HOMOMORPHISMS OF d-SIMPLE INVERSE SEMIGROUPS WITH IDENTITY

R. J. WARNE

Munn determined all homomorphisms of a regular Rees matrix semigroup S into a Rees matrix semigroup S^* [3, 2]. This generalized an earlier theorem due to Rees [7, 2].

We consider the homomorphism problem for an important class of d-simple semigroups.

Let S be a d-simple inverse semigroup with identity. Such semigroups are characterized by the following conditions [1, 4, 2].

A1: S is d-simple.
A2: S has an identity element.
A3: Any two idempotents of S commute.

It is shown by Clifford [1] that the structure of S is determined by that of its right unit semigroup P and that P has the following properties:

B1: The right cancellation law hold in P.
B2: P has an identity element.
B3: The intersection of two principal left ideals of P is a principal left ideal of P.

Two elements of P are L-equivalent if and only if they generate the same principal left ideal.

Since any homomorphic image of a d-simple inverse semigroup with identity is a d-simple inverse simigroup with identity [5], we may limit our discussion to homomorphisms of S into S^* where S^*, as well as S, is of this type.

In § 1, we consider two such semigroups S and S^* with right unit semigroups P and P^* respectively. We determine the homomorphisms of S into S^* in terms of certain homomorphism of P into P^*, and we show that S is isomorphic to S^* if and only if P is isomorphic to P^*.

In § 2, we show that if P is a semigroup satisfying B1 and B2 on which L is a congruence relation then P is a Schreier extension of its group of units U by P/L and that P/L satisfies B1, B2, and has a trivial group of units. P satisfies B3 if and only if P/L satisfies B3. The converse of this theorem is also given. In this case, we determine the homomorphisms of P into P^* in terms of the homomor-

Received June 25, 1963.

phisms of U into U^* and those of P/L into P^*/L^* and give the corresponding isomorphism theorem. In § 3, some examples are given.

It is a pleasure to acknowledge several helpful conversations with Professor A. H. Clifford.

Section 1. *The correspondence between the homomorphism of S and those of P.*

We first summarize the construction of Clifford referred to in the introduction.

Let S be any semigroup with identity element. We say that the two elements are R-equivalent if they generate the same principal right ideal: $aS = bS$. L-equivalent elements are defined analogously. Two elements a and b are called d-equivalent if there exists an element of S which is L-equivalent to a and R- equivalent to b (This implies the existence of an element of S which is R-equivalent to a and L-equivalent to b.) We shall say that S is d-simple if it consists of a single class of d-equivalent elements.

Now let P be any semigroup satisfying B1, B2 and B3. From each class of L-equivalent elements of P, let us pick a fixed representative. B3 states that if a and b are elements of P, there exists c in P such that $Pa \cap Pb = Pc$. c is determined by a and b to within L-equivalence. We define avb to be the representative of the class to which c belongs. We observe also that

$$avb = bva \; . \tag{1.1}$$

We define a binary operation x by

$$(axb)b = avb \tag{1.2}$$

for each pair of elements a, b of P.

Now let $P^{-1}oP$ denote the set of ordered pairs (a, b) of elements of P with equality defined by

$$(a, b) = (a', b') \text{ if } a' = \rho a \text{ and } b' = \rho b \text{ where } \rho \text{ is a unit in } P \text{ (}\rho\text{ has a two sided inverse with respect to 1, the identity of } P\text{).} \tag{1.3}$$

We define product in $P^{-1}oP$ by

$$(a, b)(c, d) = ((cxb)a, (bxc)d) \; . \tag{1.4}$$

Clifford's main theorem states: *Starting with a semigroup P satisfying* B1, 2, 3 *equations* (1.2), (1.3), *and* (1.4) *define a semigroup $P^{-1}oP$ satisfying* A1, 2, 3. P *is isomorphic with the right unit subsemigroup* of $P^{-1}oP$ (the right unit subsemigroup of $P^{-1}oP$ is the set of elements of $P^{-1}oP$ having a right inverse with respect to 1. This set is easily shown to be a semigroup). *Conversely, if S is a semigroup satisfying* A1, 2, 3 *its right unit subsemigroup P satisfies* B1, 2, 3 *and S is isomorphic with $P^{-1}oP$.*

The following results are also obtained:

The elements $(1, a)$ of $P^{-1}oP$ constitute a subsemigroup thereof isomorphic to P. We have

(1.5) $$(1, a)(1, b) = (1, ab) \text{ for } a, b \text{ in } P.$$

The ordered pair $(1, 1)$ is the identify of $P^{-1}oP$, i.e.

