BANACH ALGEBRAS OF LIPSCHITZ FUNCTIONS

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1. Lip (X, d) will denote the collection of all bounded complex-valued functions defined on the metric space (X, d) that satisfy a Lipschitz condition with respect to the metric d. That is, Lip (X, d) consists of all f defined on X such that both

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

and

$$\|f\|_d = \sup\{|f(x) - f(y)|/d(x, y) : x, y \in X, x \neq y\}$$

are finite. With the norm $\|\cdot\|$ defined by $\|f\| = \|f\|_\infty + \|f\|_d$, Lip (X, d) is a Banach algebra. We shall sometimes refer to such an algebra as a Lipschitz algebra. In this paper we investigate some of the basic properties of these Banach algebras.

It will be assumed throughout the paper that (X, d) is a complete metric space. There is no loss of generality in doing so: for suppose (X, d) were not complete and let (X', d') denote its completion. Since each element of Lip (X, d) is uniformly continuous on (X, d), it extends uniquely and in a norm preserving way to an element of Lip (X', d'). Thus as Banach algebras, Lip (X, d) and Lip (X', d') are isometrically isomorphic.

In § 2 we sketch briefly the main points of the Gelfand theory and observe that every commutative semi-simple Banach algebra A is isomorphic to a subalgebra of the Lipschitz algebra Lip (Σ, σ), where Σ is the carrier space of A and σ is the metric Σ inherits from being a subset of the dual space A^* of A. This representation is obtained from the Gelfand representation; instead of using the usual Gelfand (relative weak*) topology of Σ, the metric topology is used. Later, in § 4, we show that this isomorphism is onto if and only if $A = \text{Lip}(X, d)$ for a compact (X, d).

In § 3 we study the carrier space Σ of Lip (X, d). The fact that Lip (X, d) is a point separating algebra of functions on X allows us to identify X as a subset of Σ. The topologies X inherits from Σ are compared to the original d-topology; they are shown to be equivalent and in the case of the two metric topologies we show them to be equivalent in a strong sense. In Theorem 3.9 we show that the important case of $\Sigma = X$ is equivalent to (X, d) being compact,

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and also equivalent to (Σ, σ) being compact.

The Gelfand representation for $\mathrm{Lip}(X, d)$ is considered in § 4. The image of $\mathrm{Lip}(X, d)$ under the Gelfand mapping turns out to be precisely those functions in $\mathrm{Lip}(\Sigma, \sigma)$ that are continuous on Σ in the Gelfand topology.

In § 5 we identify the homomorphisms from $\mathrm{Lip}(X_1, d_1)$ into $\mathrm{Lip}(X_2, d_2)$ where the (X_i, d_i) are compact. As a corollary the automorphisms of $\mathrm{Lip}(X, d)$ for compact (X, d) are obtained.

2. Let A be a semi-simple commutative Benach algebra with identity and with norm $\|\cdot\|_A$. The collection of nonzero multiplicative linear functionals on A is called the carrier space of A and will be denoted by Σ. It is well known [1, p. 69] that these functionals are bounded so that Σ forms a subset of the dual space A^* of A. In fact, Σ lies on the unit sphere of A^*. As a subset of A^* the carrier space Σ inherits two important topologies: the relative weak* topology, which we shall refer to as the Gelfand topology of Σ, and the relative norm, or metric topology.

The Gelfand theory of commutative Banach algebras utilizes the former topology. When A has an identity, Σ with its Gelfand topology is compact. For each $f \in A$ the function $\hat{f}$ is defined on Σ by $\hat{f}(\varphi) = \varphi f$, $\varphi \in \Sigma$. Each $\hat{f}$ is continuous on Σ in the Gelfand topology; indeed, the Gelfand topology is precisely the weakest topology on Σ such that the family $\{\hat{f} : f \in A\}$ is continuous. Let $C(\Sigma)$ denote the space of complex-valued functions on Σ continuous in the Gelfand topology supplied with sup norm $\|\cdot\|_\infty$. Then the Gelfand mapping $f \to \hat{f}$ is an isomorphism of A into $C(\Sigma)$ and is norm decreasing. Details of the Gelfand theory can be found in [1, 2].

