AN ANALOGUE OF KOLMOGOROV'S THREE-SERIES THEOREM FOR ABSTRACT RANDOM VARIABLES

R. P. PAKSHIRAJAN

1. **Introduction.** $(\Omega, \mathscr{M}, P)$ is a probability space i.e. Ω is an abstract set of points w, $\mathscr{M}$ is a σ-field of subsets of Ω and P is a nonnegative countably additive set function defined on $\mathscr{M}$ such that $P(\Omega) = 1$. G is a locally compact Hausdorff abelian metric topological group. The group operation in G, as well as in the several other groups to be dealt with, will be denoted by $+$. Let e denote the identity element of G. By the Borel sets of G we mean the sets belonging to the σ-ring generated by the class $\mathscr{C}$ of compact subsets of G. Let $\mathscr{D}$ be the class of subsets of G whose intersection with every compact set is a Borel set. Notice that $\mathscr{D}$ is a σ-field containing the open subsets of G. The character group of G will be denoted by $\hat{G}$. A single valued mapping f of Ω into G will be called a generalised random variable (g.r.v.) if $f^{-1}(A) \in \mathscr{M}$ whenever $A \in \mathscr{D}$. An immediate consequence of this definition is that if f is a g.r.v. then $\eta(f)$ is an ordinary (complex valued) random variable for every $\eta \in \hat{G}$. A finite or a countably infinite collection of g.r.v.'s is said to be independent if and only if for every finite subset $\{X_i, i = 1, 2, \cdots, n\}$ of distinct members of the collection and for every choice of sets $A_j \in \mathscr{D}$, $j = 1, 2, \cdots, n$ it is true that $P\{w: X_i(w) \in A_i, i = 1, 2, \cdots, n\} = \prod_1^n P\{w: X_i(w) \in A_i\}$.

If G is the real line, $\hat{G}$ is the real line too. For $t \in \hat{G}$ and $x \in G$, $t(x) = \exp(itx)$. Given the random variable X and any real number $c > 0$ we define a new random variable $Y = t_0\alpha$ where $t_0 = c/\pi$ and α is the principal amplitude of $\exp(i\pi X/c)$. The two sets $\{w: -c < X(w) \leqq c\}$ and $\{w: X(w) \neq Y(w)\}$ are then seen to be equal. Denoting by N the interval $(-c, c]$, the classical three series theorem [2] may be stated thus: If $\{X_n, n = 1, 2, \cdots\}$ is a sequence of independent real valued random variables then $\sum_1^\infty X_n$ exists with probability 1 (a.e.) if and only if, for some $c > 0$, the following three series converge.

(i) $\sum_1^\infty P\{w: X_n(w) \notin N\}$

(ii) $\sum_1^\infty EY_n$ and

(iii) $\sum_1^\infty \operatorname{var} Y_n$.

E and var denote respectively the mathematical expectation and

Received August 1962. This research was supported in part by funds provided under contract No. DA-04-200-ORD-651 with the office of ordnance Research, U.S. Army and was carried out under the guidance of Professsor Herman Rubin at the University of Oregon, U.S.A.

variance. α_n is the principal amplitude of $\exp(i\pi X_n/c)$ and $Y_n = t_0\alpha_n$. The convergence of the above three series is easily seen to be equivalent to the convergence of

(i) $\sum_1^\infty P\{w\colon X_n(w) \notin N\}$

(ii) $\sum_1^\infty E \log t(X_n)$ and

(iii) $\sum_1^\infty \operatorname{var} \log t(X_n)$ for every $t \in \hat{G}$, $\log t(X_n)$ being defined to be equal to $i\theta_n$ where θ_n is the principal amplitude of $\exp(itX_n)$. It is in this form the classical three series theorem lends itself for extension to the case of generalised random variables. In § 2 three lemmas are proved leading to the generalisation. In § 3 we give a neccessary and sufficient condition for the convergence almost everywhere of $\sum_1^\infty X_n$ in terms only of characters and not using characterstic functions.

The following two known results are quoted for the sake of completeness and ready reference.

THEOREM A. (Cor. (2.1) [4]).

If $\{h_n, n = 1, 2, \cdots\}$ is a sequence of continuous homomorphisms on a topological group G_1 to a toplogical group G_2 which converge pointwise to h throughout some Baire set of the second category then h is continuous.

THEOREM B. (§ 2.21 [3]).

