SOME FUNCTION CLASSES RELATED TO THE CLASS OF CONVEX FUNCTIONS

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1. Introduction. A real-valued function f defined on the positive real line $[0, \infty)$ is said to be convex if for every $x \geqq 0, y \geqq 0$, and $\alpha, 0 \leqq \alpha \leqq 1, f$ satisfies the inequality

$$f[\alpha x + (1 - \alpha)y] \leqq \alpha f(x) + (1 - \alpha)f(y) . \tag{1}$$

Such functions are important in many parts of analysis and geometry and their properties have been studied in detail (see e.g. the expository article Beckenbach [1] which contains an extensive bibliography).

A related class of functions is the class of superadditive functions which satisfy the defining inequality

$$f(x + y) \geqq f(x) + f(y) . \tag{2}$$

These functions, more precisely their negatives which are subadditive, have been studied by Hille and Phillips [5] and R. A. Rosenbaum [7] among others.

In the paper we shall be concerned, in large part, with classes of functions that properly lie between these two classes and which are defined by inequalities which are weaker than (1) but stronger than (2). We obtain a strict hierarchy of classes and various characterizing properties of these classes and study a simple averaging operation that transforms each class into a smaller class.

2. Definitions and elementary properties of the classes. We shall restrict our attention generally to functions which are continuous, nonnegative, and for which $f(0) = 0$ unless the contrary is explicitly stated. The requirement of being nonnegative simplifies many proofs which could be given without this assumption by considering the sum of f with a suitably chosen linear function.

DEFINITION 1. Let f be defined on $[0, \infty)$. The average function F of f is the function defined for all $x > 0$ by

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt , \qquad F(0) = 0 .$$

DEFINITION 2. The function f is said to be starshaped if for each

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α, $0 \leqq \alpha \leqq 1$, and all x

$$f(\alpha x) \leqq \alpha f(x) .$$

It is easy to see that the set of points lying above the graph of a starshaped function is starshaped with respect to the origin in the usual sense. A function can, of course, be starshaped with respect to any other point on its graph, the definition of this phenomenon being made in an obvious way. The characterization of Lemma 3 below then applies mutatis mutandis. It is not hard to verify that a continuous function is convex if and only if it is starshaped with respect to a set of points dense in its graph.

DEFINITION 3. The function f is said to be convex on the average, starshaped on the average, or superadditive on the average if F is respectively convex, starshaped, or superadditive.

In the sequel we shall use the abbreviation COA for convex on the average. We shall also use the following notation for derivatives:

$$\underline{f}'(x_0) = \underline{\lim}_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, f'_+(x_0) = \underline{\lim}_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} ,$$

and

$$f'_-(x_0) = \underline{\lim}_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} .$$

Simple characterizations of the classes are recorded in the following series of lemmas.

LEMMA 1. *A continuous convex function f is left and right differentiable at each point, the one-sided derivatives being increasing functions. Conversely, if any one of the Dini derivatives of a continuous function f is increasing, the function is convex.*

Proof. For a proof of the first part see Hardy, Littlewood, and Polya [4]. To prove the converse, let Df denote an increasing Dini derivative of f and let G be an indefinite integral of Df. Then G is convex. If x_0 is a point of continuity of Df, then both f and G are differentiable at x_0 and $Df(x_0) = G'(x_0) = f'(x_0)$. Since Df is increasing, it is continuous except on at most a countable set of points. It follows (see Hobson [6]) that f and G differ by at most a constant. Thus f is convex.

The proofs of the next three Lemmas are straightforward and will be omitted.

LEMMA 2. *The function f is* COA *if and only if* $\underline{f}' \geqq 2F'$.

LEMMA 3. *The function f is starshaped if and only if either one the two following conditions is satisfied*:

(i) $f(x)/x$ *is increasing*,

(ii) $\underline{f}'(x) \geqq f(x)/x$ *for all* x.

LEMMA 4. *The function f is starshaped on the average if and only if* $f \geqq 2F$.

