THE STRUCTURE OF THE ORBITS AND THEIR LIMIT SETS IN CONTINUOUS FLOWS

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1. Introduction. If f is a mapping of a product space $X \times Y$ into a space Z, then the image of $(x, y) \in X \times Y$ under f is denoted by xy. A continuous flow $\mathscr{F}$ on a metric space X is a continuous mapping f of the product space $X \times R$, where R is the space of real numbers, onto X such that (1) for each $r \in R$, xr is a homeomorphism of X onto X and (2) for each $x \in X$ and $r, s \in R$, $(xr)s = x(r + s)$.

For each $x \in X$ the sets $O(x) = \{xr \mid r \in R\}$, $O_+(x) = \{xr \mid r \geqq 0\}$, $O_-(x) = \{xr \mid r \leqq 0\}$ are called the orbit, positive semi-orbit and negative semi-orbit of x under $\mathscr{F}$, respectively. The orbit $O(x)$ is either (1) a point, (2) a simple closed curve, or (3) a one-to-one and continuous image of R. In general one can not replace (3) by (3′) a homeomorphic image of R.

Bebutoff [1] has given necessary and sufficient conditions that the entire collection of orbits of a continuous flow be homeomorphic to a family of parallel lines in Hilbert space. In the second section of this paper we solve the simpler problem of describing those points of an arbitrary metric space with orbits homeomorphic to R. These will be the points which are neither positively nor negatively recurrent.

In the last section we discuss the structure of the orbit family of continuous flows on a 2-cell, with special attention being given to the α and ω limit sets of an orbit [5; 6; 7]. The author wishes to acknowledge the referee's assistance in condensing the original paper.

2. The topological nature of the orbits under a continuous flow. Consider a metric space $\{X, \rho\}$ and a continuous flow $\mathscr{F}$ on X. The following definitions are well-known in Topological Dynamics:

DEFINITION 1. A point $x \in X$ is said to be a *rest* point under $\mathscr{F}$ if

$$xr = x$$

for each $r \in R$.

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DEFINITION 2. A point $x \in X$ is said to be *periodic* under $\mathscr{F}$ and $\mathscr{F}$ is said to be periodic at x if there is a $t \in R, t \neq 0$, for which $xt = x$. If $\mathscr{F}$ is periodic at a non-rest point x, then the smallest positive number $w \in R$ for which $xw = x$ is called the *primitive period* of x.

DEFINITION 3. A point $x \in X$ is said to be positively (negatively) *recurrent* under $\mathscr{F}$ if for each $\varepsilon > 0$ there exists a strictly increasing (decreasing) sequence $\{r_i\}$ of points of R such that $\lim_{i \to \infty} r_i = +\infty(-\infty)$ and

$$\rho(xr_i, x) < \varepsilon$$

for all i.

THEOREM 1. *The point x is neither positively nor negatively recurrent if and only if ϕ: $R \to 0(x)$ defined by $\phi(t) = xt, t \in R$, is a homeomorphism.*

Proof. Since the mapping $f: X \times R \to X$ is continuous, it follows that ϕ is continuous. Assume that x is neither positively nor negatively recurrent. It follows that x is not periodic and thus ϕ is a one-to-one map of R onto $O(x)$. Let $\{xt_i \mid i = 1, 2, \cdots\}$ be a sequence of points of $O(x)$ converging to xt_0. To prove that ϕ^{-1} is continuous it is sufficient to prove that $\lim_{i \to \infty} t_i = t_0$. If this is not the case, either the sequence $\{t_i\}$ contains a subsequence which is unbounded and x is either positively or negatively recurrent or the sequence $\{t_i\}$ contains a subsequence converging to $s \neq t_0$, and x is a periodic point. We conclude that ϕ^{-1} is continuous and ϕ is a homeomorphism.

Now suppose ϕ is a homeomorphism and suppose x is positively recurrent. Then there exists a sequence $\{t_i \mid t_i \in R, i = 1, 2, \cdots\}$ with $\lim_{i \to \infty} t_i = +\infty$ and such that $\lim_{i \to \infty} xt_i = x$. But then, since ϕ^{-1} is continuous,

$$\infty = \lim_{i \to \infty} t_i = \lim_{i \to \infty} \phi^{-1}(xt_i) = \phi^{-1}(x) = 0 .$$

Thus x is not positively recurrent. Similarly, x is not negatively recurrent.

The proof is completed.