Let G be a locally compact abelian group. Let N be a compact symmetric neighbourhood of e. Let G' be the subgroup of G generated by N. Then G' contains a discrete subgroup D with a finite number of generators such that the quotient group G'/D is compact and $D \cap (N + N + N) = \{e\}$.

2. For a sequence of real or complex numbers $g_n, n = 1, 2, \cdots$ we say that $\prod_1^\infty g_n$ exists if $\prod_n^\infty g_k$ is nonzero for sufficiently large n.

LEMMA 1. *For $\eta \in \hat{G}$, a necessary and sufficient condition that $\prod_1^\infty \eta(X_n)$ exists a.e. is that $\prod_1^\infty E\eta(X_n)$ exists.*

Proof. If $\prod_1^\infty \eta(X_n)$ exists a.e. then, by the bounded convergence theorem, $\prod_1^\infty E\eta(X_n)$ exists.

Conversely let $\prod_1^\infty E\eta(X_n)$ exist. Hence $\prod_1^\infty |E\eta(X_n)|$ exists. Let $\eta(X_n(w)) = \exp(i\theta_n(w))$ where $\theta_n(w)$ is the principal value of the amplitude. Hence $\theta_1, \theta_2, \cdots$ is a bounded, independent sequence of real valued random variables. Let θ_n' be the symmetrised version of θ_n and let θ_n' (1) be θ_n' truncated at 1. One has (p. 196, [2]) var θ_n' (1) $\leqq 3\{1 - |E\eta(X_n)|^2\}$. Hence $\sum_1^\infty \operatorname{var} \theta_n'(1) < \infty$. By the classical three series theorem it follows that $\sum_1^\infty \theta_n'$ converges a.e. Consequently (p. 250, [2]) there exist constants α_n such that $\sum_1^\infty (\theta_n - \alpha_n)$ exists a.e. or equivalently $\prod_1^\infty \exp(-i\alpha_n)E\eta(X_n)$ exists. This implies the convergence of $\sum_1^\infty \alpha_n$ since $\prod_1^\infty E\eta(X_n)$ is assumed to converge. We now conclude $\sum_1^\infty \theta_n$ exists a.e. or, what is same, $\prod_1^\infty \eta(X_n)$ exists a.e.

LEMMA 2. *For a given $\eta \in \hat{G}$, the following two sets of conditions are equivalent.*

$$\prod_1^\infty E\eta(X_n) \textit{ exists}; \ \sum_1^\infty \operatorname{var} \eta(X_n) < \infty \tag{2.1}$$

$$\sum_1^\infty E\theta_n \textit{ converges}; \ \sum_1^\infty \operatorname{var} \theta_n < \infty \tag{2.2}$$

where $\eta(X_n) = \exp(i\theta_n)$, θ_n being the principal amplitude.

Proof. Suppose (2.2) holds. Therefore by the three series theorem on the line, $\sum_1^\infty \theta_n$ exists a.e. This implies that $\prod_1^\infty \eta(X_n)$ exists a.e. Hence $\prod_1^\infty E\eta(X_n)$ exists by the bounded convergence.

Let now $\alpha_n = E\theta_n$; $\beta_n = \operatorname{var}\theta_n$ and $\theta_n = \alpha_n + y_n$. As in the last lemma, $E\eta(X_n) = (1 + d_n\beta_n/2)\exp(i\alpha_n)$ where $|d_n| \leqq 1$.

$$\begin{aligned} E|\eta(X_n) - E\eta(X_n)|^2 &= E|\exp(iy_n) - (1 + d_n\beta_n/2)|^2 \\ &\leqq c\beta_n \text{ where } c \text{ is an absolute constant} \\ &= c \operatorname{var}\theta_n . \end{aligned}$$

Hence $\sum_1^\infty \operatorname{var}\eta(X_n) < \infty$.