The inequality $f \geqq 2F$ has the following simple geometric interpretation: Since

$$xF(x) = \int_0^x f(t)\, dt \leqq \frac{x}{2} f(x) ,$$

the area under the graph of f is at each point dominated by the area of the triangle with vertices $(0, 0)$, $(x, 0)$ and $(x, f(x))$.

The inequality $f(x_0) \geqq 2F(x_0)$ can be cast in the form

$$\frac{F(x_0)}{x_0} \leqq \frac{1}{2} \frac{f(x_0)}{x_0} .$$

Since $F(x)/x$ is increasing, we actually obtain the slightly more general result,

$$\frac{F(a)}{a} \leqq \frac{F(x_0)}{x_0} \leqq \frac{1}{2} \frac{f(x_0)}{x_0}$$

for all $a \leqq x_0$. This means geometrically that for $a < x_0$, the area of the triangle cut off from the above mentioned triangle by the line $x = a$ is no smaller than the area under the graph of f from 0 to a.

LEMMA 5. *If f is respectively convex, convex on the average, starshaped, or superadditive, then f is a nondecreasing function.*

Proof. We have restricted ourselves to nonnegative functions for which $f(0) = 0$. If f is superadditive, then $f(y) = f[x + (y - x)]$

$$\geqq f(x) + f(y - x) \geqq f(x) \quad \text{for } y \geqq x .$$

As we show in Theorem 5, f satisfying any of the other conditions implies that f is superadditive.

If f is merely starshaped on the average, it is clear from the geometric interpretation of $f \geqq 2F$ that f need not be increasing.

Since f is an increasing function provided f belongs to one of the function classes of Lemma 5, f has a finite derivative almost everywhere. For all these classes, F has a continuous derivative for $x > 0$

since $xF'(x) = f(x) - F(x)$. We consider the behavior of F' at the origin in Theorem 7 below.

We now investigate various operations under which our function classes are closed. We have first of all

THEOREM 1. *Let f and g be respectively convex,* COA, *starshaped, starshaped on the average, superadditive, superadditive on the average; then for $a \geqq 0$, $b \geqq 0$, $af + bg$ belongs to the same class.*

The proof involves a trivial computation.

The next two theorems consider the behavior of our classes under the operation of pointwise limits.

THEOREM 2. *Let $\{f_n\}$ be a sequence of convex, starshaped, or superadditive functions converging pointwise to a limit function f. Then f is respectively convex, starshaped, or superadditive. Moreover, the average functions F_n converge to the average function F.*

Proof. It is clear that the defining inequalities of these classes are preserved in the limit. The proof of the second statement parallels the proof of the corresponding part of Theorem 3.

THEOREM 3. *Let $\{f_n\}$ be a sequence of* COA *functions converging pointwise to a continuous limit f. The limit function is then* COA *and the average functions F_n converge to the average function F.*

Proof. Let $b > 0$. The sequence $\{f_n\}$ is uniformly bounded on $[0, b]$ by $\sup \{f_n(b)\} = M$. M is finite for $f_n(b) \to f(b)$ and M is a uniform bound because each f_n is an increasing function. By the Lebesgue bounded convergence theorem,

$$\frac{1}{x}\int_0^x f_n(t)\,dt \to \frac{1}{x}\int_0^x f(t)\,dt$$

for each $x \in [0, b]$, that is $F_n(x) \to F(x)$. Since b was arbitrary, this last relation holds for all x. The convexity of F follows from the convexity of F_n.

In general, however, it is not true that the limit of the average functions is equal to the average of the limit function. If $f_n \to f$ and the averages $F_n \to G$, an easy calculation shows that $F \leqq G$. For functions which are starshaped on the average, we do have the following theorem.