THEOREM 2. *Let $x \in X$ and let $O(x)$ be homeomorphic to R. Then x is neither positively nor negatively recurrent.*

Proof. By assumption there exists a homeomorphism h of R onto $O(x)$. Then x is not a periodic point. For if x is a periodic point, $O(x)$ is either a point or a simple closed curve which is homeomorphic to a circle. It follows that $xt_1 = xt_2$ implies $t_1 = t_2$.

Let $r \in R$ and let $h(r) = xt$. Then t is uniquely determined. Let $\psi: R \to R$ be defined by $\psi(r) = t$. Since h is an onto homeomorphism, ψ is an onto map and ψ is one-to-one. Let $\phi: R \to O(x)$ be defined by $\phi(t) = xt$, $t \in R$. Then ϕ is continuous, onto and one-to-one, and $\psi^{-1} = h^{-1}\phi$ and thus ψ^{-1} is continuous. Now ψ^{-1} is a continuous, one-to-one, onto map of R onto R and hence is a homeomorphism. Since $\phi = h\psi^{-1}$, it follows that ϕ is a homeomorphism and from Theorem 1 we infer that x is neither positively nor negatively recurrent.

3. The stucture of the α-and ω-limit sets in a continuous flow on a 2-cell. Let X be an open or closed 2-cell, that is, a homeomorphic image of the interior of the unit circle or of the unit disk. Let $\mathscr{F}$ be a continuous flow on X and let $A \subset X$ be the set of rest points under $\mathscr{F}$.

We recall the following definition due to Whitney [9] (cf., also, [8]).

DEFINITION 4. A closed set $S \subset X$ is a *local section* of $\mathscr{F}$ if there exists a $\tau \in R, \tau > 0$, such that for each $x \in S$

$$\{xt \mid |t| \leqq \tau\} \cap S = x .$$

If $x \in S$, then S is called a *local section through* x.

Whitney [9] (cf., also, [5]) proved, for the spaces under discussion, that for each $x \in X - A$ there is an arc $S \subset X$ such that S forms a local section of $\mathscr{F}$ through x. Using this Whitney [9] (cf., also, [5]) proved the following:

LEMMA 1. *If $x \in X - A$, then there exists a local section S of $\mathscr{F}$ through x such that the set*

$$E = \{yt \mid y \in S, |t| \leqq \tau\}$$

can be mapped homeomorphically onto the closed rectangle $|u| \leqq 1$, $|v| \leqq 1$ in such a way that the arcs $\{yt \mid |t| \leqq \tau\}$, for $y \in S$, become the lines $v =$ constant of the rectangle while S has image $u = 0$, $|v| \leqq 1$.

The local section S of Lemma 1 divides the interior of the set E into two disjoint subregions.

DEFINITION 5. Let x, S, and E be as in Lemma 1. That one of the two regions into which S separates E, which the orbit $O(x)$ of x enters under increasing values of r, will be termed (after Bendixson [3]) the *positive side* S^+ of S. The other region will be termed the *negative side* S^- of S.

LEMMA 2. *Let x, S, and E be as in Definition 5. Then each orbit which enters E crosses S from S^- to S^+ under increasing values of r.*

Proof. Suppose the contrary. Then there exists a sequence $\{y_i\}$ of distinct points of S converging to $y \in S$ such that the orbit $O(y_i)$ of each y_i enters one of the two regions S^+ or S^- under increasing values of r, while the orbit $O(y)$ of y enters the other region under increasing values of r. Thus for any t such that $0 < t < \tau$ the points $y_i t$ and yt lie in disjoint subregions of E. This is impossible since $\lim_{i\to\infty} y_i t = yt$.

Let S be any local section of $\mathscr{F}$ and let $y \in S$. Let S^- and S^+ be as in Definition 5, it follows from Lemma 2 that S^- and S^+ are independent of y. Thus if $O(x)$ is any orbit such that $O(x) \cap S \neq 0$, then each crossing of S by $O(x)$ is from S^- to S^+ under increasing values of r. Let the orbit $O(x)$ meet S in successive points x' and x'' in the positive direction on $O(x)$, then x is an interior point of X and $(x'x'') \cup S_1$, where $(x'x'')$ and S_1 denote the subarcs joining x' and x'' of $O(x)$ and S, respectively, is a simple closed curve lying in the interior of X. Let $C = (x'x'') \cup S_1$, it follows that $X - C$ consists of exactly two components. Denote by C^+ that component of $X - C$ which lies on the positive side S^+ of S along S_1 and by C^- the other component of $X - C$. Any simple closed curve C determined in this manner will be termed a harbor [7].