(1.6) $$(a, b)(1, 1) = (1, 1)(a, b) = (a, b) \text{ for } a, b \text{ in } P.$$

The right inverse of $(1, a)$ is $(a, 1)$, i.e.

(1.7) $$(1, a)(a, 1) = (1, 1) \text{ for a in } P.$$

(1.8) $$(a, c) = (a, 1)(1, c) \text{ for all } a \text{ and } c \text{ in } P.$$

We identity S with $P^{-1}oP$ and P with $\{(1, a) : a \text{ in } P\}$.

(1.9) $(avb)c = \rho(acvbc)$ where a, b, and c are in P and ρ is a unit in P.

(1.10) The idempotent elements of $P^{-1}oP$ are just those elements of the form (a, a) where a in P.

(1.11) $$(a, a)(b, b) = (avb, avb) \text{ for all } a, b \text{ in } P.$$

(1.12) $aLb(a, b$ in $P)$ if and only if $a = \rho b$ where ρ is a unit in P.

Let P and P^* be semigroups satisfying B1, and B2 and B3. Let v and u be the 'join' operations on P and P^* respectively defined on page 2. Let N be a homomorphism of P into P^*. N is called a *semilattice homomorphism* (or sl-homomorphism) if

(1.13) $$P^*((avb)N) = P^*(aN) \cap P^*(bN)$$

i. e. $(avb)N\ LaNubN$ in P^*.

It is easily seen that we always have $P^*((avb)N) \subseteq P^*(aN) \cap P^*(bN)$. However, the reverse inclusion is not generally valid. For example, we might have $P = G^+$, $P^* = G^{*+}$, where G and G^* are lattice-ordered groups. An order-preserving homomorphism of G into G^* need not preserve the lattice operations.

THEOREM 1.1. *Let S and S^* be semigroups satifying* A1, A2, *and*

A3, *and let P and P^* be their right unit subsemigroups, Let N, be a* sl-*homomorphism of P into P^*, and let k be an element of P^*.*

For each element (a, b) of S, define

$$(a, b)M = [(aN)k, (bN)k] \tag{1.14}$$

the square brackets indicating an element of S^*. Then M is a homomorphism of S into S^*. Conversely, every homomorphism of S into S^* is obtained in this fashion.

PROOF. To show that M is single valued, let $(a, b) = (a', b')$. Then, $a' = \rho a$ and $b' = \rho b$ where ρ is a unit in P by (1.3). Thus, $a'N = \rho NaN$ and $b'N = \rho NbN$. Thus, since ρN is a unit of P^*, $(a, b)M = (a', b')M$ by (1.3). To show that M is a homomorphism let $\times$ and $\otimes$ be the operations defined on P and P^* respectively by (1.2). Thus, using (1.2), (1.9), (1.13), and (1.12) obtain $((rN)k \otimes (nN)k)(nN)k = (rN)k \quad u(nN)k = w(rNunN)k = w\rho^* \quad ((rvn)N)k = w\rho^*(((r \times n)n)N)k = w\rho^*((r \times n)N)(nN)k$ where w and ρ^* are units in P^*. Thus, from B1,

$$(rN)k \otimes (nN)k = w\rho^*((r \times n)N) . \tag{1.15}$$

Now, from (1.2), (1.1), and (1.15), we have $((nN)k \otimes (rN)k) \ (rN)k = (nN)k \ u(rN)k = (rN)k \ u(nN)k = w\rho^* \ ((rvn)N)k = w\rho^* \ ((nvr)N)k = w\rho^* \ (((n \times r)r)N)k = w\rho^* \ ((n \times r)N) \ (rN) \ k$. Therefore, by B1,