THEOREM 4. *If $\{f_n\}$ is starshaped on the average and $f_n \to f$, then f is starshaped on the average.*

Proof. For each $x > 0$, let T_n^x and T^x be the linear functions determined by the origin and the points $(x, f_n(x))$ and $(x, f(x))$. Since $f_n \to f$, $T_n^x \to T^x$. Moreover, the inequality $2F_n \leqq f_n$ is equivalent to

$$\int_0^x f_n(t)\,dt \leqq \int_0^x T_n^x(t)\,dt\ ;$$

by Fatou's theorem,

$$\int_0^x f(t)\,dt \leqq \varliminf_{n\to\infty} \int_0^x f_n(t)\,dt \leqq \varliminf_{n\to\infty} \int_0^x T_n^x(t)\,dt = \int_0^x T^x(t)\,dt\ .$$

Thus,

$$\frac{1}{x}\int_0^x f(t)\,dt \leqq \frac{1}{x}\int_0^x T^x(t)\,dt = \frac{1}{2}f(x)\ ,$$

i.e.

$$F(x) \leqq \frac{1}{2}f(x)\ ,$$

so f is starshaped on the average.

3. **The hierarchy.** We now consider the inclusion relationships among the six classes.

THEOREM 5. *Let f be a nonnegative continuous function which vanishes at the origin.*

Consider the following six conditions on f:

(i) *f is convex,*
(ii) *f is* COA,
(iii) *f is starshaped,*
(iv) *f is superadditive,*
(v) *f is starshaped on the average,*
(vi) *f is superadditive on the average.*

Then the following chain of implications is valid but none of the reverse implications holds: (i) $\to$ (ii) $\to$ (iii) $\to$ (iv) $\to$ (v) $\to$ (vi).

Proof. (i) $\to$ (ii). This will be a consequence of Theorem 10.

(ii) $\to$ (iii).

$$\frac{f(x)}{x} = F'(x) + \frac{F(x)}{x}\ .$$

Since F is convex, both F' and $F(x)/x$ are increasing. Thus $f(x)/x$ is increasing. It follows from Lemma 3, condition (i), that f is starshaped.

(iii) $\to$ (iv). For $x > 0$ and $y > 0$, we have

$$\frac{f(x)}{x} \leqq \frac{f(x+y)}{x+y}$$

and

$$\frac{f(y)}{y} \leqq \frac{f(x+y)}{x+y} .$$

These inequalities are equivalent to

$$(x+y)f(x) \leqq xf(x+y)$$

and

$$(x+y)f(y) \leqq yf(x+y)$$

which on addition yield $f(x) + f(y) \leqq f(x+y)$.

(iv) $\rightarrow$ (v). We first consider the case in which f is a *polygonal* superadditive function. The general case then follows by a limit argument.

Let $x > 0$ and let f be polygonal of n segments with vertices over the equidistantly spaced points $0, v, 2v, \cdots, nv = x$. Let T be the linear function determined by the origin and the point $(x, f(x))$, i.e. $T(t) = (f(x)/x)t$ for all t. Furthermore, let $q(t) = f(t) - T(t)$. The function q is polygonal and superadditive, having its vertices over the same points as f, and $q(0) = q(x) = 0$. We will show that $\int_0^x q(t)\,dt \leqq 0$ which suffices for F to be starshaped. Using the linearity of f on the intervals $[kv, (k+1)v]$, we obtain

$$\begin{aligned}\int_0^x q(t)\,dt &= v\sum_{k=1}^{n-1} q(kv) \\ &= \begin{cases} v\displaystyle\sum_{k=1}^{(n-1)/2}[q(kv) + q(n-k)v)] & \text{if } n \text{ is odd ,} \\ vq((n/2)v) + v\displaystyle\sum_{k=1}^{n/2-1}[q(kv) + q((n-k)v)] & \text{if } n \text{ is even .}\end{cases}\end{aligned}$$

Now $q(kv) + q((n-k)v) \leqq q(nv) = q(x) = 0$ for q is superadditive. In either case $\int_0^x q(t)\,dt \leqq 0$.

In the general case let $\{p_n\}$ be a sequence of polygonal functions over equidistantly spaced points such that $p_n \rightarrow f$. Let T be the linear function defined as above related to f. Since $\{p_n\}$ is superadditive for each n (see Bruckner [2, THEOREM 8] and $p_n(x) \leqq f(b)$ for all n and all $x \leqq b$, where b is arbitrary, it follows for each x that

$$\int_0^x p_n(t)\,dt \rightarrow \int_0^x f(t)\,dt .$$

Since

$$\int_0^x p_n(t)\,dt \leqq \int_0^x T(t)\,dt ,$$

the limit result

$$\int_0^x f(t)\,dt \leqq \int_0^x T(t)\,dt$$

follows.