Conversely, suppose (2.1) holds.

$$\begin{aligned} \operatorname{var}\eta(X_n) &= E|\exp(iy_n) - (1 + d_n\beta_n/2)|^2 \\ &= 1 + |1 + d_n\beta_n/2|^2 - 2 \text{ real part of } E\overline{(1 + d_n\beta_n/2)}\exp(iy_n) \\ &= 1 - |1 + d_n\beta_n/2|^2 . \end{aligned}$$

Hence $\sum_1^\infty \{1 - |1 + d_n\beta_n/2|^2\} < \infty$. Now, $|1 + d_n\beta_n/2|$ is the absolute value of the expectation $E\exp(iy_n)$ and hence is less than or equal to 1. It follows therefore that $\sum_1^\infty \{1 - |1 + d_n\beta_n/2|\} < \infty$ As $1 - |1 + d_n\beta_n/2| \geqq \beta_n/2$, this implies that

$$\sum_1^\infty \beta_n < \infty \quad \text{i.e.} \quad \sum_1^\infty \operatorname{var}\theta_n < \infty .$$

From the convergence of $\prod_1^\infty E\eta(X_n)$ and $\sum_1^\infty \beta_n$ and the relation $E\eta(X_n) = (1 + d_n\beta_n/2)\exp(i\alpha_n)$, we see that $\sum_1^\infty E\theta_n = \sum_1^\infty \alpha_n$ converges. Thus (2.1) implies (2.2).

LEMMA 3. *A necessary and sufficient condition that $\sum_1^\infty X_n$ exist a.e. is that $\prod_1^\infty \eta(X_n)$ exists a.e. for every $\eta \in \hat{G}$, and for some compact neighbourhood N of e*

$$\sum_{1}^{\infty} P(w\colon X_n(w) \notin N) < \infty \ . \tag{2.3}$$

Proof. Suppose $\sum_1^\infty X_n$ exists a.e. Consequently, for every compact neighbourhood N of e, $P(w\colon X_n(w) \notin N \text{ i.o.}^1) = 0$ or, equivalently, $\sum_1^\infty P\{w\colon X_n(w) \notin N\} < \infty$ by the Borel-Cantelli lemma. That $\prod_1^\infty \eta(X_n)$ exists a.e. for each $\eta \in \hat{G}$ follows from the continuity of the characters η.

Conversely, let N be any compact neighbourhood of e for which (2.3) is satisfied. Since $N - N \supseteqq N$, we have $P\{w\colon X_n(w) \notin N - N\} \leqq P\{w\colon X_n(w) \notin N\}$. Hence the symmetric neighbourhood $N - N$ of e also satisfies (2.3). Without loss of generality we may therefore assume that N in (2.3) is symmetric.

Denote by G^* the closed subgroup generated by N. Necessarily G^* is σ-compact. Further, by Theorem B, G^* contains a discrete subgroup D with a finite number of generators such that $G_1 = G^*/D$ is compact and $D \cap (N + N - N) = \{e\}$. Hence by the Borel-Cantelli lemma, (2.3) implies that $P\{w\colon X_n(w) \notin N \text{ i.o.}\} = 0$; that is, if $A_1 = \{w\colon X_n(w) \in N \text{ for all } n \geqq n_0(w)\}$ then $P(A_1) = 1$. Let σ be the natural mapping of G^* onto G_1 and write $Y_n(w) = \sigma X_n(w)$.

As G_1 is a compact, metric group, G_1 (and consequently $\hat{G}_1$) satisfies the second axiom of countablity. Also $\hat{G}_1$ is discrete, since G_1 is compact. Further $\hat{G}_1$ consists precisely of those elements of $\hat{G}$ which are identically one on D (cf: Theorem 34 [5]). In view of (2.3), we have $\prod_1^\infty \xi(Y_n)$ exists a.e. for each $\xi \in \hat{G}_1$. As $\hat{G}_1$ is countable we conclude that, with probability 1, $\prod_1^\infty \xi(Y_n)$ exists for *all* $\xi \in \hat{G}_1$. Observe that G_1, being a compact metric space, is a Baire set of the second category. It is now immediate from Theorem A that $\sum_1^\infty Y_n$ exists a.e.

Let A_2 be a set of probability 1 on which $\sum_1^\infty Y_n$ exists. If $A = A_1 \cap A_2$ then $P(A) = 1$. Let $w \in A$ and $n \geqq n_0(w)$. Hence

$$X_n(w) + X_{n+1}(w) \in N + N \ . \tag{2.4}$$

As $\sigma(N)$ is a neighbourhood of the identity in G_1 and since $\sum_1^\infty Y_n(w)$ exists, it is clear that $Y_n(w) + Y_{n+1}(w) \in \sigma(N)$, if n is larger than a certain $n_1(w)$. That is

$$X_n(w) + X_{n+1}(w) \in N + D \qquad \text{if } n \geqq n_1(w) \ . \tag{2.5}$$

From (2.4) and (2.5) and the property $D \cap (N + N - N) = \{e\}$, we conclude that $X_n(w) + X_{n+1}(w) \in N$ if $n \geqq \max(n_0, n_1)$. Repeating the argument a finite number of times it is seen that all finite tails of the series $\sum_1^\infty X_n(w)$ lie in N. By exactly similar reasoning, all finite tails lie in any preassigned neighbourhood M of e with $M \subseteqq N$. As N is compact, we can show (by arguments similar to the ones