Let us now consider the metric topology of Σ. The norm $\|\cdot\|_A^*$ of the dual space A^* is defined by

$$\|\varphi\|_A^* = \sup\{|\varphi f| : f \in A, \|f\|_A \leqq 1\} \qquad \varphi \in A^*.$$

The metric σ on Σ induced by this norm is given by

$$\sigma(\varphi, \psi) = \|\varphi - \psi\|_A^* \qquad \varphi, \psi \in \Sigma.$$

In terms of the functions $\hat{f}, f \in A$, we may express the metric σ by

$$(2.1) \quad \sigma(\varphi, \psi) = \sup\{|\hat{f}(\varphi) - \hat{f}(\psi)| : f \in A, \|f\|_A \leqq 1\} \quad \varphi, \psi \in \Sigma.$$

The metric topology of Σ is stronger than the Gelfand topology. Therefore, since Σ is closed in A^* in the weak* topology, it is also closed in the metric topology. Hence, (Σ, σ) is a complete metric space. Also each $\hat{f}$, $f \in A$, is continuous on (Σ, σ) since it is continuous on Σ in the Gelfand topology. The metric σ is bounded because Σ lies on the unit sphere of A^*.

With this metric space (Σ, σ) we can form the Lipschitz algebra $\mathrm{Lip}(\Sigma, \sigma)$ with norm $\|\cdot\|$ given by

$$\|g\| = \|g\|_\infty + \|g\|_\sigma \qquad g \in \mathrm{Lip}(\Sigma, \sigma).$$

We now show that the Gelfand mapping takes A into this Lipschitz algebra to yield what might be termed a "Lipschitz representation".

PROPOSITION 2.1. *Let A be a semi-simple commutative Banach algebra with identity. Then the Gelfand mapping is a continuous isomorphism of A onto a subalgebra of* $\mathrm{Lip}(\Sigma, \sigma)$. *Furthermore, for each $f \in A$,*

$$\|\hat{f}\|_\sigma \leqq \|f\|_A \text{ and } \|\hat{f}\|_\infty \leqq \|f\|_A .$$

Proof. For $g \in A$ with $\|g\|_A \leqq 1$ we have from (2.1) that

$$|g(\varphi) - g(\psi)| \leqq \sigma(\varphi, \psi) \qquad \varphi, \psi \in \Sigma. \tag{2.2}$$

Then for any nonzero $f \in A$, the element $\dfrac{1}{\|f\|_A}f$ has norm 1, so that from (2.2) we obtain

$$|\hat{f}(\varphi) - \hat{f}(\psi)| \leqq \|f\|_A \sigma(\varphi, \psi) \qquad \varphi, \psi \in \Sigma.$$

Hence, $\|\hat{f}\|_\sigma \leqq \|f\|_A$, for all $f \in A$. From the Gelfand theory we have $\|\hat{f}\|_\infty \leqq \|f\|_A$, all $f \in A$. Thus, the mapping $f \to \hat{f}$ takes A continuously into $\mathrm{Lip}(\Sigma, \sigma)$. From the Gelfand theory we know that the mapping is an isomorphism, hence the image of A is a subalgebra of $\mathrm{Lip}(\Sigma, \sigma)$.

In § 4 we consider the Gelfand mapping for the case $A = \mathrm{Lip}(X, d)$.

3. The Banach algebra $\mathrm{Lip}(X, d)$ is an algebra of functions defined on a set X. The function identically 1 is its identity and it is evidently a self-adjoint (closed under complex conjugation) algebra. We now observe that it separates the points of X.

For fixed $s \in X$ define the function f on X by $f(x) = d(x, s)$, $x \in X$. An application of the triangle inequality for d shows that $\|f\|_d \leqq 1$. However, if the metric is unbounded, then the function f so defined is not an element of $\mathrm{Lip}(X, d)$. This is remedied by truncation. The following lemma is easily verified.

LEMMA 3.1. *For each $s \in X$, the function f_s defined on X by*

$$f_s(x) = \min\{d(x, s), 1\} \qquad x \in X \tag{3.1}$$

belongs to $\mathrm{Lip}(X, d)$ *and has norm $\|f_s\| \leqq 2$. The family of functions $\{f_s : s \in X\}$ separates the points of X.*

Let Σ denote the carrier space of Lip (X, d). Since Lip (X, d) is a point separating algebra of functions on X each $x \in X$ can be identified with the evaluation functional φ_x in Σ where $\varphi_x(f) = f(x)$. More precisely, since the algebra separates points the injection mapping $x \to \varphi_x$ is one-to-one from X to Σ. Thus we may regard X as a subset of Σ.