(v) $\rightarrow$ (vi). This is just the case (iii) $\rightarrow$ (iv) for F.

That none of the reverse implications hold is shown by the following examples:

(ii) $\rightarrow$ (i): $f(x) = x^2 - x^3$ is COA on $[0, 4/9]$ but convex only on $[0, 1/3]$.

(iii) $\rightarrow$ (ii):

$$f(x) = \begin{cases} x^2 & 0 \leqq x \leqq 1 \\ x & 1 < x \end{cases}$$

is starshaped on $[0, \infty)$ but COA only on $[0, 1]$.

(iv) $\rightarrow$ (iii): $f(x) = n + (x - n)^2$ for $n \leqq x < n + 1$, $(n = 0, 1, 2, \cdots)$ is superadditive on $[0, \infty)$ but starshaped only on $[0, 1]$.

(v) $\rightarrow$ (iv): Let f be any function that is starshaped on the average without being increasing.

(vi) $\rightarrow$ (v): Let F be any superadditive function which is not starshaped such that F' is continuous. Then $xF(x)$ has a continuous derivative $f(x)$ and F is the average function of f.

4. Behavior for large and small x. Our first theorem in this section shows that superadditive functions are differentiable at the origin. Actually, a weaker hypothesis suffices to give this result.

THEOREM 6. *Let f be a continuous nonnegative function on $[0, c]$, $f(0) = 0$, such that $f((1/n)x) \leqq (1/n)f(x)$ for all $n = 1, 2, 3, \cdots$, and for all $x \in [0, c]$. Then f is differentiable at $x = 0$.*

Proof. The hypothesis $f((1/n)x) \leqq (1/n)f(x)$ implies that

$$\overline{\lim_{x\to 0}} \frac{f(x)}{x} < \infty .$$

Suppose f is not differentiable at the origin. Then there exists an $\varepsilon > 0$ such that $\bar{f}'(0) - \underline{f}'(0) = 3\varepsilon$. Choose x_0 so that $f(x_0) < (\underline{f}'(0) + \varepsilon)x_0$ and let $\{y_k\}$ be a sequence such that $y_k \rightarrow 0$ and $f(y_k) > (\bar{f}'(0) - \varepsilon)y_k$ $(k = 1, 2, 3, \cdots)$. Since f is continuous at x_0, there is a $\delta > 0$ such that if $|x - x_0| < \delta$, then $f(x) < (\underline{f}'(0) + \varepsilon)x$. Let y^* be a member of the sequence $\{y_k\}$ such that $y^* < \delta$. There is then an integer N such that

$$|Ny^* - x_0| < \delta; \text{ hence } f(Ny^*) < (\underline{f}'(0) + \varepsilon)Ny^* .$$

However

$$f(y^*) \leqq \frac{1}{N} f(Ny^*) < (\underline{f}'(0) + \varepsilon)y^* < (\bar{f}'(0) - \varepsilon)y^*$$

which contradicts the fact y^* is a member of the sequence $\{y_k\}$. Thus f is differentiable at the origin.

COROLLARY. *If f is superadditive, in particular if f is starshaped,* COA, *or convex, then $f'(0)$ exists.*

THEOREM 7. *Let f be superadditive on the average, and let F be its average function. If $f'(0)$ exists, then F' is continuous at $x = 0$ and $f'(0) = 2F'(0)$.*

Proof.

$$F'(x) = \frac{f(x)}{x} - \frac{F(x)}{x}, \qquad x > 0 .$$

The right member of this equality approaches $f'(0) - F'(0)$ for $F'(0)$ exists by Theorem 6; hence $\lim_{x\to 0} F'(x)$ exists, and because F' is a derivative, this limit must be $F'(0)$. Thus, F' is continuous at $x = 0$ and $2F'(0) = f'(0)$.

Theorem 7 indicates that $2F(x)/x$ is approximately the same as $f(x)/x$ for x near 0, provided f behaves sufficiently well near the origin. The next theorem shows that under suitable hypotheses the same behavior holds for large x.