[1] infinitely often

on p. 193 [1]) that $\sum_1^\infty X_n(w)$ exists. Thus on A, which is a set of probability 1, $\sum_1^\infty X_n$ exists. Combining these results, we have

THEOREM 1. *If $\{X_n, n = 1, 2, \cdots\}$ is an independent sequence of generalised random variables then $\sum_1^\infty X_n$ exists a.e. if and only if the series*

(i) $\sum_1^\infty P\{w: X_n(w) \notin N\}$, *$N$ being any preassigned compact neighbourhood of e,*

(ii) $\sum_1^\infty E \log \eta(X_n)$ *and*

(iii) $\sum_1^\infty \operatorname{var} \log \eta(X_n)$ *converge for all $\eta \in \hat{G}$. Here $\log(X_n)$ is taken to be $i\theta_n$ where θ_n is the principal amplitude of $\eta(X_n)$.*

3. DEFINITION. The measure μ induced in $\mathscr{D}$ by a generalised random variable f will be called the distribution function of f. The distribution μ will be said to be symmetric if $\mu(A) = \mu(-A)$ for every $A \in \mathscr{D}$. It will be called regular if for every $A \in \mathscr{D}$, $\mu(A) = \sup\{\mu(C): C \subseteqq A, C \in \mathscr{C}\}$.

THEOREM 2. *If $\{X_n, n = 1, 2, \cdots\}$ is an independent sequence of generalised random variables with regular distributions, then $\sum_1^\infty X_n$ exists a.e. if and only if $\prod_1^\infty \eta(X_n)$ exists a.e. for every $\eta \in \hat{G}$.*

Proof. If $\sum_1^\infty X_n$ exists a.e. then $\prod_1^\infty \eta(X_n)$ exists a.e. for every $\eta \in \hat{G}$ by the continuity property of η.

Conversely, let $\prod_1^\infty \eta(X_n)$ exist a.e. for each $\eta \in \hat{G}$. The assertion is established through the following steps.

(i) Let G be compact. That the assertion is true in this case is seen by the same reasoning as for G_1 in Lemma 3.

(ii) Let G be discrete. The compact subsets of G are therefore only those subsets with a finite number of elements. As the distribution of each X_n is regular we can find a countable subgroup G_1 such that $P\{w: X_n(w) \in G_1, n = 1, 2, \cdots\} = 1$. Observe that $\hat{G}_1$ is the same as $\hat{G}$ restricted to G_1. Now let the X_n's have symmetric distributions. Hence, if $\varphi_n(\eta) = E\eta(X_n)$ then the φ_n's are real and $\varphi_n(-\eta) = \varphi_n(\eta)$. Now by Lemma 1, $\prod_1^\infty \eta(X_n)$ exists a.e. for each $\eta \in \hat{G}$, implies that $\prod_1^\infty \varphi_n(\eta)$ exists. Therefore $g(\eta) = \sum_1^\infty \{1 - \varphi_n(\eta)\}$ exists for every $\eta \in \hat{G}$. If $g_n(\eta) = \sum_1^n \{1 - \varphi_k(\eta)\}$ then the g_n's are continuous and $g_n(\eta)$ converges monotonically up to $g(\eta)$ as $n \to \infty$ for each η. Hence $\{\eta: g(\eta) \leqq a\} = \bigcap_1^\infty \{\eta: g_n(\eta) \leqq a\}$ is a closed set. $\hat{G}$ is a compact metric space and so is complete. Hence it is a set of the second category. Further, $\hat{G} = \bigcup_{n=1}^\infty \{\eta: g(\eta) \leqq n\}$ i.e. $\hat{G}$ is the union of a countable number of closed sets. Therefore by the Baire category theorem, one of these closed sets in the union, say the set $A = \{\eta: g(\eta) \leqq k\}$, has a nonnull interior V. Trivially g is bounded on V. By the positive definiteness and symmetry of ϕ_k,

$$1 - \phi_k^2(\xi) - \phi_k^2(\eta) + 2\phi_k(\xi)\phi_k(\eta)\phi_k(\xi + \eta) - \phi_k^2(\xi + \eta) \geqq 0 .$$