$$(nN)k \otimes (rN)k = w\rho^* \ ((n \times r)N) . \tag{1.16}$$

Thus, by (1.14), (1.4), (1.15), (1.16), and (1.3), $(m, n)M(r, s)M = [(mN)k, (nN)k] \ [(rN)k, (sN)k] = [((rN)k \otimes (nN)k) \ (mN)k, \ ((nN)k \otimes (rN)k) \ (sN)k] = [w\rho^*((r \times n)N) \ (mN)k, \ w\rho^* \ ((n \times r)N) \ (sN)k] = [((r \times n)m)Nk, ((n \times r)s)Nk] = ((r \times n)m, (n \times r)s)M = ((m, n) \ (r, s))M$. Conversely, let M be a homomorphism of S into S^*. Then, by (1.6) and (1.10),

$$(1, 1)M = [k, k] \tag{1.17}$$

for some k in P^*. Now suppose that $(1, n)M = [a, b]$ and $(n, 1)M = [c, d]$ for n in P. It thus follows from (1.7) and (1.6) that $[a, b] [c, d] [a, b] = [a, b]$ and $[c, d] [a, b] [c, d] = [c, d]$. From (1.8) and (1.7), it easily follows that $[a, b] [b, a] [a, b] = [a, b]$ and $[b, a] [a, b] [b, a] = [b, a]$. Hence, $[b, a]$ and $[c, d]$ are inverses of $[a, b]$ (2, p. 27). Therefore, it follows from a theorem of Munn and Penrose (4; 2, p. 28, Theorem 1.17) that $[b, a] = [c, d]$. Thus

$$\begin{aligned} (1, n)M &= [a, b] \\ (n, 1)M &= [b, a] \end{aligned} \tag{1.18}$$

Now, from (1.7), (1.17), and (1.18), $[a, b][b, a] = [k, k]$. Thus, from (1.8) and (1.7), we have $[a, a] = [k, k]$. Hence, by (1.3), $a = \rho k$ where ρ is a unit of P^*. Therefore, by (1.18) and (1.3),

$$\begin{aligned}(1.19) \qquad (1, n)M &= [\rho k, b] = [k, \rho^{-1}b] = [k, c]\\ (n, 1)M &= [b, \rho k] = [\rho^{-1}b, k] = [c, k]\end{aligned}$$

where $c = \rho^{-1}b$. Now, again using (1.8) and (1.7), $[c, k][k, c] = [c, c]$. Thus, by (1.11), $[k, k]\ [c, c] = [kuc, kuc] = [c, c]$. Therefore, by (1.3) (1.12), $P^*(kuc) = P^*c$. Hence, by the definition of u, $P^*k \cap P^*c = P^*c$ and $P^*c \subseteq P^*k$. Thus, we may write $c = B_n k$ where B_n in P^*. Thus, from (1.19), we have

$$\begin{aligned}(1.20) \qquad (1, n)M &= [k, B_n k]\\ (n, 1)M &= [B_n k, k]\ .\end{aligned}$$

It follows easily from (1.8), (1.20) and (1.7) that

$$(1.21) \qquad (m, n)M = [B_m k, B_n k]\ .$$

Thus, to complete the proof, we must show that $n \rightarrow B_n$ is a homomorphism of P into P^* and that $P^*\ (B_m\ u\ B_n) \subseteq P^*B_{mvn}$. It follows from (1.20), (1.3), and (B1) that $n \rightarrow B_n$ is single valued. To show that $n \rightarrow B_n$ is a homomorphism we first note that from (1.5) and (1.20), $[k, B_m k][k, B_n k] = [k, B_{mn}k]$. Thus, by (1.4)

$$(1.22) \qquad [(k \otimes B_m k)k, (B_m k \otimes k)B_n k] = [k, B_{mn}k]\ .$$

From (1.2), the definition of u, and (1.12)

$$(1.23) \qquad (k \otimes B_m k)\ B_m k = ku\ (B_m k) = wB_m k$$

where w is a unit of P^*. Thus, by (B1)

$$(1.24) \qquad k \otimes (B_m k) = w\ .$$

By virtue of (1.2), (1.1), and (1.23), $((B_m k \otimes k)k = (B_m k)\ uk = ku\ (B_m k) = wB_m k$. Hence, by (B1),

