# A GEOMETRIC CHARACTERIZATION FOR A CLASS OF DISCONTINUOUS GROUPS OF LINEAR FRACTIONAL TRANSFORMATIONS

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Let $\mathfrak{V} = \{V_i \mid V_i z = (a_i z + b_i)/(c_i z + d_i);\ a_i d_i - b_i c_i = 1, i = 1, 2, \cdots\}$ be a group of linear fractional transformations, where $a_i, b_i, c_i, d_i$ ($i = 1, 2, \cdots$) denote complex numbers. As indicated we use $V_i$ and $V_i z$ to denote transformations and we use (linear) transformation in short for linear fractional transformation. A point $z$ of the plane (by plane we mean, of course, plane of complex numbers) is called a limit point of $\mathfrak{V}$ if there exists a point $z_0$ and an infinite sequence of distinct transformations of $\mathfrak{V}$, say, $\{U_i\}$ such that $U_i z_0 \to z$ as $i \to \infty$. A point of the plane which is not a limit point is called an ordinary point of the group. A discontinuous group is one for which there exists an ordinary point. If $c_i \neq 0$, we define $I(V_i) = \{z \mid |c_i z + d_i| = 1\}$ and $K(V_i) = \{z \mid |c_i z + d_i| < 1\}$, called the isometric circle and isometric disk of $V_i$, respectively. The main result is contained in the following theorem which is proved in Part I of this paper.

THEOREM 1. *Let $\mathfrak{V}$ be a group of linear fractional transformations all of whose elements (except the identity) possess isometric circles whose radii are bounded. Then $\mathfrak{V}$ is discontinuous if and only if there exists an open set of points in the plane that is exterior to the union of all isometric circles.*

According to the theorem discontinuity for the class of groups in question could be defined in terms of the geometry of the isometric circles. In addition, it will be shown that the set of points exterior to the union of all isometric circles could be used to construct a fundamental region for these groups. This last result removes certain restrictions on a known result which is found in [1] (p. 39–49). There Ford shows that if a group is discontinuous and if infinity is an ordinary point, then the radii of all isometric circles are bounded and some neighborhood of infinity is exterior to the union of all isometric circles. The set of points exterior to the isometric circles he uses to construct a fundamental region for the group. In Ford's proof the fact that infinity is an ordinary point is crucial. For the

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class of discontinuous groups characterized by Theorem 1 we remove that distinguished role of infinity. We would like to mention that Ford uses the concept of 'proper discontinuity' rather than discontinuity as defined here. However the results carry over, since the two concepts are equivalent ([2]).

In Part II of this paper we give an example for a group $\mathfrak{V}$ of linear transformations for which the closed disks $\bar{K}(V_i)$ $(i = 1, 2, \cdots)$ cover the plane. By Theorem 1 it follows that $\mathfrak{V}$ is not discontinuous. This shows that the set of groups of linear transformations which satisfy the hypotheses of Theorem 1 and whose isometric circles cover the plane is not empty. This group is also discrete; it is therefore, like the Picard group, an example of a discrete group which is not discontinuous.

## Part I

We say that the plane is almost covered by closed disks if the points that are exterior to the union of all the disks do not comprise an open set of the plane. In the following for cover or almost cover we write in short 'cover'. First we prove

Theorem 2. *Let $\mathfrak{V}$ be a group of linear transformations all of whose elements save the identity possess isometric circles. If the isometric disks $\bar{K}$ 'cover' the plane and if their radii are bounded, then $\mathfrak{V}$ is not discontinuous.*

It is no restriction to assume that the number of transformations in $\mathfrak{V}$ is denumerable, since this is a necessary condition for discontinuity of $\mathfrak{V}$. First we prove six lemmas about the group $\mathfrak{V}$. We assume throughout that the hypotheses of Theorem 2 hold.

Lemma 1. *If $z_0$ is an elliptic fixed point of order $n > 1$, then every point $z$ equivalent to $z_0$ under $\mathfrak{V}$ is an elliptic fixed point of order $n$.*

The proof is easy and we omit it here.