An algebra of functions defined on a set X is called *inverse-closed* if for every function f in the algebra satisfying $|f(x)| \geqq \varepsilon > 0$, all $x \in X$, the inverse f^{-1} is also in the algebra. It is not difficult to check that Lip (X, d) is an inverse-closed algebra.

The next lemma is a consequence of the general theory of function algebras and holds for any algebra of functions on a set that is self-adjoint, separates points and is inverse-closed. See [1, p. 55].

LEMMA 3.2. *Let Σ be the carrier space of* Lip (X, d). *Then X is dense in Σ in the Gelfand topology. If (X, d) is compact, then $X = \Sigma$ and the Gelfand topology coincides with the d-topology of X.*

As a subset of Σ, X inherits two more topologies—the Gelfand and the metric topologies of Σ. The remainder of this section is concerned with the comparison of these inherited topologies of X to its original d-topology. We first look at the relative Gelfand topology of X. A basic neighborhood of $x_0 \in X$ in the relative Gelfand topology is of the form

$$\begin{aligned} &N(x_0, f_1, \cdots, f_n, \varepsilon) \\ &\qquad = \{x \in X : |f_i(x) - f_i(x_0)| < \varepsilon,\ i = 1, 2, \cdots, n\} \end{aligned}$$

where the f_i are elements of Lip (X, d) and ε is a positive number.

PROPOSITION 3.3. *The relative Gelfand topology of X and the d-topology of X are equivalent.*

Proof. It is clear from the definition of the Gelfand topology that the relative Gelfand topology of X is weaker than the d-topology. To show that every d-open set of X is also open in the relative Gelfand topology, it suffices to show that given $x_0 \in X$ and $\varepsilon > 0$ the sphere $S(x_0, \varepsilon) = \{x \in X : d(x, x_0) < \varepsilon\}$ is open in the relative Gelfand topology. Define the function f on X by (3.1) for $s = x_0$.

Then f belongs to Lip (X, d) and the neighborhood of x_0 in the relative Gelfand topology determined by f and ε is precisely (we assume $\varepsilon < 1$)

$$N(x_0, f, \varepsilon) = \{x \in X : |f(x) - f(x_0)| < \varepsilon\} = S(x_0, \varepsilon).$$

Hence, the spheres $S(x_0, \varepsilon)$ are open in the relative Gelfand topology and the proof is finished.

We now turn to the comparison of the two metric topologies on X. The metric σ on Σ is defined by (2. 1). When restricted to the subset X of Σ, the metric σ can be expressed by

$$(3.2)\quad \sigma(x, y) = \sup\{|f(x) - f(y)| : f \in \mathrm{Lip}\,(X, d),\ \|f\| \leqq 1\} \qquad x, y \in X.$$

The next few propositions are concerned with the relation between d and σ on X. We first define the notions of equivalence of metrics which will be the pertinent ones in this discussion.

DEFINITION. Two metrics d_1 and d_2 on a space X are called *boundedly equivalent* if and only if there exist positive numbers K_1 and K_2 such that

$$K_1 d_1(x, y) \leqq d_2(x, y) \leqq K_2 d_1(x, y) \qquad x, y \in X.$$

They are called *uniformly equivalent* if and only if the identity mapping from (X, d_1) to (X, d_2) is a uniform homeomorphism.

Bounded equivalence implies uniform equivalence, but not conversely. For example, the metrics $d_1(x, y) = |x - y|$ and $d_2(x, y) = |x - y|^{1/2}$ on $[0, 1]$ are uniformly but not boundedly equivalent. An unbounded metric can never be boundedly equivalent to a bounded metric. Therefore if d is an unbounded metric it can not be boundedly equivalent to the metric $d/(1 + d)$ although it is well know that these two metrics are uniformly equivalent.

Since Σ lies on the unit sphere of the dual space of Lip (X, d), the diameter of (Σ, σ) is at most two. Thus σ is always a bounded metric. If the original metric d on X is boundedly equivalent to the inherited metric σ, then d must also be a bounded metric. It turns out that the converse is also true.

PROPOSITION 3.4. *If the diameter of (X, d) is finite, then the metric σ on X defined by* (3.2) *is boundedly equivalent to d.*

Proof. If $f \in \mathrm{Lip}\,(X, d)$ satisfies $\|f\| \leqq 1$, then $|f(x) - f(y)| \leqq d(x, y)$, all x, y in X. Hence, forming the supremum over all f with $\|f\| \leqq 1$, we obtain $\sigma(x, y) \leqq d(x, y)$, all x, y in X.

