroup of index p in B.*

Proof. Suppose G is a group of minimal order for which the lemma is false. Then $C\subset G$, so there is a subgroup M of index p in G which contains C. It follows that M contains an elementary abelian normal subgroup B_1 such that $[B_1\colon A]=p$. Set $D=M\cap C_G(A)$. Then $B_1\lhd D\lhd G$.

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Since $(D, B_1) \subseteqq A$ and $(D, A) = 1$, we have $B_1 \subseteqq Z_2(D) \lhd G$. Therefore $B_1^G \subseteqq Z_2(D)$. But $Z_2(D)$ is a regular p-group for $p > 2$, so B_1^G has exponent p. Let B be a subgroup of B_1^G which is normal in G and which contains A as a subgroup of index p. Clearly B is elementary abelian, so the lemma is true for G.

THEOREM 1. *If G has property A_3 then every subgroup of G has property A_3.*

Proof. Suppose G is a group of minimal order for which the theorem is false. Then G contains an elementary abelian normal subgroup A of order p^3, and there is a subgroup M of index p in G which does not have property A_3. It follows that M contains an elementary abelian normal subgroup D of order p^4. Let N be a subgroup of order p^2 in A which is contained in M and which is normal in G. If we let $C = C_G(N)$, then $[G\colon C] \leqq p$, hence $[D\colon D \cap C] \leqq p$. Thus we may suppose that $N \subset D$, since otherwise we could choose a new subgroup D_1 in $(C \cap D)N$ such that $N \subset D_1 \lhd M$ and D_1 is elementary abelian of order p^4.

Since G has property A_3 it follows from Lemma 1 that A contains the only elements of order p in $C_G(A)$. Therefore $N = D \cap C_G(A)$. It is easy to see that $[C\colon C_G(A)] \leqq p^2$, thus $C = DC_G(A)$. Therefore, if $d \in D$, $g \in G$, then $g^{-1}dg = d_1c$ for some $d_1 \in D$, $c \in C_G(A)$. We recall that D is an abelian normal subgroup of M, and that $M \lhd G$. Thus D and $g^{-1}Dg$ generate a group of class at most two; hence for $p > 2$ the group generated by D and $g^{-1}Dg$ has exponent p. Thus it follows from $g^{-1}dg = d_1c$ that $c^p = 1$, whence $c \in A$. Therefore $AD \lhd G$. But $A \cap D = N$, so $[AD\colon D] = p$. Since D is not normal in G, we must have $AD = D(g^{-1}Dg)$ for some element $g \in G$. Therefore $D \cap g^{-1}Dg$ has order at least p^3 and is contained in $Z_1(AD)$ which is normal in G. Thus AD must contain an element of order p which centralizes A and which does not belong to A. This is a contradiction.

THEOREM 2. *If G has property $A_n(G_n)$ then G_n can be generated by n elements.*

Proof. Suppose G is a group of minimal order for which the theorem is false. Then G_n is not abelian, so $\phi(G_n) \neq 1$. Let Z be a group of order p in $Z_1(G) \cap \phi(G_n)$. Then G_n and $(G/Z)_n$ have the same number of generators, so $(G/Z)_n$ must contain an elementary abelian subgroup B/Z of order p^{n+1} which is normal in G/Z. Let B be the preimage of B/Z in G. Then $B \lhd G$, B has order p^{n+2}, and $B^{(1)} \subseteqq Z$. Thus B has class at most two, hence is regular for $p > 2$. But $\mho(B) \subseteqq Z$, so $\Omega(B)$ is a group of order at least p^{n+1} which is normal in G. Thus there is a subgroup A of $\Omega(B)$ such that $A \lhd G$, $\mho(A) = 1$, and A has order p^{n+1}. Let N be a subgroup of index p in A which is normal in G. Then $|N| = p^n$ and $N \lhd G$ imply $N \subseteqq Z_n(G)$, whence $N \subseteqq Z_1(G_n)$. Therefore A is abelian, a contradiction.

COROLLARY. *Suppose G has property $A_n(G_n)$, where G_n has exponent p^m. Let k be an integer such that $2^k \geqq n$. Then $G^{(k+m)} = 1$.*

Proof. By Theorem 2, G_n can be generated by n elements. Therefore [3, Theorem 2] $\phi(G_n) = \Omega(G_n)$. It follows that $G_n^{(m)} = \langle 1 \rangle$ [4, Theorem 2]. In any p-group, $G^{(t)} \subseteqq G_{2^t}$. Therefore $G^{(k)} \subseteqq G_n$, whence $G^{(k+m)} = \langle 1 \rangle$.

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