$$(1.25) \qquad (B_m k) \otimes k = wB_m\ .$$

If we substitute (1.24) and (1.25) in (1.22), we obtain $[wk, wB_m B_n k] = [k, B_{mn}k]$. Hence, from (1.3) and (B1), we have $B_m B_n = B_{mn}$. We now show that $P^*(B_m u B_n) = P^*B_{mvn}$. From (1.4), $(1, m)\ (n, 1) = (n \times m, m \times n)$. Hence, it follows from (1.21), (B1), and (B2) that $[k, B_m k]$ $[B_n k, k] = [B_{n\times m}k, B_{m\times n}k]$. Thus, by virtue of (1.4), $[((B_n k) \otimes (B_m k))k, ((B_m k) \otimes (B_n k))k] = [B_{n\times m}k, B_{m\times n}k]$. Hence, by (1.3) and (B1), $(B_n k) \otimes (B_m k) = \rho^*{}_1 B_{n\times m}$ where $\rho^*{}_1$ is a unit of P^*. Thus, by (1.2), $B_n k u B_m k = ((B_n k) \otimes (B_m k))\ B_m k = \rho^*{}_1 B_{n\times m} B_m k = \rho^*{}_1 B_{(n\times m)m} k = \rho^*{}_1 B_{nvm} k$. Therefore, by (B1) and (1.9), ρ' $(B_n u B_m) = \rho^*{}_1 B_{nvm}$ where ρ' is a unit of P^*. Hence $P^*(B_n u B_m) = P^* B_{nvm}$.

THEOREM 1.2. *Let S, P, S^*, and P^* be as in Theorem 1.1. Let Ω be the set of isomorphisms of P onto P^*. Define $(m, n)M_N = [mN, nN]$ for N in Ω. Then $\{M_N : N \text{ in } \Omega\}$ is the complete set of isomorphisms of S onto S^*. Hence, $N \to M_N$ is a one-to-one correspondence between the isomorphisms of P onto P^* and those of S onto S^* and S is isomorphic to S^* if and only if P is isomorphic to P^*. The group of automorphisms of P is isomorphic to the group of automorphisms of S.*

PROOF. We first show that P^* $(aNubN) \subseteq P^*$ $((avb)N)$ for a, b in P and for any isomorphism N of P onto P^*. It is easy to see that $Pa \subseteq Pb$ if and only if $P^*(aN) \subseteq P^*(bN)$. Since $aNubN = zN$ for some z in P, $P^*zN = P^*(aN) \cap P^*(bN) \subseteq P^*(aN)$, $P^*(bN)$ by the definition of u. Thus, $Pz \subseteq P(avb)$ by the definition of v and the desired result follows. Therefore, by Theorem 1.1, M_N is a homomorphism of S into S^*. To show it is one-to-one let $(m, n)M_N = (p, q)M_N$, i. e. $[mN, nN] = [pN, qN]$. Thus, using (1.3), we may show that $mN = (\rho' p)N$ and $nN = (\rho' q)N$ where ρ' is a unit of P. Thus, by (1.3), $(m, n) = (p, q)$. Clearly, M_N maps S onto S^*. Conversely, let M be an isomorphism of S onto S^*. By Theorem 1.1, $(m, n)M = [(mN)k, (nN)k]$ where k in P^* and N is a homomorphism of P into P^*. Now, it follows from (1.6), (B1), and (B2) that $(1, 1)$ $M = [k, k] = [1^*, 1^*]$ where 1^* is the identity of P^*. Thus, by (1.3), k is a unit of P^*. Now, let $nA = k^{-1}$ $(nN)k$ for all n in P. It is easily seen that A is a homomorphism of P into P^*. Now, by (B1), (B2), and (1.3), we have

$$
\begin{aligned}
(1.26) \qquad (m, 1)M &= [(mN)k, k] = [k^{-1}(mN)k, 1^*] = [mA, 1^*] \\
(1, m)M &= [k, (mN)k] = [1^*, k^{-1}(mN)k] = [1^*, mA] \,.
\end{aligned}
$$

Thus, from (1.26) and (1.3), we have $mA = nA$ implies $m = n$. Let a be in P^*. Then, by the remarks on page 3, it follows that $[1^*, a] = (1, m)M$ for some m in P. Hence, by (1.26) and (1.3), $a = mA$. Therefore A is an isomorphism of P onto P^*. From (1.26) and (1.8), we have $(m, n)M = [mA, nA]$. Thus, $M = M_A$.