Lemma 2. *Let $\{I_n\}$ be a sequence of isometric circles of infinitely many distinct transformations of $\mathfrak{V}$ with radii $r_n$ $(n = 1, 2, \cdots)$. If the centers of the $I_n$ converge to the finite point $\delta$, then the sequence $\{r_n\}$ is a nullsequence.*

*Proof.* Let $U_n z = (a_n z + b_n)/(c_n z + d_n)$ be the elements of $\mathfrak{V}$ for which $I(U_n) = I_n$ $(n = 1, 2, \cdots)$. Since the radii are bounded, the sequence of positive constants $\{r_n\}$ has at least one limit point. Let $r$ be a limit point, and let us assume that $r > 0$. Then, on a subsequence, we have $\lim_{j\to\infty} r_j = \lim_{j\to\infty} (1/|c_j|) = r$. The sequence $\{c_j\}$, where $c_j = (1/r_j)e^{i\varphi_j}$, has a finite limit point $c \neq 0$; and, on a subsequence, $\lim_{k\to\infty} c_k = c$. To every $c_k$ of the last subsequence corresponds a $U_k$ whose isometric circle has center $-d_k/c_k$. By hypothesis we have $\lim_{k\to\infty}(-d_k/c_k) = \delta$. On noting that $U_k U_{k+1}^{-1} \neq I$ and that the matrix of the transformation $U_k U_{k+1}^{-1}$ is of the form $\begin{pmatrix} x & x \\ c_k d_{k+1} - c_{k+1} d_k & x \end{pmatrix}$, where the $x$ stands for certain complex numbers, we deduce that for sufficiently large $k$ $|c_k d_{k+1} - c_{k+1} d_k| = |(d_{k+1}/c_{k+1} - d_k/c_k)c_k c_{k+1}| < \varepsilon$, where $\varepsilon$ is arbitrarily small and positive. But this is impossible, since all elements of $\mathfrak{V}$ (except I) possess isometric circles whose radii are bounded.

LEMMA 3. *If $z_0$ lies within infinitely many isometric circles, then it is a limit point of $\mathfrak{V}$.*

*Proof.* Since the radii of the isometric circles are bounded, every neighborhood of infinity contains centers of isometric circles. Thus infinity as an accumulation point of such centers is a limit point of $\mathfrak{V}$. This in turn implies that the centers of all isometric circles are limit points of the group.

Let $g_n$ denote the center of $I_n$ $(n = 1, 2, \cdots)$. Let $K$ be a positive real number such that $r_n < K$ for $n = 1, 2, \cdots$, and let $\{g_j\}$ be a sequence of centers of those isometric circles that satisfy the hypotheses of the lemma. Since $|z_0 - g_j| < K$ or $|g_j| < K + |z_0|$, the sequence $\{g_j\}$ is bounded. We pick a limit point $\delta$. Then, on a subsequence, we have $\lim_{k\to\infty} g_k = \delta$. Let the sequence $\{I_k\}$ correspond to the last subsequence. Since $|z_0 - g_k| < r_k$ and since by Lemma 2 $\lim_{k\to\infty} r_k = 0$, we deduce $\lim_{k\to\infty} g_k = z_0$. Thus $z_0$ as accumulation point of limit points of $\mathfrak{V}$ is itself a limit point.

LEMMA 4. *If every neighborhood of a point $z_0$ contains arcs of infinitely many isometric circles, then $z_0$ is a limit point of $\mathfrak{V}$.*

*Proof.* Since the lemma certainly holds when $z_0$ is an accumulation point of centers of isometric circles, we assume that these centers are bounded away from $z_0$. Let $C$ be a circle with center $z_0$ and of radius $\rho$ so that the centers of all isometric circles lie outside $C$; let $C'$ be a circle with center $z_0$ and of radius $\rho/2$. We consider the infinite set of isometric circles $\varphi = \{I_n \mid I_n \cap C' \neq \Lambda; n = 1, 2, \cdots\}$, where $\Lambda$ denotes the empty set. Their radii $r_n > \rho/2$ $(n = 1, 2, \cdots)$. The sequence $\{g_n\}$ consisting of the centers of the isometric circles in $\varphi$ is bounded (see proof of Lemma 3). If $\delta$ denotes a limit point of $\{g_n\}$

then $|\delta - z_0| \geqq \rho$, and, on a subsequence, we have $\lim_{k\to\infty} g_k = \delta$. To this subsequence corresponds the sequence of isometric circles $\{I_k\}$ whose centers accumulate at $\delta$ only. By Lemma 2 the sequence $\{r_k\}$ is a null sequence, which contradicts $r_n > \rho/2$ for all $n$.