LEMMA 3. *If C is a harbor, then the positive semi-orbit $O_+(y)$ of each $y \in C^+$ lies in C^+, and the negative semi-orbit $O_-(y)$ of each $y \in C^-$ lies in C^-.*

Proof. If $y \in C^+$ and $O_+(y) \cap C^- \neq 0$, then $O_+(y)$ must first cross S on S_1 and hence cross S from S^+ to S^- under increasing values of r which is impossible. If $y \in C^-$ and $O_-(y) \cap C^+ \neq 0$, then $O_-(y)$ must first enter C^+ on S_1 crossing from S^- to S^+ under decreasing values of r which is also impossible.

Using Lemma 3 one can construct a very short proof of the following result proved by Bohr and Fenchel ([4], Vol. II, C38).

If x is a positively or negatively recurrent point of X under $\mathscr{F}$, then x is periodic under $\mathscr{F}$.

Since the only points of X with orbits not homeomorphic to R are those which are either positively or negatively recurrent, it follows that (3) may be replaced by (3′) for continuous flows on 2-cell. Thus if $x \in X$, then the orbit $O(x)$ is either a point, a simple closed curve, or a homeomorphic image of R.

DEFINITION 6. A point $y \in X$ is said to be an *ω-limit* (*α-limit*) *point* of an orbit $O(x) \subset X$ if there exists a strictly increasing (decreasing) sequence $\{r_i\}$ of points of R such that $\lim_{i\to\infty} r_i = +\infty(-\infty)$ and $\lim_{i\to\infty} xr_i = y$. The set of all ω-limit (α-limit) points of an orbit $O(x)$ will be denoted by $w(x)\,(\alpha(x))$.

THEOREM 3. *If x is a nonperiodic point of X under $\mathscr{F}$, then $\omega(x) \cap \alpha(x) \subset A$.*

Proof. Suppose there exists a point y in the set $\omega(x) \cap \alpha(x) - A$. Choose a local section S of $\mathscr{F}$ through y. Then, since $y \in \omega(x) \cap \alpha(x)$, $O_+(x)$ and $O_-(x)$ must both cross S an infinite number of times near y. Thus an arc $(x'x'')$ of $O(x)$ and a subarc S_1 of S form a harbor C. Let p and q denote the end-points of S and assume the labeling so that the order p, x', x'', q holds on S. Then the half-open subarc $(px') - x'$ of S lies in C^- while the half-open subarc $(x''q) - x''$ of S lies in C^+. Now $y \notin S_1$ since $O(x)$ can not cross S on S_1. If $y \in (px')$, then $y \notin \omega(x)$ since the positive semi-orbit $O_+(x'')$ from x'' on lies in C^+. Thus $y \notin (px')$. If $y \in (x''q)$, then $y \notin \alpha(x)$ since the negative semi-orbit $O_-(x')$ from x' on lies in C^-. Thus $y \notin (x''q)$. This is a contradiction of the fact that $y \in S$ and $y \in \omega(x) \cap \alpha(x)$. Hence the theorem is proved.

Throughout the remainder of this section X shall denote a closed 2-cell. Then X contains at least one point a such that a is a rest point under the continuous flow $\mathscr{F}$ [2]. Thus $A \neq 0$. Let F denote the family of orbits $\{O(x) \mid x \in X - A\}$. Then each member of F is either an open arc or a simple closed curve. Since X is compact each of the sets $\omega(x)$, $\alpha(x)$ for any $x \in X$ is a non-null closed and connected subset of X and is the union of points of A and curves of F [8]. It follows from a theorem due to Kaplan [5] that $\omega(x)(\alpha(x))$ is identical with any nondegenerate periodic orbit contained in $\omega(x)$ $(\alpha(x))$. Thus the set $\omega(x)(\alpha(x))$ is either the union of points of A and open arcs of F or a simple closed curve of F.

THEOREM 4. *Let A be a totally disconnected set. If z is a nonperiodic point in $\overline{O(x)} - O(x)$, then $O(z)$ is an open arc whose closure, $\overline{O(z)}$, is either a closed arc with end-points in A or a simple closed curve consisting of the orbit $O(z)$ together with a point of A.*

Proof. The theorem will be proved when it is shown that $\omega(z) \subset A$ and $\alpha(z) \subset A$, since each of the sets $\omega(z)$, $\alpha(z)$ is connected and A is totally disconnected.