Section 2. *A reduction of the homomorphism problem by an application of Schreier extensions.*

We first will briefly review the work of Rédei [6] on the Schreier extension theory for semigroups (we actually give the right-left dual of his construction.). Let G be a semigroup with identity e. We consider a congruence relation n on G and call the corresponding division of G into congruence classes a *compatible class division* of G. The class H containing the identity is said to be the *main class* of the division. H is easily shown to be a subsemigroup of G. The division is called *right normal* it and only if the classes are of the form,

$$Ha_1, Ha_2, \cdots \ (a_1 = e) \tag{2.1}$$

and $h_1 a_i = h_2 a_i$ with h_1, h_2 in H implies $h_1 = h_2$. The system (2.1) is shown to be uniquely determined by H. H is then called a *right normal divisor* of G and G/n is denoted by G/H.

Let G, H, and S be semigroups with identity. Then, if there exists a right normal divisor H' of G such that $H \cong H'$ and $S \cong G/H'$, G is said to be a Schreier extension of H by S.

Now, let H and S be semigroups with identities E and e respectively. Consider $H \times S$ under the following multiplication:

$$(A, a)\ (B, b) = (AB^a a^b, ab)\ (A, B \text{ in } H;\ a, b \text{ in } S) \tag{2.2}$$

in which

$$a^b,\ B^a \text{ (in } H)$$

designate functions of the arguments a, b and B, a respectively, and are subject to the conditions

$$a^e = E,\ e^a = E,\ B^e = B,\ E^a = E\,. \tag{2.3}$$

We call $H \times S$ under this multiplication a Schreier product of H and S and denote it by HoS.

Redéi's main theorem states:

THEOREM 2.1 (Rédei). *A Schreier product $G = HoS$ is a semigroup if and only if*

$$(AB)^c = A^c B^c\ (A, B \text{ in } H:\ c \text{ in } S) \tag{2.4}$$

$$(B^a)^c c^a = c^a B^{ca}\ (B \text{ in } H;\ a, c \text{ in } S) \tag{2.5}$$

$$(a^b)^c c^{ab} = c^a (ca)^b\ (a, b,\ c \text{ in } S) \tag{2.6}$$

are valid. These semigroups (up to an isomorphism) are all the Schreier extensions of H by S and indeed the elements (A, e) form a right normal divisor H' of G for which

$$\begin{gathered} G/H' \cong S\ (H'(E,\ a) \to a) \\ H' \cong H\ ((A,\ \mathrm{e}) \to A) \end{gathered} \tag{2.7}$$

are valid.

THEOREM 2.2 *Let U be a group with identity E and let S be a semigroup satisfying* B1 *and* B2 (*denote its identity by e*) *and suppose S has a trivial group of units. Then every Schreier extension $P = UoS$ of U by S satisfies* B1 *and* B2 (*the identity is* (E, e)) *and the group of units of P is $U' = \{(A, e): A \text{ in } U\} \cong U$. Furthermore L is a congruence relation on P and $P/L \cong S$. P satisfies* B3 *if and only if S satisfies* B3.

Conversely, let P be a semigroup satisfying B1 *and* B2 *on which L is a congruence relation. Let U be the group of units of P. Then U is a right normal divisor of P and $P/U \cong P/L$. Thus, P is a Schreier extension of U by P/L. P/L satisfies* B1 *and* B2 *and has a trivial group of units.*

REMARK. Hence if P is any semigroup satisfying B1 and B2 with group of units U such that L is a congruence relation on P, we will write $P = (U, P/L, a^b, A^b)$ in conjunction with Theorem 2.1 and 2.2. (We note that L is a right regular equivalence relation on any semigroup) a^b, A^b will be called the function pair belonging to P.