When every neighborhood of a point $z_0$ contains arcs of isometric circles we say that $z_0$ is an accumulation point of arcs of isometric circles. We observe that Lemma 4 includes the case where infinitely many of the circles pass through $z_0$. In view of the hypothesis that the isometric circles 'cover' the plane, a consequence of Lemma 4 is

LEMMA 5. *If $z_0$ is exterior to all isometric circles, then it is a limit point of $\mathfrak{B}$.*

LEMMA 6. *If $z_0$ is not an accumulation point of arcs of isometric circles and if it does not lie within an isometric circle, then $z_0$ lies on at least three isometric circles.*

*Proof.* Clearly, $z_0$ cannot be exterior to all isometric circles. Thus it lies on at least one isometric circle. If only one or two circles were to pass through $z_0$ we could construct a neighborhood of $z_0$ sufficiently small so that all other isometric circles lie outside this neighborhood. In either case the neighborhood contains an open set that is exterior to all isometric circles. Observing that three circles passing through $z_0$ can be arranged so that all points in a sufficiently small deleted neighborhood of $z_0$ lie within a circle the lemma follows.

These preliminary results we use now in the proof of Theorem 2. Let $\mathscr{L}$ be the set of limit points of $\mathfrak{B}$ on the Riemann sphere. Then $\mathscr{L}$, which is a closed set ([1], p. 43), contains

( i ) the centers of all isometric circles (see proof of Lemma 3),

(ii) the nonelliptic fixed points of all transformations of $\mathfrak{B}$ (we assume that $\mathfrak{B}$ contains elliptic transformations of finite order only, since, if that were not the case, the theorem would be trivial),

(iii) the points that lie outside all isometric circles (see Lemma 5),

(iv) the points which are accumulation points of arcs of isometric circles (see Lemma 4).

Suppose that there is a point $z \notin \mathscr{L}$. Since then every point in a sufficiently small neighborhood of $z$ is an ordinary point of $\mathfrak{B}$ (the set of ordinary points on the Riemann sphere is open, since $\mathscr{L}$ is closed) and since the number of elliptic fixed points is at most denumerable, it is no restriction to assume that $z$ lies within an isometric circle and that $z$ is not an elliptic fixed point. The isometric circle within which $z$ lies we denote by $I(U_1)$. $U_1$ carries $z$ into $z_1$, where $z_1 \neq z$ and where $z_1$ lies outside $I(U_1^{-1})$. Furthermore, $z_1 \notin \mathscr{L}$, since an ordinary point is not mapped on a limit point. Either $z_1$ lies within isometric circles, in which case we pick one of them and call it $I(U_2)$, or, according to Lemma 6, it lies on at least three isometric circles. In the latter case $I(U_2)$ denotes any one of them. Certainly $U_2 \neq U_1^{-1}$. In the same manner we proceed with $z_2 = U_2 z_1 = (U_2 U_1)z$. Again, because of Lemma 1 $z_2 \neq z_1$, and $z_2 \notin \mathscr{L}$. If $z_2$ does not lie within an isometric circle and if $z_1$ lies on $I(U_2)$, then $z_2$ lies on $I(U_2^{-1})$. By Lemma 6, it is then possible to pick $I(U_3) \neq I(U_2^{-1})$, and hence $U_3 \neq U_2^{-1}$. Continuing in this manner we obtain an infinite number of transformations $W_n = U_n U_{n-1} \cdots U_1$ $(n = 1, 2, \cdots)$, where $U_i U_{i-1} \neq I$ $(i = 2, 3, \cdots)$. Because of Lemma 3 the proof of Theorem 2 will be complete if we can show that $z$ lies within $I(W_n)$ for $n = 1, 2, \cdots$.