THEOREM 8. *Let f be increasing and starshaped on the average and let F be its average function. Then $\lim_{x\to\infty} f(x)/x$ exists and is equal to $2\lim_{x\to\infty} F(x)/x$.*

Proof. Since F is starshaped, the $\lim_{x\to\infty} F(x)/x$ exists.

Let α be such that $0 < \alpha < 1$ and let $M = \overline{\lim}_{x\to\infty}(f(x)/x)$. Then

$$\begin{aligned}\frac{F(x)}{x} &= \frac{1}{x^2}\int_0^x f(t)\,dt \\ &= \frac{1}{x^2}\int_0^{\alpha x} f(t)\,dt + \frac{1}{x^2}\int_{\alpha x}^x f(t)\,dt\end{aligned}$$

$$\geqq \frac{\alpha F(\alpha x)}{x} + \frac{1-\alpha}{x} f(\alpha x) .$$

It follows that

$$\lim_{x\to\infty} \frac{F(x)}{x} \geqq \alpha^2 \lim_{x\to\infty} \frac{F(x)}{x} + \alpha(1-\alpha)M .$$

This last inequality holds for all α, $0 < \alpha < 1$ so

$$\lim_{x\to\infty} \frac{F(x)}{x} \geqq \sup_{0<\alpha<1} \frac{\alpha M}{1+\alpha} = \frac{M}{2} .$$

On the other hand, since F is starshaped, $f(x) \geqq 2F(x)$ for all x so that

$$\lim_{x\to\infty} \frac{F(x)}{x} \leqq \frac{1}{2} \varliminf_{x\to\infty} \frac{f(x)}{x} .$$

It follows that $\lim_{x\to\infty} (f(x)/x)$ exists and equals $2 \lim_{x\to\infty} (F(x)/x)$.

COROLLARY. *Let f be increasing and starshaped on the average with average function F. Then the three functions $f(x)/x$, $F'(x)$, and $F(x)/x$ simultaneously are bounded or unbounded.*

Proof. This follows directly from the identity

$$\frac{f(x)}{x} = F'(x) + \frac{F(x)}{x}$$

and the preceding theorem.

5. Minimal extensions. We suppose in this section that f is defined initially on an interval $[0, c]$. We shall consider in this section the problem of extending f in a minimal way to $[0, \infty)$ while staying within the same class. We start with

DEFINITION 4. Let f be convex (COA, starshaped, superadditive) on $[0, c]$. Suppose $\hat{f}$ is a function defined on $[0, \infty)$ with the following properties:

(i) $\hat{f} = f$ on $[0, c]$,

(ii) $\hat{f}$ is convex (COA, starshaped, superadditive) on $[0, \infty)$,

(iii) if g is any function on $[0, \infty)$ satisfying (i) and (ii), then $g(x) \geqq \hat{f}(x)$ for all x;

then $\hat{f}$ is said to be the minimal convex (COA, starshaped, superadditive) extension of f.

We restrict our definition to functions which are at least superadditive for minimal extensions of functions in the larger two classes are not, in general, continuous.

It is well known that if f is convex on $[0, c]$, there exists a convex extension of f to $[0, \infty)$ precisely when $f'_-(c) < \infty$. In this case, the minimal convex extension of f is linear on $[c, \infty)$ with slope $f'_-(c)$. When f is starshaped, it is clear that the minimal starshaped extension of f to $[0, \infty)$ is the linear function with slope $f(c)/c$. For superadditive functions the situation is much more complicated and has been studied in detail in Bruckner [2], where it is shown that the minimal extension does exist and, roughly speaking, behaves about as well as f.

The following theorem states the corresponding result for functions that are COA on $[0, c]$.