REMARK. A theorem of Rees [8, Theorem 3.3] is a special case of the above theorem.

Proof. It follows easily from (2.2) and (2.3) that P satisfies B1 and has identity (E, e). From Theorem 2.1, $U' \cong U$. Now, suppose (A, a) is a unit of P. Then, (A, a) $(B, b) = (E, e)$ for some (B, b) in P. Hence by (2.2), $ab = e$. Thus, by (B1), (B2), and the fact that the group of units of S is e, $a = b = e$, and (A, a) in U'. From (2.2) and (2.3), every element of U' is a unit of P.

Next, we determine the principal left ideals of P. From (2.2), we have

(2.8) $$P(A, a) = \{(BA^b b^a, ba): B \text{ in } U, b \text{ in } S\}$$

$= \{(C, ba): C \text{ in } U, b \text{ in } S\}$.

Since $P(A, a)$ just depends on a, we may write $P(A, a) = P_a$ for all A in U.

Next, we show that

(2.9) $$(A, a)\ L\ (B, b) \text{ if and only if } a = b\ .$$

Now, from (2.8), (A, a) L (B, b) implies $b = xa$ and $a = yb$ for some x, y in S Thus, by B1, $xy = yx = e$, and since S has a trivial group of units, $x = y = e$. Thus, $a = b$. The converse is evident from (2.8). It follows easily from (2.9) and (2.2) that L is a congruence relation. $L_{(E,a)}$ will denote the L-class of P containing (E, a). It is easily seen that the mapping $L_{(E,a)} \to a$ is an isomorphism of P/L onto S. Now suppose S satisfies B3, i.e. a, b in S implies there exists c in S such that

$$Sa \cap Sb = Sc \,. \tag{2.10}$$

From (2.10) and (2.8),

$$P_a \cap P_b = P_c \tag{2.11}$$

and P satisfies B3. If P satisfies B3, it follows from (2.8) and (2.11) that S satisfies B3.

Now, let P be a semigroup satisfying B1 and B2 with group of units U on which L is a congruence relation. By (1.12) (this is shown without using B3) U is the congruence class mod L containing the identity 1 of P, i.e. U is the main class of the compatible class division of P given by L. If a in P, $L_a = Ua$ from (1.12). If $\rho_1 a = \rho_2 a$ a where ρ_1, ρ_2 in U, then $\rho_1 = \rho_2$ by B1. Thus, U is a right normal divisor of P and $P/U \cong P/L$. Hence, P is a Schreier extension of U by P/L. By virtue of (1.12) and (B1), P/L satisfies B1.

Let $a \to \bar{a}$ be the natural homomorphism of P onto P/L. Then, $\bar{1}$ is the identity of P/L. Let $\bar{a}$ be a unit of $\bar{P}$. Then, by (1.12), (B1), and (B2), a is in U. Hence, $\bar{a} = \bar{1}$. Therefore, P/L has a trivial group of units.

THEOREM 2.3. *Let* $P = (U,\ P/L,\ a^b,\ A^b)$ *and* $P^* = (U^*,\ P^*/L^*,\ b^c,\ B^c)$ *be semigroups satisfying* B1 *and* B2 *on which* L *and* L^* *are congruence relations.* U *and* a^b, A^b *denote the unit group and function pair of* P. U^* *and* b^c, B^c *denote the unit group and function pair of* P^*. P/L *is the factor semigroup of* P *mod* L *and* P^*/L^* *is the factor semigroup of* P^* *mod* L^*. *Let* f *be a homomorphism of* U *into* U^*, g *be a homomorphism of* P/L *into* P^*/L^*, *and* h *be a function of* P/L *into* U^*. *Suppose* f, g *and* h *are subject to the following conditions:*

$$(ah)\ (bh)^{(ag)}(ag)^{(bg)} = (a^b f)(ab)h \tag{2.12}$$

$$(bh)(Af)^{(bg)} = (A^b f)(bh) \,. \tag{2.13}$$

For each $(A,\ a)$ *in* P *define*

$$(A,\ a)M = [(Af)(ah),\ ag] \tag{2.14}$$

where the square brackets denote elements of P^*. *Then* M *is a homomorphism of* P *into* P^* *Conversely, every homomorphism of* P *into* P^* *is obtained in this fashion.* M *is an isomorphism if and* *only if f and g are isomorphisms.*