Let $U_i z = (a_i z + b_i)/(c_i z + d_i)$, and let $\delta(U_i, z) = |c_i z + d_i|^{-2}$, called the deformation of $U_i$ ([2]). Then $\delta(U_i, z)$ is greater than, equal to, or less than one according as $z$ lies within, on, or outside $I(U_i)$. It is readily verified that

$$\delta(U_j U_i, z) = \delta(U_j, U_i z)\delta(U_i, z)\,, \tag{1}$$

and by an induction argument the formula can be extended to a product of any number of transformations. For $n \geqq 1$ we have $\delta(W_n, z) = \delta(U_n \cdots U_1, z) = \delta(U_1, z)\delta(U_2, U_1 z) \cdots \delta(U_n, U_{n-1} \cdots U_1 z) > 1$, since $\delta(U_1, z) > 1$ and every other factor $\delta(U_k, U_{k-1} \cdots U_1 z) \geqq 1 (k = 2, \cdots, n)$. This implies that $z$ lies within $I(W_n)$ $(n = 1, 2, \cdots)$. Hence $z \in \mathscr{L}$; a contradiction.

If, however, the isometric disks $\bar{K}$ of $\mathfrak{B}$ do not 'cover' the plane we have the following theorem.

THEOREM 3. *Let $\mathfrak{B}$ be a group of linear fractional transformations all of whose elements (except the identity) possess isometric circles. If there exists an open set of points that is exterior to all isometric circles, then $\mathfrak{B}$ is discontinuous.*

*Proof.* Let $\mathscr{O}$ be the open set in the hypothesis. Pick $z_0$ in $\mathscr{O}$, where $z_0$ is finite. There exists a $\varepsilon$-neighborhood $N_\varepsilon$ of $z_0$ such that $z \in N_\varepsilon$ implies $z \in \mathscr{O}$. For any transformation $V$ in $\mathfrak{B}$ $(V \neq I)$ $Vz_0$ lies within $I(V^{-1})$, or $|Vz_0 - z_0| > \varepsilon$. Thus $z_0$ is a standard point of $\mathfrak{B}$ ([3], p. 38). Since every standard point is an ordinary point ([3], p. 47), $\mathfrak{B}$ is discontinuous.

This completes the proof of Theorem 1, since Theorems 2 and 3 imply the former.

Next, the remark about the fundamental region in the introduction calls for further elucidation. Let $\mathfrak{B}$ be a discontinuous group that satisfies the hypothesis of Theorem 1 and for which infinity is a limit point (for the case in which infinity is an ordinary point the following is well known). Let $K_i$ $(i = 1, 2, \cdots)$ be the isometric disks of $\mathfrak{B}$, let $\mathscr{O}$ be the set of points exterior to the set $\bigcup_i K_i$, and let $\mathscr{M}$ denote the set of ordinary points of $\mathfrak{B}$. Then $\mathscr{O}$ and $\mathscr{M}$ are open sets. Furthermore, $\mathscr{O} \subset \mathscr{M}$ is an immediate consequence of the proof of Theorem 3. Here, we do not intend to give a definition for a fundamental region. However, we want to show that $\mathscr{O}$ has the two properties that are customarily used in any definition; namely,

( i ) no two points of $\mathscr{O}$ are equivalent under $\mathfrak{B}$ and

(ii) every point in $\mathscr{M}$ is equivalent to some point in $\bar{\mathscr{O}}$, the closure of the set $\mathscr{O}$.

As for (i) we note that if $z \in \mathscr{O}$ and $V \in \mathfrak{B}$ $(V \neq 1)$, then $Vz$ lies within $I(V^{-1})$ and hence is exterior to $\mathscr{O}$.

The gist of our proof of (ii) is the same as that of the corresponding proof in [2], where infinity is considered to be an ordinary point of the group. In our proof we make use of the following lemma, where we use primes to denote derivatives.

LEMMA 7. *Let $f_i(z)$ $(i = 1, \cdots, k)$, where $k$ is an integer greater than 1, be nonvanishing holomorphic functions in a domain $\mathscr{D}$, and let for $j \neq i$ $f_i'(z_0)f_j(z_0) - f_j'(z_0)\ f_i(z_0) \neq 0$ and $|f_i(z_0)| = |f_j(z_0)|$ $(i, j = 1, \cdots, k)$ for some point $z_0$ in $\mathscr{D}$. Then every neighborhood of $z_0$ in $\mathscr{D}$ contains a point $z^*$ such that $|f_i(z^*)| \neq |f_j(z^*)|$ for $j \neq i$ $(i, j = 1, \cdots, k)$.*

*Proof*. For $i, j = 1, \cdots, k$ and $i < j$ we define the functions $f_{ij}(z) = f_i(z)/f_j(z)$. We observe that $|f_{ij}(z_0)| = 1$ and $f_{ij}'(z_0) \neq 0$. We choose the (circular) neighborhood $N(z_0)$ of $z_0$ so small that the mappings $f_{ij}(z)$ are one-to-one. For each function $f_{ij}(z)$ the level curve $|f_{ij}(z)| = 1$ consists of a finite number of disjoint analytic arcs in $N(z_0)$. If we pick a point $z^*$ in $N(z_0)$ that does not lie on any level curve the conclusion of the lemma holds.