Let D denote the diameter of (X, d). Let x and y in X be given. Define f by $f(u) = d(u, x)$, $u \in X$. Then $f \in \mathrm{Lip}\,(X, d)$ and $\|f\| \leqq 1 + D$, so that $g = Kf$ where $K = 1/(1 + D)$ has norm at most 1. Hence

$$\sigma(x, y) \geqq |g(x) - g(y)| = Kd(x, y).$$

Thus for all x, y in X we have

$$Kd(x, y) \leqq \sigma(x, y) \leqq d(x, y)$$

and the proposition is proved.

In particular, if (X, d) is compact, then the diameter is finite and σ and d are boundedly equivalent.

We show next as a corollary that bounded equivalence of metrics on a space X is the appropriate notion when dealing with Lipschitz functions in the sense that two metrics on X yield the same class of Lipschitz functions if and only if the metrics are boundedly equivalent.

COROLLARY 3.5. *Let d_1 and d_2 be bounded metrics on X. Then $A_1 = \mathrm{Lip}(X, d_1)$ and $A_2 = \mathrm{Lip}(X, d_2)$ have the same elements if and only if d_1 and d_2 are boundedly equivalent.*

Proof. It is clear that bounded equivalence of metrics preserves Lipschitz functions. For the converse, suppose $A_1 = A_2$. By the uniqueness of norm theorem for semisimple commutative Banach algebras [2, p. 75], the norms on $A_1 = A_2$ determined by d_1 and d_2 are equivalent. Therefore the norms on the dual space $A_1^* = A_2^*$ are equivalent. Then the metrics σ_i on $\Sigma(A_i)$, $i = 1, 2$, are boundedly equivalent. Since the d_i are bounded metrics it follows from Proposition 3.4 that d_i is boundedly equivalent to σ_i on $X, i = 1, 2$. The relation of bounded equivalence is evidently transitive. Hence d_1 is boundedly equivalent to d_2.

As remarked previously Proposition 3.4 is false for spaces (X, d) of inifinite diameter. But the next proposition shows that from the viewpoint of Banach space theory there is no loss in generality in assuming that d is always a bounded metric. We use the fact that for a given metric d on $X, d/(1 + d)$ is also a metric on X and is uniformly equivalent to d.

PROPOSITION 3.6. *Given the metric space (X, d), the Banach algebras $\mathrm{Lip}(X, d)$ and $\mathrm{Lip}(X, d')$ where $d' = d/(1 + d)$ have the same elements and their norms are equivalent.*

Proof. Let $\|\cdot\|'$ denote the norm on $\mathrm{Lip}(X, d')$. Since $d'(x, y) \leqq d(x, y)$ for all x, y in X we have $\|\cdot\|_d \leqq \|\cdot\|_{d'}$. Sup norms are unaffected by a change of metrics, so we have $\|\cdot\| \leqq \|\cdot\|'$. On the other hand,

$$
\begin{aligned}
\|f\|_{d'} &= \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)}(1+d(x,y)) : x, y \in X\right\} \\
&= \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} + |f(x)-f(y)| : x, y \in X\right\} \\
&\leqq \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} + |f(x)| + |f(y)| : x, y \in X\right\} \\
&\leqq \|f\|_d + 2\|f\|_\infty .
\end{aligned}
$$

Thus $\|\cdot\|' \leqq 3\|\cdot\|$. Hence Lipschitz functions are preserved so that $\mathrm{Lip}(X, d) = \mathrm{Lip}(X, d')$, the norms are equivalent, and the proof is complete.

COROLLARY 3.7. *The metrics d and σ on X are always uniformly equivalent.*

Proof. That the metrics d and $d' = d/(1+d)$ are uniformly equivalent is well known. We have shown in Proposition 3. 6 that the norms $\|\cdot\|$ and $\|\cdot\|'$ determined by d and d' respectively are equivalent. Therefore, the corresponding norms on the dual space are equivalent. Thus the metrics σ and σ' on Σ corresponding to d and d' respectively are boundedly equivalent. Since d' is bounded by 1, the metrics d' and σ' on X are boundedly equivalent by Proposition 3.4. Hence, d is uniformly equivalent to d', which is boundedly equivalent to σ', which in turn is boundedly equivalent to σ. It follows that d and σ are uniformly equivalent.