Thus suppose $y \in \omega(z) - A$ and let S be a local section of $\mathscr{F}$ through y. Then $O_+(z)$ must cross S an infinite number of times near y. Thus there exists successive points z', z'' in the positive direction on $O_+(z)$ such that $z \in O_-(z') - z'$ and $z', z'' \in S$. Let C be the harbor formed by the arc $(z'z'')$ of $O(z)$ and the subarc S_1 of S between z' and z''. As in the proof of Theorem 3, y and $O_+(z'') - z''$ lie together in C^+ while $O_-(z) \subset C^-$. Since $\overline{O(x)} - O(x) \neq 0$, x is a non-periodic point. Thus, by Theorem 3, z is in exactly one of the sets $\omega(x)$, $\alpha(x)$. $z \in \omega(x)$ implies $\overline{O(z)} \subset \omega(x)$ and $z \in \alpha(x)$ implies $\overline{O(z)} \subset \alpha(x)$. If $z \in \omega(x)$, then $O_+(x)$ must cross S entering C^+ under increasing values of r. By Lemma 3, $O_+(x)$, from where it enters C^+ on, lies in C^+. Then $\omega(x) \subset C^+$ which is impossible since $\overline{O(z)} \subset \omega(x)$, $O_-(z) \subset C^-$ and $\alpha(z) \neq 0$. Thus $z \in \alpha(x)$. Let $U(z)$ be a neighborhood of z such that $U(z) \subset C^-$, and let x' be a point on $O(x)$ in $U(z)$. Then, by Lemma 3, $O_-(x') \subset C^-$. This together with $y \in C^+$ implies $y \notin \alpha(x)$ which is a contradiction of $\overline{O(z)} \subset \alpha(x)$. Hence $z \notin \alpha(x)$. But z is in one or the other of the sets $\omega(x)$, $\alpha(x)$. Thus the assumption that $\omega(z) - A \neq 0$ is false and $\omega(z) \subset A$. In a like manner $\alpha(z) \subset A$. It follows that $\lim_{t\to+\infty} zt = \omega(z) \in A$ and $\lim_{t\to-\infty} zt = \alpha(z) \in A$. Thus $\overline{O(z)} = O(z) \cup \omega(z) \cup \alpha(z)$ is a closed arc with end-points in A or a simple closed curve consisting of $O(z)$ and a point of A according as $\omega(z) \neq \alpha(z)$ or $\omega(z) = \alpha(z)$.

The proof of the theorem is completed.

THEOREM 5. *Let A be a totally disconnected set and let $x \in X - A$ be such that $\overline{O(x)} \cap A \neq 0$. Let $a \in \omega(x) \cap A$ $(\alpha(x) \cap A)$, and suppose $\omega(x) \neq a$ $(\alpha(x) \neq a)$. Let $G(a) = \{O(z) \mid \overline{O(z)} = O(z) \cup a, z \in \overline{O(x)} - O(x), z \neq a\}$. Then $G(a)$ is an at most countable set of open arcs, and if $G(a) = \bigcup_{n=1}^{\infty} D_n$ is infinite, and if $\{y_n\}$ is any sequence of points with $y_n \in D_n$, then $\lim_{n\to\infty} y_n = a$.*

Proof. Let $D \in G(a)$. It follows that D is the orbit of a point $z \in \overline{O(x)} - O(x) - A$, and $\bar{D} = D \cup a$ is a simple closed curve. For each $D \in G(a)$, let D^i denote the interior of $\bar{D}$. Then if D_j and D_k are distinct members of $G(a)$, the sets D_j^i and D_k^i are either disjoint or one is a proper subset of the other. If $D_j^i \subset D_k^i$, then D_k^i must contain $G(a) - D_k$, since then $O(x) \subset D_k^i$. Such a member of $G(a)$ will be termed a boundary arc of $G(a)$. Clearly $G(a)$ can contain at most one boundary arc. Also, if D_j and D_k are distinct members of $G(a)$, neither of which is a boundary arc of $G(a)$, then D_j^i and D_k^i are disjoint. It follows that $G(a)$ consists of an at most countable set of open arcs.