Proof. Clearly, M is single valued. From (2.14), (2.2), (2.4), (2.13) and (2.12), we have

$$(A, a)M\ (B, b)M = [Af)(ah),\ ag]\ [(Bf)(bh),\ bg] =$$
$$= [(Af)(ah)((Bf)(bh))^{(ag)}(ag)^{(bg)}, ag.\ bg] = [(Af)(ah)(Bf)^{ag}(bh)^{ag}(ag)^{bg}, (ab)_g]$$
$$= [(Af)(B^a f)(ah)(bh)^{ag}(ag)^{bg},\ (ab)_g] = [(Af)(B^a f)(a^b f)(ab)h,\ (ab)_g]$$
$$[(AB^a a^b)f(ab)h,\ (ab)_g] = (AB^a a^b,\ ab)M = ((A, a)(B, b))M\ .$$

Thus, M is a homomorphism of P into P^*. Conversely, let M be any homomorphism of P into P^*. It follows from B1 and B2 that $UM \subseteq U.^*$ Thus, by Theorem 2.2, we may let

$$(2.15) \qquad (A, e)M = [Af,\ e^*]$$

where e and e^* denote the identities of P/L and P^*/L^* respectively. Clearly, f is a mapping of U into U^*. It follows easily from (2.15), (2.2) and (2.3) that f is a homomorphism of U into U^*. Let E be the identity of U. Then,

$$(2.16) \qquad (E, a)M = [ah,\ ag]\ .$$

Clearly, h is a function of P/L into U^* and g is a function of P/L into P^*/L^*. From (2.2) and (2.3), $(A, a) = (A, e)(E, a)$. Thus, by (2.15), (2.16), (2.2), and (2.3)

$$(2.17) \quad (A, a)M = (A, e)M(E, a)M = [Af,\ e^*][ah,\ ag] = [(Af)(ah),\ ag]\ .$$

From (2.2) and (2.3), we have $(E, a)(E, b) = (a^b, ab)$. Thus, by (2.17), we have $[ah, ag]\,[bh, bg] = [(a^b f)(ab)h, (ab)g]$. Therefore, by (2.2)

$$(2.18) \qquad [(ah)(bh)^{ag}(ag)^{bg},\ (ag)(bg)] = [(a^b f)(ab)h,\ (ab)g]\ .$$

From (2.18), it follows that g is a homomorphism and (2.12) is satisfied. From (2.2) and (2.3), we have $(E, b)(A, e) = (A^b, b)$. Thus, from (2.17) and (2.15), $[bh, bg]\,[Af, e^*] = [(A^b f)(bh), bg]$. Hence, (2.13) follows from (2.2) and (2.3).

Suppose M is an isomorphism of P onto P^*. Therefore, by (2.14) $(A, a)M = [(Af)(ah), ag]$ where f is a homomorphism of U into U^*, h is a single valued mapping of P/L into U^* and g is a homomorphism P/L into P^*/L^*. It is easy to see that $UM = U^*$. Thus, by virtue of theorem 2.2, if B in U^*, there exists A in U such that $(A, e)M = [B, e^*]$. Thus, by (2.15), $Af = B$ and f maps U onto U^*. By (2.15), f is one-to-one and hence is an isomorphism of U onto U^*. To show g is one-to-one, let

$$ag = bg \ . \tag{2.19}$$

There exists x in U^* such that

$$x(bh) = ah \ . \tag{2.20}$$

Now, by (2.2) and (2.3), $(xf^{-1}, e)(E, b) = (xf^{-1}, b)$. Hence, by (2.15), (2.14), (2.2), (2.3), (2.19) and (2.20), $(xf^{-1}, b)M = [x, e^*][bh, bg] = [x(bh), bg] = [ah, ag] = (E, a)M$. Hence, $a = b$. It follows immediately from (2.14) that g maps P/L onto P^*/L^* and hence g is an isomorphism of P/L onto P^*/L^*.