We have seen that when Σ has its Gelfand topology, X appears as a dense subset. In contrast to this, the following lemma shows that when Σ has its metric topology X is a closed subset of Σ. Note that the standing hypothesis that (X, d) be complete is used explicitly for the first time.

LEMMA 3.8. *The subset X of Σ is closed in the metric topology of Σ.*

Proof. Let $\{x_n\}$ be a sequence X such that $\sigma(x_n, \xi) \to 0$ where $\xi \in \Sigma$. We must show that $\xi \in X$. The sequence $\{x_n\}$ is σ-Cauchy since it converges in (Σ, σ). Uniform equivalence of metrics preserves Cauchy sequences. Therefore, since d and σ are uniformly equivalent on X by Corollary 3.7 and since $\{x_n\} \subset X$, the sequence $\{x_n\}$ is d-Cauchy. The completeness of (X, d) then implies that $\lim x_n = \xi$ belongs to X. Hence, X is closed in (Σ, σ).

Although certain of the implications in the next theorem have been established, we state them here for the sake of unity. The point of interest here is that the set equality $X = \Sigma$ alone is enough to imply compactness of the spaces under consideration. For an arbitrary Banach algebra of functions defined on a space X, the fact that the carrier space is just X does not in general have topological ramifications. In the case of $\mathrm{Lip}(X, d)$, however, we have the following.

THEOREM 3.9. *The following statements are equivalent:*
(i) $X = \Sigma$
(ii) *The Gelfand and metric topologies on Σ coincide.*
(iii) *(Σ, σ) is compact.*
(iv) *(X, d) is compact.*

Proof. (i) $\rightarrow$ (ii) follows from Proposition 3.3. (ii) $\rightarrow$ (iii) is a triviality. To see (iii) $\rightarrow$ (iv), note that if (Σ, σ) is compact, then by Lemma 3.8, X is a closed, hence compact subset of (Σ, σ). Since σ and d are equivalent on X, we have that (X, d) is compact. Finally, (iv) $\rightarrow$ (i) follows from Lemma 3.2.

4. We now turn to the Gelfand representation of $\mathrm{Lip}(X, d)$ The general Gelfand theory was mentioned in § 2. So that no confusion of norms will arise here let the norm in $\mathrm{Lip}(\Sigma, \sigma)$ be denoted by $|||\cdot|||$; then

$$|||g||| = |||g|||_\infty + |||g|||_\sigma \qquad g \in \mathrm{Lip}(\Sigma, \sigma)$$

where $|||g|||_\infty$ and $|||g|||_\sigma$ denote the sup norm and Lipschitz norm respectively of g on (Σ, σ). Proposition 2.1 tells us that the Gelfand mapping $f \rightarrow \hat{f}$ takes $\mathrm{Lip}(X, d)$ isomorphically into $\mathrm{Lip}(\Sigma, \sigma)$ and satisfies $|||\hat{f}|||_\infty \leqq \|f\|$ and $|||\hat{f}|||_\sigma \leqq \|f\|$, all $f \in \mathrm{Lip}(X, d)$. These statements followed from general considerations. But in the particular case of $\mathrm{Lip}(X, d)$ this can be strengthened.

THEOREM 4.1. *The Gelfand mapping $f \rightarrow \hat{f}$ is an isomorphism of* $\mathrm{Lip}(X, d)$ *onto the closed subalgebra of* $\mathrm{Lip}(\Sigma, \sigma)$ *consisting of those functions in* $\mathrm{Lip}(\Sigma, \sigma)$ *that are continuous in the Gelfand topology of Σ.*

Proof. If $f \in \mathrm{Lip}(X, d)$ and $\|f\| \leqq 1$, then $\|f\|_d \leqq 1$ so that $|f(x) - f(y)| \leqq d(x, y)$, all $x, y \in X$. Thus we have $\sigma(x, y) \leqq d(x, y)$ for all x, y in X. Hence for any $f \in \mathrm{Lip}(X, d)$

$$\begin{aligned}\|f\|_d &= \sup\left\{\frac{|f(x)-f(y)|}{d(x, y)} : x, y \in X\right\} \\ &\leqq \sup\left\{\frac{|f(x)-f(y)|}{\sigma(x, y)} : x, y \in X\right\} \\ &\leqq \sup\left\{\frac{|\hat{f}(\varphi)-\hat{f}(\psi)|}{\sigma(\varphi, \psi)} : \varphi, \psi \in \Sigma\right\} \\ &= |||\hat{f}|||_\sigma .\end{aligned}$$