Suppcse that $G(a) = \bigcup_{n=1}^{\infty} D_n$ is an infinite set of open arcs. Let $\{y_n\}$ be any sequence of points with $y_n \in D_n$, and let $\lim_{n\to\infty} y_n = z$. The prcof of the theorem is completed by showing $z = a$. In order to show $z = a$, it is first shown that $z \in A$. If $z \notin A$, let S be a local section of $\mathscr{F}$ through z. If D_k is not a boundary arc of $G(a)$, then $z \notin D_k^i$. Since $G(a)$ has at most one boundary arc, the removal of a boundary are will not alter $\lim_{n\to\infty} y_n$. Thus suppose $G(a)$ has no boundary arc. Then $z \in X - H$, where $H = \bigcup_{n=1}^{\infty} D_n^i$. No D_n can cross S more than once, since D_n is an orbit and $\bar{D}_n$ is a simple closed curve. For each n, let D_n^e denote the exterior of $\bar{D}_n$. Then if S crosses D_n, S must pass from D_n^i to D_n^e. But then S can cross at most two D_ns. Hence $z \in A$, for $\lim_{n\to\infty} y_n = z$ implies that an infinite number of $O(y_n) = D_n$ intersect S. That $z = a$ is shown next. Consider the subarcs of D_n joining y_n and a. From this sequence of arcs we can choose a converging snbfequence converging to a set B. It follows that both z and a are in B. If $b \in B, a \neq b \neq z$, then b is the limit of a sequence $\{y'_{n_k}\}$ with $y'_{n_k} \in D_{n_k}$. Thus, by the same argument used to show $z \in A$, it can be shown that $b \in A$. But the set B is connected [10] and A is totally disconnected, hence $z = a = B$.

This theorem is a generalization of a theorem due to Kaplan [7].

THEOREM 6. *Let A be a totally disconnected set and let $x \in X - A$ be such that $\omega(x) \cap A = \bigcup_{n=1}^{k} a_n$ ($\alpha(x) \cap A = \bigcup_{n=1}^{k} a_n$). Then $\omega(x)$ ($\alpha(x)$) consists of a finite number of open arcs, each of which is an orbit joining distinct elements of $\bigcup_{n=1}^{k} a_n$ together with $\bigcup_{n=1}^{k} G(a_n)$.*

Proof. Consider the sets $G(a_n), n = 1, 2, \cdots, k$. Let $G^*(a_n)$ denote the point set union of all open arcs in $G(a_n)$. One can easily show that the point set closure of $G^*(a_n)$ is $G^*(a_n) \cup a_n$ and that $G^*(a_i) \cap G^*(a_j) = 0$ for $a_i \neq a_j$. The set $w(x)$ ($\alpha(x)$) is connected. By Theorem 4, the orbit of any point in $\omega(x) - \bigcup_{n=1}^{k} G(a_n)$ ($\alpha(x) - \bigcup_{n=1}^{k} G(a_n)$) is an open arc terminating at distinct points of $\bigcup_{n=1}^{k} a_n$. Thus each a_i must be joined to some a_j ($i \neq j$) by the orbit of some point in $\omega(x)$ ($\alpha(x)$). Clearly, no two a_n's are connected by more than two such arcs. Hence, $w(x)$ ($\alpha(x)$) contains only a finite number of arcs joining distinct a_n's. Thus the theorem is proved.

REFERENCES

1. M. V. Bebutoff, *Sur la repr´sentation des trajectoires d'un systèms dynamiques sur un syst me de droites parall.les*, Bull. Math. Univ. Moscow, **2** (1939), No. 3, 1-22.

2. A. Beck, *On invariant sets,* Ann. of Math., (2) **67** (1958), 99-103.
3. I. Bendixson, *Sur les courbes défines par des équations differentielles,* Acta Math., **24** (1901), 1-88.
4. H. Bohr, *Collected Mathematical Works, Vol.* II, Danish Math. Soc., Univ. of Copenhagen (1952).
5. W. Kaplan *Regular curve-families filling the plane,* I, Duke Math. J., **7** (1940), 154-185.
6. ———, *Regular curve-families filling the plane,* II, Duke Math J., **8** (1941), 11-46.
7. ———, *The structure of a regular curve-family on a surface in the neighborhood of an isolated singularity,* Amer, J. Math., **64** (1942), 1-35.
8. V. V. Nemytskii and V. V. Stepanov, *Qualitative Theory of Differential Equations,* (Second ed.) Translated by S. Lefschetz, Princeton Math. Ser., **22** (1960).
9. H. Whitney, *Regular families of curves,* Ann. of Math., (2), **34** (1933), 244-270.
10. G. T. Whyburn, *Analytic Topology,* Amer, Math. Soc. Coll. Publ., **28** (1942).

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