Let $a_k^2 = 1 - \phi_k(\xi)$, $b_k^2 = 1 - \phi_k(\eta)$ and $c_k^2 = 1 - \phi_k(\xi + \eta)$. Then the above inequality implies that

$$c_k^2 \leqq a_k^2 + b_k^2 - a_k^2 b_k^2 + a_k b_k \sqrt{(2 - a_k^2)(2 - b_k^2)} \leqq (a_k + b_k)^2 .$$

Consequently,

$$g(\xi + \eta) \leqq \{[g(\xi)]^{1/2} + [g(\eta)]^{1/2}\}^2 . \tag{3.1}$$

For any $\xi \in \hat{G}$ consider the open set $\xi - V$. From (3.1) it is immediate that g is bounded on $\xi - V$. The family $\xi - V$, $\xi \in \hat{G}$ is an open covering for the compact $\hat{G}$. Therefore there exists a finite subcover from this. As g is bounded on each member of this subcover it follows that g is bounded on $\hat{G}$.

Let m be the Haar measure of $\hat{G}$ with $m(\hat{G})=1$. As $P\{w: X_n(w) \neq e\} = \int_{\hat{G}} \{1 - \varphi_n(\eta)\} dm(\eta)$, we obtain $\sum_1^\infty P(w: X_n(w) \neq e\} = \int_{\hat{G}} g(\eta) dm(\eta) < \infty$. Since G is discrete this means that for the compact neighbourhood $N = \{e\}$ of e, $\sum_1^\infty P\{w: X_n(w) \notin N\} < \infty$. That $\sum_1^\infty X_n$ exists a.e. follows from Lemma 3.

(iii) Let G be discrete but the distributions of the X_n's need not be symmetric.

Let Y_n, $n = 1, 2, \cdots$ be another independent sequence of g.r.v.'s and independent of the X_n's; let Y_n have the same distribution as X_n, $n = 1, 2, \cdots$.

Write $Z_n = X_n - Y_n$. The Z_n's therefore have symmetric distributions. Also the hypothesis yields that $\prod_1^\infty \eta(Z_n)$ exists a.e. for every $\eta \in \hat{G}$. Hence by (ii) above

$$\sum_1^\infty P\{w: Z_n(w) \neq e\} < \infty . \tag{3.2}$$

The distribution of each X_n is assumed to be regular. Hence there exists a countable set A such that $P\{w: Z_n(w) \in A$ for all $n\} = 1$. Now, if $p_n(a) = P\{w: X_n(w) = a\}$, we have

$$\begin{aligned} P\{w: Z_n(w) = e\} &= \sum_{a \in A} P\{w: X_n(w) = a\} P\{w: Y_n(w) = a\} . \\ &= \sum_{a \in A} p_n^2(a) \leqq \sup_{a \in A} p_n(a) \end{aligned}$$

Since there can only be a finite number of 'values' of X_n for which the associated probability is larger than any preassigned number, the supremum is attained. Let a_n be any one of the values taken by X_n with probability equal to this supremum. Therefore $P\{w: X_n(w) \neq a_n\} \leqq$

$P\{w: Z_n(w) \neq e\}$. Consequently, using (3.2), we obtain

$$\sum_1^\infty P\{w: X_n(w) \neq a_n\} < \infty \tag{3.3}$$

$$\text{or} \quad \sum_1^\infty P\{w: X_n(w) - a_n \notin N\} < \infty . \tag{3.4}$$

Where N is the compact neighbourhood of e consisting only of itself. From (3.3) we conclude that, with probability 1, $X_n = a_n$ except for a finite number of n's. This fact together with the hypothesis implies that $\prod_1^\infty \eta(a_n)$ exists for every $\eta \in \hat{G}$. That $\prod_1^\infty \eta(X_n - a_n)$ exists a.e for every $\eta \in \hat{G}$ is then immediate. Now using (3.4) we see by lemma 3 that $\sum_1^\infty (X_n - a_n)$ exists a.e. By Theorem A or by applying Lemma 3 to the random variables a_n we see however that $\sum_1^\infty a_n$ exists since $\prod_1^\infty \eta(a_n)$ exists, for every $\eta \in \hat{G}$. Hence $\sum_1^\infty X_n$ exists a.e., as was to be proved.