Since each $\hat{f}$, $f \in \mathrm{Lip}\,(X, d)$, is continuous on Σ in the Gelfand topology and since X is dense in Σ in the Gelfand topology, we have the sup norm preserved: $\|f\|_\infty = |||\hat{f}|||_\infty$. Thus for all $f \in \mathrm{Lip}\,(X, d)$,

$$\|f\| = \|f\|_\infty + \|f\|_d \leqq |||\hat{f}|||_\infty + |||\hat{f}|||_\sigma = |||\hat{f}|||.$$

This together with the inequality from Proposition 2.1 yields $\|f\| \leqq |||\hat{f}||| \leqq 2\,\|f\|$. Hence the mapping $f \to \hat{f}$ is a bicontinuous isomorphism, and the image of $\mathrm{Lip}\,(X, d)$ is therefore a closed subalgebra of $\mathrm{Lip}\,(\Sigma, \sigma)$.

Let $g \in \mathrm{Lip}\,(\Sigma, \sigma)$ be continuous on Σ in the Gelfand topology; let $f = g \mid X$ denote the restriction of g to X. Then $f \in \mathrm{Lip}\,(X, d)$, since $\sigma(x, y) \leqq d(x, y)$ for all x, y in X; and $\hat{f} = g$, since both are continuous on Σ in the Gelfand topology and agree on the dense subset X. Thus those $g \in \mathrm{Lip}\,(\Sigma, \sigma)$ which are continuous in the Gelfand topology lie in the range of the mapping $f \to \hat{f}$ from $\mathrm{Lip}\,(X, d)$. Since every $\hat{f}$ is continuous on Σ in the Gelfand topology, we see that the image of $\mathrm{Lip}\,(X, d)$ under the mapping $f \to \hat{f}$ is exactly the set of functions in $\mathrm{Lip}\,(\Sigma, \sigma)$ which are continuous on Σ in the Gelfand topology. This completes the proof.

We remark that if (X, d) is not compact, then there do exist functions in $\mathrm{Lip}\,(\Sigma, \sigma)$ which are not continuous in the Gelfand topology. One such function g on Σ is

$$g(\varphi) = \sigma(\varphi, X) = \inf\{\sigma(\varphi, x) : x \in X\} \qquad \varphi \in \Sigma .$$

It is readily checked that g is a bounded Lipschitz function on (Σ, σ). Since X is a proper closed subset of (Σ, σ) by Theorem 3.9 and Lemma 3.8, g is not identically zero. But since X is dense in Σ in the Gelfand topology and $g = 0$ on X, we see that g cannot be continuous on Σ in the Gelfand topology. Hence there can be no $f \in \mathrm{Lip}\,(X, d)$ such that $g = \hat{f}$. Thus for non-compact (X, d) the Gelfand mapping does not take $\mathrm{Lip}\,(X, d)$ onto $\mathrm{Lip}\,(\Sigma, \sigma)$.

The Banach algebra A is called *regular* if for each proper subset K of Σ closed in the Gelfand topology and each point $\varphi \in \Sigma - K$, there exists an $f \in A$ such that $\hat{f}(\varphi) = 1$ and $\hat{f}(K) = 0$.

That Lip (X, d) is regular follows from the fact that a Lipschitz condition is preserved under truncation. The following proposition, due to J. Lindberg, shows that such algebras are regular in general. Call an algebra of functions A on a space X *closed under truncation* if the function $\min(f, 1)$ belongs to A for all real-valued $f \in A$.

PROPOSITION 4.2. *Let X be a compact Hausdorff space and A be a self-adjoint subalgebra of $C(X)$ which separates the points of X and contains the constant functions. If A is closed under truncation, then A is regular.*

Proof. Let $x \in X$ and let V be a neighborhood of x. Choose $f \in C(X)$ such that $f(x) = 0$, $f(X - V) = 3/2$ and $0 \leqq f \leqq 3/2$. Since A is dense in $C(X)$, there exists $g \in A$ with $\|f - g\|_\infty < 1/2$; we may take g to be real-valued. Then $g \geqq 1$ on $X - V$. Let $h = \min(g, 1)$, so that $h = 1$ on $X - V$. Also, $|h(x)| < 1/2$. Set

$$f = (h - 1)/(h(x) - 1).$$

Then $f \in A$ and $f(x) = 1$ while $f(X - V) = 0$. Thus A is regular.