THEOREM 9. *Suppose f is* COA *on $[0, c]$ with average function F. Define $\hat{f}$ by the equations*

$$\hat{f}(x) = \begin{cases} f(x) & 0 \leqq x \leqq c \\ 2F'_-(c)x + f(c) - 2F'_-(c)x & x > c\,; \end{cases}$$

then $\hat{f}$ is the minimal COA *extension of f to $[0, \infty)$. If $\hat{F}$ is the average function of $\hat{f}$, then $\hat{F}$ is the minimal convex extension of F to $[0, \infty)$.*

Proof. For $x \geqq c$, we have

$$\hat{F}(x) = \frac{1}{x}\int_0^c f(t)\,dt + \frac{1}{x}\int_c^x [2F'_-(c)t + f'(c) - 2F'_-(c)c]\,dt\,.$$

It is easy to check that $\hat{F}(c) = F(c)$ and that for $x > c$, $\hat{F}''(x) = 0$ and $\hat{F}'(x) = \hat{F}'_-(c)$ so that $\hat{F}$ is the minimal convex extension of F to $[0, \infty)$. Thus $\hat{f}$ is a COA extension of f to $[0, \infty)$. Let now g, with average function G, be any COA extension of f to $[0, \infty)$ and let $x > c$. Since G is convex, G' is increasing so

$$G'(x) \geqq G'_-(c) = F'_-(c) = \hat{F}'(x)\,.$$

Thus

$$g'(x) \geqq 2G'(x) \geqq 2\hat{F}'(x) = \hat{f}'(x)\,.$$

Since $\hat{f}$ and g agree at c and $g' \geqq \hat{f}'$, $g(x) \geqq \hat{f}(x)$ so $\hat{f}$ is indeed the minimal COA extension of f.

If a function is convex on $[0, c]$, then it has extensions of each of the four types mentioned above. It is interesting to compare these various extensions. As an example, consider the function $f(x) = x^2$ on $[0, 1]$. Its minimal convex extension is linear with slope 2, the minimal COA extension is linear with slope 4/3, and the minimal starshaped extension is linear with slope 1. In contrast, the minimal superadditive extension is not linear. It is given by the function $\hat{f}(x) = n + (x - n)^2$ for $n \leqq x < n + 1$, $n = 1, 2, 3, \cdots$ (see Bruckner [2], p 1155).

6. Tests for convexity on the average. In this section we shall consider conditions that are necessary and/or sufficient that a function be COA. Similar tests concerning superadditive functions are found in Bruckner [3]. We begin with the following lemma.

LEMMA 6. *Let f_c be the function such that*

$$f_c(x) = \begin{cases} 0 & 0 \leqq x \leqq c \\ f(x - c) & x > c \,. \end{cases}$$

If f is COA, *then f_c is* COA.

Proof. Let F_c be the average function of f_c. We shall show that $\underline{f}'_c(x) \geqq 2F'_c(x)$, $x \geqq c$. Since $\underline{f}'_c(x) = \underline{f}'(x - c)$ for $x \geqq c$, it suffices to show that $\underline{f}'(x - c) \geqq 2F'_c(x)$. This last inequality will be a consequence of the inequality $F'(x - c) \geqq F'_c(x)$.

Defining

$$A(x) = \int_0^{x-c} f(t)\,dt\,,$$

we have that

$$F(x - c) = \frac{1}{x - c}\int_0^{x-c} f(t)\,dt = \frac{A(x)}{x - c}$$

and

$$\begin{aligned} F_c(x) &= \frac{1}{x}\int_0^x f_c(t)\,dt = \frac{1}{x}\int_c^x f(t - c)\,dt \\ &= \frac{1}{x}\int_0^{x-c} f(t)\,dt = \frac{A(x)}{x}\,. \end{aligned}$$

It thus suffices to show that

$$[A(x)(x - c)^{-1}]' \geqq [A(x)x^{-1}]',$$

the "$'$" denoting differentiation with respect to x. This last inequality is equivalent to

$$A'(x) \geqq \frac{2x - c}{x(x - c)} A(x)\,,$$

which is, on replacing $A(x)$ by $\int_0^{x-c} f(t)\,dt$ and simplifying, equivalent to the relation

$$f(x-c) \geqq \frac{2x-c}{x} F(x-c) .$$

Since f is starshaped, f is superadditive; hence f is starshaped on the average. Thus, by Lemma 4,

$$f(x-c) \geqq 2F(x-c) \geqq \frac{2x-c}{x} F(x-c)$$

which proves the lemma.