Let $z_0 \in \mathscr{M}$ and $z_0 \notin \bar{\mathscr{O}}$. In view of Lemma 3 $z_0$ lies within or on a finite number of isometric circles. Let $U_i$ $(i = 1, \cdots, n)$ denote the transformations in $\mathfrak{B}$ whose isometric circles $I(U_i)$ have this property. Since every element of $\mathfrak{B}$ (save the identity) possesses an isometric circle, no two of the $I(U_i)$'s coincide.

We divide the proof into two parts.

( i ) We assume $\delta(U_1, z_0) > \delta(U_i, z_0)(i = 2, \cdots, n)$. Then $U_1z_0$ lies in $\mathscr{O}$. For suppose that $U_1z_0$ lies within or on some isometric circle $I(V)$. Using (1) for the deformation of the transformation $VU_1$ we deduce $\delta(VU_1, z_0) = \delta(V, U_1z_0)\ \delta(U_1, z_0) \geqq \delta(U_1, z_0)$. This would imply that $z_0$ lies within or on $I(VU_1)$. Hence $VU_1 = U_i$ for some $i$ with $1 < i \leqq n$; which contradicts the maximum property of $\delta(U_1, z_0)$. We remark that the proof still holds for $n = 1$.

(ii) We assume $\delta(U_1, z_0) = \delta(U_2, z_0) = \cdots = \delta(U_k, z_0) > \delta(U_i, z_0)$ for $k < i \leqq n (1 < k \leqq n)$. Let $N(z_0)$ be a neighborhood of $z_0$ containing only ordinary points of $\mathfrak{V}$ and being so small that $\delta(U_j, z) > \delta(U_i, z)$ $(j = 1, \cdots, k; i = k+1, \cdots, n)$ holds for all $z$ in $N(z_0)$ and that $N(z_0)$ does not intersect any isometric disk other than the $K(U_i)$ $(i = 1, \cdots, n)$. As one readily verifies the functions $U_j'(z)$ $(j = 1, \cdots, k)$, where the prime denotes the derivative, satisfy the hypothesis of Lemma 7 with $N(z_0)$ in place of $\mathscr{D}$. Thus we conclude that every neighborhood of $z_0$ contains a point $z^*$ having the property that, for some integer $m$ with $1 \leqq m \leqq k$, $\delta(U_m, z^*) > \delta(U_j, z^*)$ $(j = 1, \cdots, k; j \neq m)$. By part (i) of this proof it follows that $U_m(z^*) \in \mathscr{O}$; and by continuity we have $U_{m^*}(z_0) \in \mathscr{O}$ for some suitable $m^*$ with $1 \leqq m^* \leqq k$. This completes the proof about the two properties of $\mathscr{O}$.

We conclude this part with a remark. Let $\mathfrak{V}$ be a nondiscontinuous group of linear transformations satisfying the hypotheses of Theorem 2. For $V \in \mathfrak{V}$ we denote by $V^*$ the $2 \times 2$ matrix that can be associated with $Vz$. Then $\mathfrak{V}^* = \{\pm V^* \mid V \in \mathfrak{V}\}$ is a group under matrix multiplication. Since every element of $\mathscr{V}$ (save the identity) possesses an isometric circle and since all the radii are bounded, in every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathfrak{V}^*$ (save the two identity matrices) $c \neq 0$ and all the $c$'s are bounded away from 0. This implies that $\mathfrak{V}^*$ is discrete; that is, $\mathfrak{V}^*$ does not contain a sequence of distinct matrices $\{V_n^*\}$ such that $V_n^* \to I^*$ as $n \to \infty$, where $I^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus all the nondiscontinuous groups characterized in Theorem 2 are such that the corresponding groups of matrices are discrete. That nondiscontinuous groups, as considered here, exist is not a trivial fact. We dovote the second part of this paper to the construction of a group of this type.