Conversely, suppose there exists an isomorphism f of U onto U^*, an isomorphism g of P/L onto P^*/L^* and a single valued mapping h of P/L into U^* such that (2.12) and (2.13) are satisfied. Therefore, by (2.14), $(A, a)M = [(Af)(ah), ag]$ is a homomorphism of P into P^*. It is easily seen that M is one-to-one. Let $[B, b]$ be in P^*. Now there exists a in P/L such that $b = ag$ and A in U such that $(Af)(ah) = B$. Hence $(A, a)M = [B, b]$ by (2.14).

REMARK. If $ah = E^*$, where E^* is the identity of U^*, then (2.12) and (2.13) simplify greatly:

$$(ag)^{bg} = a^b f \ , \tag{2.12$'$}$$

$$(Af)^{bg} = A^b f \ . \tag{2.13$'$}$$

Professor Clifford remarks that we can bring this about by making a new choice of representative elements in P or in P^*, respectively, in the following two cases: if the range of h is contained in the range of f; or if $ag = a'g$ (a, a' in P/L) implies $ah = a'h$.

Section 3. Examples. We give some examples to illustrate the theory.

EXAMPLE 1. The bicyclic semigroup "C" [2, p. 43] consists of all pairs of nonnegative integers with multiplication given by

$$(i, j)(k, s) = (i + k - \min\ (j, k)\, j + s - \min\ (j, k)) \ , \tag{3.1}$$

A complete set of endomorphisms of "C" is given by

$$(i, j)M_{(t,\, k)} = (ti + k, tj + k)(i, j \text{ are nonnegative integers}) \tag{3.2}$$

where (t, k) runs through all ordered pairs of nonnegative integers.

The only automorphism of 'C' is the identity.

EXAMPLE 2. Let G be any group of order greater than or equal to two with identity E. Let I_0 be the nonnegative integers under the usual addition. Consider $P = GxI_0$ under the following multiplication.

$$(A, a)(B, b) = (AB^a, a + b) \tag{3.3}$$

where $B^a = B$ if $a = 0$
$B^a = E$ if $a \neq 0$.

P is a semigroup satisfying (B1), (B2), (B3) which is not left cancellative. Let S be the semigroup corresponding to P in Clifford's main theorem. Let h be a mapping of I_0 into G such that $oh = E$ and $ah = (a + b)h$ for all $a \neq 0$. Let f be an automorphism of G. Then,

(3.4) $((A, a), (B, b))M = (((Af)(ah), a), ((Bf)(bh), b))$ where (A, a) , (B, b) in P is an automorphism of S. Conversely every automorphism of S is obtained in this fashion.

One obtains similar results if I_0 is replaced by the positive part of any lattice ordered group.

EXAMPLE 3. Let G^+ be the positive part of any lattice ordered group G. Let S be the semigroup corresponding to G^+ in Clifford's main theorem. Then there exists a one-to-one correspondence between the automorphisms M of S and the order preserving automorphisms N of G. This correspondence is given by

$$(m, n)M = (mN, nN) \ (m \text{ and } n \text{ in } G^+) .$$

REFERENCES

1. A. H. Clifford, *A class of d-simple semigroups*, Amer. J. Math., **75** (1953), 547-556.
2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1962.
3. W. D. Munn, *Semigroups and their algebras*, Dissertation, Cambridge University, 1955.
4. W. D. Munn and R. Penrose, *A note on inverse semigroups*, Proc. Cambridge Philos. Soc., **51** (1955), 369-399.
5. G. B. Preston, *Inverse semi-groups*, J. London, Math. Soc., **29** (1954), 396-403.
6. L. Redéi, *Die Verallgemeinerung der Schreierschen Erweiterungstheorie*, Acta Sci. Math., Szeged **14** (1952), 252-273.
7. D. Rees, *On semi-groups*, Proc. Cambridge Phil. Soc., **36** (1940), 387-400.
8. ———, *On the ideal structure of a semi-group satisfying a cancellation law*, Quarterly J. Math. Oxford Ser., **19** (1948), 101-108.

WEST VIRGINIA UNIVERSITY
MORGANTOWN, WEST VIRGINIA