DEFINITION 5. Let f be defined on [0, a]. The functions $f_1, f_2, \cdots, f_n$ defined on $[0, a_1], [0, a_2], \cdots, [0, a_n]$ respectively form a decomposition of f provided

(i) $f_i(0) = 0 \qquad i = 1, \cdots, n$

(ii) $a_1 + a_2 + \cdots + a_n = a$ and $a_i > 0$ for $i = 1, \cdots, n$

(iii) $$f(x) = \begin{cases} f_1(x) & 0 \leqq x \leqq a_1 \\ f_2(x - a_1) + f_1(a_1) & a_1 < x \leqq a_1 + a_2 \\ \cdots\cdot & \\ f_n(x - a_1 - a_2 - \cdots - a_{n-1}) + f_1(a_1) + \cdots + f_{n-1}(a_{n-1}) & \\ & a - a_n < x \leqq a . \end{cases}$$

In this case we write $f = f_1 \wedge f_2 \wedge \cdots \wedge f_r$.

THEOREM 10. *Let f_1 and f_2 be* COA *on $[0, a_1]$ and $[0, a_2]$ respectively and let $f = f_1 \wedge f_2$ on $[0, a_1 + a_2]$. Let $\hat{f}_1$ be the minimal* COA *extension of f_1. A necessary and sufficient condition that f be* COA *is that $f \geqq \hat{f}_1$ on $[0, a_1 + a_2]$.*

Proof. The necessity is obvious. As to the sufficiency let $\hat{F}_1$ be the average function of $\hat{f}_1$. For $x \in [0, a_1 + a_2]$, write

$$F(x) = \hat{F}_1(x) + [F(x) - \hat{F}_1(x)] = \hat{F}_1(x) + \frac{1}{x}\int_0^x [f(t) - \hat{f}_1(t)]\, dt .$$

Consider $g(t) = f(t) - \hat{f}_1(t)$. $g(t) = 0$ on $[0, a_1]$ so there is an h defined on $[0, a_2]$ such that $g(t) = h_{a_1}(t)$ for $t \in [0, a_1 + a_2]$. On $[a_1, a_1 + a_2]$, $-\hat{f}_1$ is linear. Since f_2 is COA on $[0, a_2]$, h is COA on $[0, a_2]$ being the sum of COA functions. It follows from Lemma 6 that g is COA on $[0, a_1 + a_2]$. Its average function is therefore convex and so F is the sum of convex functions; hence convex.

THEOREM 11. *Let $f_1, \cdots, f_n$ be* COA *on $[0, a_1], \cdots, [0, a_n]$ respectively and let $f = f_1 \wedge f_2 \wedge \cdots \wedge f_n$. Furthermore let $\hat{f}_k$ be the mimimal* COA *extension of f_k, $(k = 1, \cdots, n)$. Then f is* COA *on $[0, a_1 + \cdots + a_n]$*

if $f_k \wedge f_{k+1} \wedge \cdots \wedge f_n \geqq \hat{f}_k$ *for each* $k = 1, 2, \cdots, n$.

Proof. The proof is an induction argument using the sufficiency part of Theorem 10.

We now return to the proof of the first part of Theorem 5, namely the proof of the statement: if f is convex, then f is COA.

Proof. Let us assume first that f is a polygonal function on $[0, c]$. If f has only one segment, then f is linear so the theorem is trivially true.

Supposing, by induction, that the theorem holds for polygonal functions with n segments, let f be polygonal with $(n+1)$ segments. Let f_n be the polygonal function which agrees with f on the first n segments of f and let $\hat{f}_n$ be the minimal convex extension of f_n to $[0, c]$. Thus $\hat{f}_n$ is convex and polygonal with n segments and so is COA. On the last segment f is linear and $f \geqq \hat{f}_n$. By Theorem 10, f is COA on $[0, c]$.

The general situation follows immediately by Theorem 3. Let $\{p_n\}$ be a sequence of convex polygonal functions approximating f. The $\{p_n\}$ are thus COA and so their limit function f is COA on $[0, c]$. Since c is arbitrary, this concludes the proof.

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