## PART II

Here we give an example of a group of linear transformations that contains only elements with isometric circles which have the property that the closed isometric disks cover the plane. We divide the construction into three parts.

1. We construct a covering of the plane by closed disks $\bar{K}$ such that the open disks $K$ are mutually disjoint. To begin with, we draw circles of radius unity and with centers at the points with coordinates $(2m+1, 2n+1)$ $(m, n = 0, \pm 1, \pm 2, \cdots)$. After drawing circles of radius $(\sqrt{2} - 1)$ units and with centers $(2m, 2n)$ $(m, n = 0, \pm 1, \pm 2, \cdots)$, there remain the interiors of congruent triangles whose sides are circular arcs uncovered. Within every triangle we construct a circle touching all three sides, and we continue in this manner. The follow-

ing proof[1] shows that by this construction every point of the plane lies within or on a circle.

Diagram 1 shows a triangle we encounter in our construction,

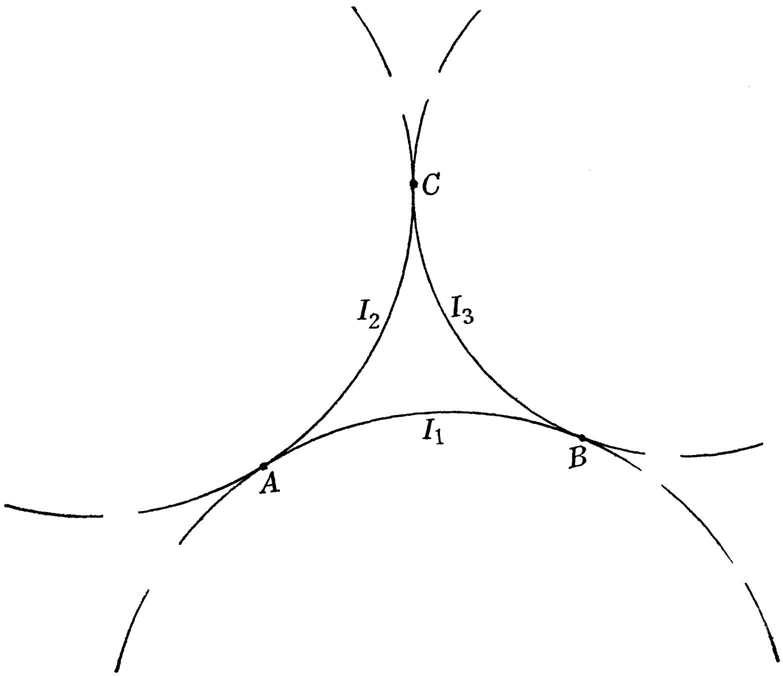


Diagram 1.

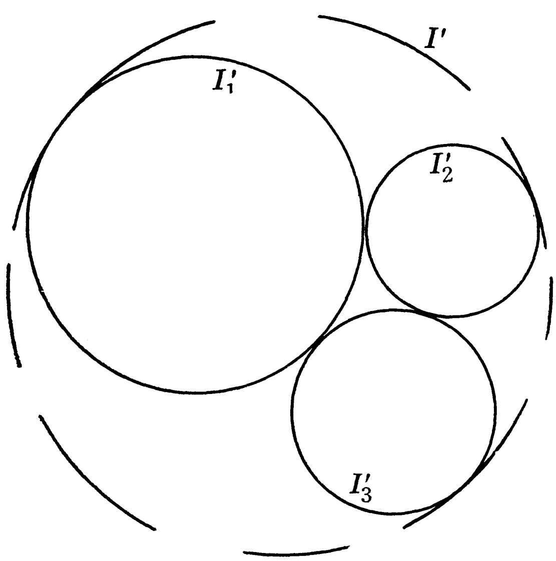


Diagram 2.

[1] This proof I owe to F. Herzog.

where $I_1$, $I_2$ and $I_3$ denote the circles whose arcs form the sides of triangle $ABC$. Let $z_0$ be any point in the interior of the triangle, and let $S$ be a linear fractional transformation such that $Sz_0 = \infty$. Depending on the relative sizes of the image circles $I_k'$ of $I_k$ under $S\,(k = 1, 2, 3)$, we distinguish two cases. Either, (i) we can construct a circle $I'$ such that the $I_k'\,(k = 1, 2, 3)$ are tangent internally to $I'$ as indicated in Diagram 2. Then $S^{-1}$ maps $I'$ on a circle $I$ that touches the three sides of triangle $ABC$ and contains $z_0$ in its interior.