COROLLARY 4.3. Lip (X, d) *is regular.*

Proof. Let $f \in \mathrm{Lip}(X, d)$ be real and set $Tf = \min(f, 1)$. Then

$$|(Tf)(x) - (Tf)(y)| \leqq |f(x) - f(y)| \qquad x, y \in X$$

which may be seen by comparing the graphs of f and Tf, or by checking each of the possible cases for a given x and y. It is immediate from this that Lip (X, d) is closed under truncation.

Also Lip (X, d) is self-adjoint, point-separating and contains the constants. Since X is dense in Σ in the Gelfand topology, it follows that Lip (X, d) is closed under truncation if and only if $\{\hat{f} : f \in \mathrm{Lip}(X, d)\}$ is closed under truncation. Hence, Lip (X, d) is regular.

5. We now consider the problem of describing the homomorphisms from one Lipschitz algebra into another. For Lipschitz algebras on compact metric spaces we are able to identify the homomorphisms and the description is given in the proposition below. We first make a few general comments on homomorphisms of Banach algebras. To avoid technical trivialities we shall always assume that homomorphisms carry the identity of one algebra into the identity of the other.

It is well known [2] that if T is a homomorphism of a semi-simple commutative Banach algebra A_1 into another A_2, then T is automatically continuous and induces a dual mapping $\tau : \Sigma_2 \rightarrow \Sigma_1$ of the carrier spaces. This dual mapping τ is defined as follows: given $\varphi \in \Sigma_2$ define the multiplicative linear functional $\tau^{\varphi} \in \Sigma_1$ by

$$(5.1) \qquad (\tau\varphi)(f) = \varphi(Tf) \qquad f \in A_1.$$

The homomorphism T is in turn induced by the mapping τ by means of (5.1). The dual mapping τ is always continuous. If T maps A_1 onto A_2, then τ is one-to-one; if τ is onto, then T is one-to-one. If T is an isomorphism of A_1 onto A_2, then the dual mapping is a homeomorphism of Σ_2 onto Σ_1. For a detailed discussion see [2, p. 75, p. 116].

Now let (X_1, d_1) and (X_2, d_2) be compact metric spaces and let $A_1 = \mathrm{Lip}\,(X_1, d_1)$ and $A_1 = \mathrm{Lip}\,(X_2, d_2)$. By Theorem 3.9 the carrier spaces of A_1 and A_2 are X_1 and X_2 respectively. Let T be a homomorphism of A_1 into A_2. Then the dual mapping τ takes X_2 into X_1 and equation (5.1) can be written

$$(5.2) \qquad f(\tau x) = (Tf)(x) \qquad f \in A_1, x \in X_2.$$

The converse situation does not hold in general. That is, given a continuous mapping $\tau: X_2 \to X_1$, the mapping T defined on A_1 by (5.2) will not in general take A_1 into A_2. The following proposition identifies those mappings $\tau: X_2 \to X_1$ which are dual to homomorphisms of the Lipschitz algebras.

THEOREM 5.1. *Let* $A_i = \mathrm{Lip}\,(X_i, d_i)$ *where* (X_i, d_i) *is compact,* $i = 1, 2$. *Then every homomorphism* $T: A_1 \to A_2$ *is of the form*

$$(5.3) \qquad (Tf)(x) = f(\tau x) \qquad f \in A_1, x \in X_2$$

where $\tau: X_2 \to X_1$ *satisfies*

$$(5.4) \qquad d_1(\tau x, \tau y) \leqq K d_2(x, y) \qquad x, y \in X_2$$

for some positive constant K. *Conversely, if* T *is defined on* A_1 *by* (5.3) *where* $\tau: X_2 \to X_1$ *satisfies* (5.4), *then* T *is a homomorphism of* A_1 *into* A_2. T *is one-to-one if and only if* $\tau(X_2) = X_1$. T *takes* A_1 *onto* A_2 *if and only if* τ *satisfies the additional condition*

$$(5.5) \qquad K' d_2(x, y) \leqq d_1(\tau x, \tau y) \qquad x, y \in X_2.$$

for some positive constant K'.