(iv) Let G be any metric abelian locally compact group. Let N be a compact symmetric neighbourhood of e and G^* the closed subgroup generated by N. Necessarily G^* is σ-compact and open. Let σ_1 be the natural mapping of G onto $G_1 = G/G^*$. As G^* is open, G_1 is discrete. Further $\hat{G}_1$ consists precisely of those elements of $\hat{G}$ which are identically one on G^*. Hence $\prod_1^\infty \eta(X_n)$ exists a.e. for each $\eta \in \hat{G}$ implies that $\prod_1^\infty \xi(Y_n)$ exists a.e. for each $\xi \in \hat{G}_1$, where $Y_n = \sigma_1 X_n$. By part (iii) above, $P\{w: Y_n(w) \neq e_1 \text{ i.o.}\} = 0$ where e_1 is the identity of G_1. That is

$$P\{w: X_n(w) \notin G^*\} = 0 . \tag{3.5}$$

In other words, there is probability 1 that all except a finite number of the X_n's lie in G^*.

As G^* is generated by a compact symmetric neighbourhood of e there exists, by Theorem B, a discrete group D with a finite number exists, by Theorem B, a discrete group D with a finite number of generators such that $G_2 = G^*/D$ is compact and $D \cap (N - N) = \{e\}$. Let e_2 be the identity element of G_2 and σ_2 the natural mapping of G^* onto G_2. Write $Z_n = \sigma_2 X_n$ if $X_n \in G^*$ and $= e_2$ if $X_n \notin G^*$. Hence Z_n, $n = 1, 2, \cdots$ is an independent sequence of g.r.v.'s in G_2. Recall that $\hat{G}^*$ consists of all the elements of $\hat{G}$ restricted to G^* and that $\hat{G}_2$ consists precisely of those elements of $\hat{G}^*$ which are identically 1 on D. Using the hypothesis and the equation (3.5) we get $\prod_1^\infty \xi(Z_n)$ exists a.e. for every $\xi \in \hat{G}_2$. Therefore we have

$$P\{w: Z_n(w) \notin \sigma_2(N) \text{ i.o.}\} = 0 \quad \text{i.e.} \quad P\{w: X_n(w) \notin N + D \text{ i.o.}\} = 0 .$$

Define $s_n = X_n$ if $X_n \in N + D$ and $s_n = e$ if $X_n \notin N + D$. Then for each s_n we have the unique decomposition $s_n = u_n \pm v_n$ where

$u_n \in N$ and $v_n \in D$. The u_n's form an independent sequence of g.r.v.'s and so do the v_n's. It is immediate from the hypothesis that $\prod_1^\infty \eta(s_n)$ exists a.e. for each $\eta \in \hat{G}$. Also, since $\prod_1^\infty \xi(Z_n)$ exists a.e. for each $\xi \in \hat{G}_2$, $\prod_1^\infty \eta(u_n)$ exists a.e. for each $\eta \in \hat{G}$. Hence $\prod_1^\infty \xi(v_n)$ exists a.e. for each $\xi \in \hat{D}$. As D is discrete we have, by part (iii), $P\{w: X_n(w) \neq e \text{ i.o.}\} = 0$. This is equivalent to saying $P\{w: s_n(w) \neq u_n(w) \text{ i.o.}\} = 0$. Or $P\{w: X_n(w) \notin N \text{ i.o.}\} = 0$ i.e. $\sum_1^\infty P\{w: X_n(w) \notin N\} < \infty$. That $\sum_1^\infty X_n$ exists a.e. follows now by Lemma 3.

I thank the referee for his suggestions leading to a shorter proof of Lemma 1.

REFERENCES

1. J. L. Kelly, *General Topology*, New York, D. Van Nostrand Co., 1955.
2. M. Loève, *Probability Theory*, New York, D. Van Nostrand Co., 1955.
3. D. Montgomery, and L. Zippin, *Topological transformation groups*, New York, Interscience Publishers, 1955.
4. B. J. Pettis, *On the continuity and openness of homomorphisms in topological groups*, Ann. of Math., **51** (1950), 293-308.
5. L. Pontrjagin, *Topological Groups*, Princeton, 1939.

OSMANIA UNIVERSITY, INDIA
AND
UNIVERSITY OF OREGON