Or (ii) no circle $I'$ as assumed in (i) exists. Then the configuration of the circles $I_k'$ $(k = 1, 2, 3)$ resembles the one in Diagram 3, where triangle $A'B'C'$ is the image of triangle $ABC$ under $S$.

Let $t_1$ be the common tangent to $I_1'$ and $I_3'$ through $B'$, and let $t_2$ be the common tangent to $I_2'$ and $I_3'$ through $C'$. Clearly, the two tangents intersect in a point that lies within triangle $A'B'C'$. Let $t$ be that common tangent to $I_1'$ and $I_2'$ which is shown in Diagram 3. We construct the circle $K_1'$ which is tangent externally to the $I_k'$

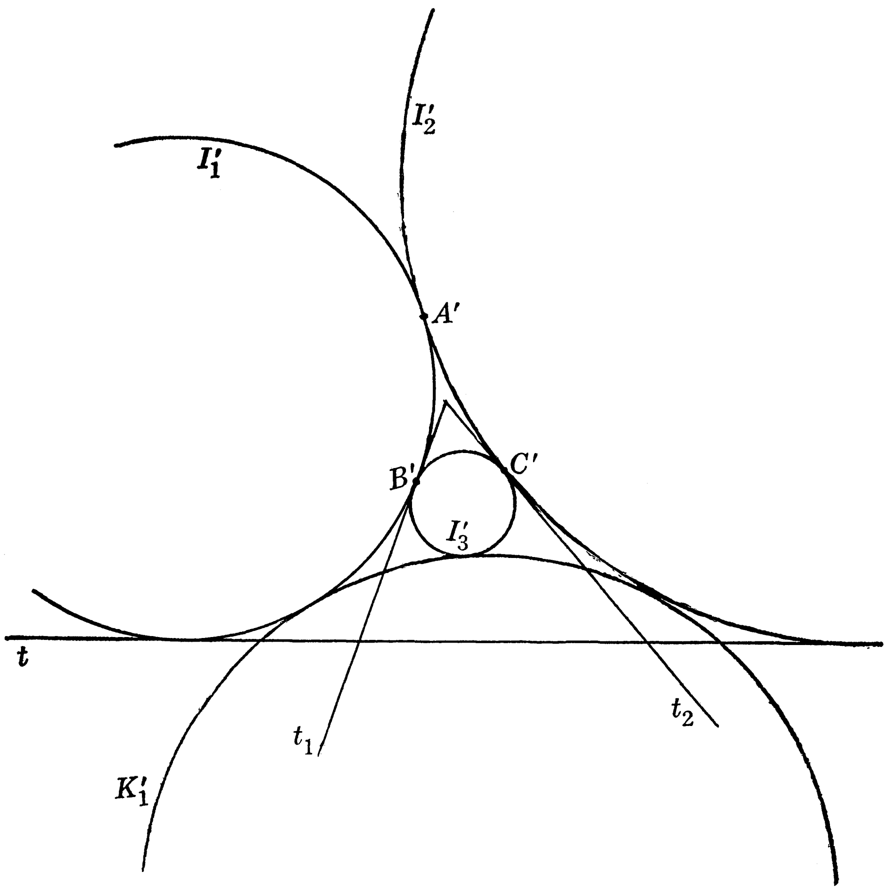


Diagram 3.

$(k = 1, 2, 3)$ and which lies outside triangle $A'B'C'$. Next we construct the circle $K_2'$ which is tangent externally to $I_1'$, $I_2'$ and $K_1'$. We continue this process until we come across the first circle, say, $K_s'$ that is tangent externally to $I_1'$, $I_2'$ and $K_{s-1}'$ and that intersects or touches the line $t$. That the $K_s'$ exists follows from the fact that the radii of successive circles $K_i'$ $(i = 1, \cdots, s)$ increase. Rigorous arguments for this we omit, in order not to lengthen the paper unduly. Finally we construct the circle $I'$ which contains the circles $I_1'$, $I_2'$ and $K_s'$ in its interior and which is tangent with each of them. In the case when $K_s'$ touches the tangent line $t$, $I'$ is the degenerate circle $t$. Under the mapping $S^{-1}$ we obtain a chain of circles within triangle $ABC$, each $K_i$, where $K_i = S^{-1}K_i'$, being tangent externally to $I_1$, $I_2$ and $K_{i-1}$ $(i = 2, \cdots, s)$. $S^{-1}I' = I$ is the circle that is tangent externally to $I_1$, $I_2$ and $K_s$ and that contains $z_0$ in its interior. In the degenerate case $I$ will pass through $z_0$. This completes the proof.