Proof. Suppose $T: A_1 \to A_2$ is a homomorphism with dual mapping $\tau: X_2 \to X_1$. For each $s \in X_1$ define the function f_s on X_1 by $f_s(t) = d_1(t, s)$, $t \in X_1$. Then $f_s \in A_1$ and

$$\|f_s\|_\infty + \|f_s\|_{d_1} \leqq \text{diameter}\,(X_1, d_1) + 1$$

for all $s \in X_1$. Thus the set $\{f_s : s \in X_1\}$ is bounded in A_1. Since T is continuous, the set $\{Tf_s : s \in X_1\}$ is bounded in A_2. Then there exists a positive constant K such that $\| Tf_s \|_{d_2} \leqq K$, all $s \in X_1$. Thus for all points $x, y \in X_2$ and $s \in X_1$ we have

$$\frac{|(Tf_s)(x) - (Tf_s)(y)|}{d_2(x, y)} = \frac{|d_1(s, \tau x) - d_1(s, \tau y)|}{d_2(x, y)} \leqq K.$$

Taking $s = \tau y$ in this inequality, we obtain

$$d_1(\tau x, \tau y) \leqq K d_2(x, y)$$

for all $x, y \in X_2$. Hence, the mapping τ satisfies (5.4).

Conversely, if T is defined by (5.3) in terms of a τ satisfying (5.4), then

$$\begin{aligned} \| Tf \|_{d_2} &= \sup \left\{ \frac{|(Tf)(x) - (Tf)(y)|}{d_2(x, y)} : x, y \in X_2 \right\} \\ &= \sup \left\{ \frac{|f(\tau x) - f(\tau y)|}{d_2(x, y)} : x, y \in X_2 \right\} \\ &\leqq K \sup \left\{ \frac{|f(\tau x) - f(\tau y)|}{d_1(\tau x, \tau y)} : x, y \in X_2 \right\} \\ &\leqq K \| f \|_{d_1}. \end{aligned}$$

Also,

$$\begin{aligned} \| Tf \|_\infty &= \sup \{ |f(\tau x)| : x \in X_2 \} \\ &\leqq \sup \{ |f(y)| : y \in X_1 \} = \| f \|_\infty. \end{aligned}$$

Hence T carries A_1 into A_2 and it is easily seen to be a homomorphism.

We know in general that if τ is onto, then T is one-to-one. Suppose T is one-to-one. If τ is not onto, then there exists $s \in X_1$ with $s \notin \tau(X_2)$. The continuity of τ implies that $\tau(X_2)$ is compact since (X_2, d_2) is compact. But then by the regularity of A_1 we can choose $f \in A_1$ with $f(s) = 1$ and $f = 0$ on $\tau(X_2)$. Then $(Tf)(x) = f(\tau x) = 0$, all $x \in X_2$. This contradicts the fact that T is one-to-one. Hence, τ is onto.

Suppose now that T is onto. Then τ is one-to-one and we can define a new metric d' on X_2 by

$$d'(x, y) = d_1(\tau x, \tau y) \qquad x, y \in X_2. \tag{5.6}$$

Let $A' = \mathrm{Lip}\,(X_2, d')$. Let $f \in A_2$ and choose $g \in A_1$ such that $Tg = f$. Then since

$$\frac{|f(x) - f(y)|}{d'(x, y)} = \frac{|g(\tau x) - g(\tau y)|}{d_1(\tau x, \tau y)}$$

for all $x, y \in X_2$, we see that $\|f\|_{d'} \leqq \|g\|_{d_1}$. Hence $A_2 \subset A'$. In particular, the function f defined by $f(x) = d_2(x, u)$, $x \in X_2$, belongs to A' for each $u \in X_2$. It follows that

$$d_2(x, y) \leqq K''d'(x, y) \qquad x, y \in X_2$$

for some constant K''. This yields (5.5) with $K' = 1/K''$.

Finally suppose that τ satisfies (5.5) in addition to (5.4). Then τ is one-to-one and the metric d' on X_2 defined by (5. 6) is boundedly equivalent to the metric d_2. By Corollary 3.5, A_2 and A' must have the same elements. But if $f \in A'$, then the function g defined on X_1 by $g(s) = f(\tau^{-1}s)$, $s \in X_1$, belongs to A_1 and $Tg = f$. Thus T maps A_1 onto A', and hence onto A_2. This completes the proof.

COROLLARY 5.2. *Every automorphism T of* Lip(X, d) *where (X, d) is compact is of the form*

$$(Tf)(x) = f(\tau x) \qquad f \in \mathrm{Lip}\,(X, d),\ x \in X$$

where $\tau: X \to X$ is a homeomorphism satisfying

$$K_1 d(x, y) \leqq d(\tau x, \tau y) \leqq K_2 d(x, y) \qquad x, y \in X$$

for some positive constants K_1 and K_2.

REFERENCES

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