2. In order to associate linear transformations with the covering circles we group them in pairs of circles of equal radii. The circles with centers $(2n, 0)$ $(n = 0, \pm 1, \pm 2, \cdots)$ and with radii $(\sqrt{2} - 1)$ units we pair in some way so that all are used up. The remaining circles whose interiors intersect the $y$-axis are mapped under reflection in the $x$-axis on congruent circles. Each circle we pair with its image under this reflection. Under reflection in the $y$-axis all the remaining circles are mapped on congruent circles, and we pair them accordingly.

With every pair of circles $I$ and $I'$, with centers $\alpha$ and $\beta$, respectively, and with radii $r$, we associate a linear fractional transformation $V$ such that $I$ is the isometric circle of $V$ and $I'$ that of $V^{-1}$. It is readily verified that $V$ is of the form

$$Vz = [(\beta/r)e^{i\varphi}z - ((\alpha\beta/r)e^{i\varphi} + re^{-i\varphi})]/[(1/r)e^{i\varphi}z - (\alpha/r)e^{i\varphi}] ,$$

where $\varphi$ may be chosen arbitrarily and where we used the normalisation det. $V = 1$.

Since every circle contains a point with rational coordinates, the number of transformations is denumerable. We denote them by $V_1$, $V_1^{-1}$, $V_2$, $V_2^{-1}$, $\cdots$. Let $G_i = \{V_i^k \mid k = 0, \pm 1, \cdots\}$ $(V_i^0 = I)$ and let $\mathfrak{V} = *\prod_i G_i$, the free product of the cyclic groups $G_i$. If $\bigcup_i G_i$ denotes the union of the $G_i$, then $\bigcup_i G_i = \mathfrak{V}$ ([2]).

3. There remains to be shown that each element $(\neq I)$ of $\mathfrak{V}$ has an isometric circle and that the radii of all circles are bounded. These properties are consequences of the following lemma.

LEMMA 8. *Let $\{T_n\}$ be an infinite sequence of linear transformations which with every $T_n$ contains $T_n^{-1}$, and all of whose elements* *possess isometric circles. Let* $K_n$ *denote the isometric disk* $K(T_n)$ *and let* $\wedge$ *denote the empty set. If* $K_i \cap K_k = \wedge$ *for* $k \neq i$ $(i, k = 1, 2, \cdots)$, *then* $K_{n_1} \supset K$, *the isometric disk of* $W = T_{n_s} T_{n_{s-1}} \cdots T_{n_1}$, *where* $n_1, \cdots, n_s$ *are arbitrary positive integers, not necessarily distinct, except that* $T_{n_{i+1}} T_{n_i} \neq I$.

*Proof.* For $i = 1, \cdots, s$ we put $T_{n_i} = S_i$, $I_i = I(S_i)$, $I_i' = I(S_i^{-1})$, and $K_i' = K(S_i^{-1})$. Let $z$ be any point outside $I_1$. Then $S_1 z$ lies in $K_1'$, and hence outside $I_2$. $S_2(S_1 z)$ lies in $K_2'$, and hence outside $I_3$. By an induction argument it follows that $Wz$ lies in $K_s'$. Since at each step lengths in the neighborhood of $z$ or its images are decreased the lemma follows.

Let $\{U_n\}$ be a sequence comprising the transformations $V_1$, $V_1^{-1}$, $V_2$, $V_2^{-1}$, $\cdots$. Then $\{U_n\}$ satisfies the hypotheses of Lemma 8 and, in addition, the radii of the isometric circles of the $U_n$ do not exceed unity. Hence we conclude that the radii of all elements of the cyclic groups $G_i$ $(i = 1, 2, \cdots)$ as well as those of $\mathfrak{V}$ do not exceed unity.

We close with a remark. That the free product $* \prod_i G_i$ is an isomorphic image of $\bigcup_i G_i$ is an easy consequence of Lemma 8.

## References


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