ON THE STRUCTURE OF INFRAPOLYNOMIALS WITH PRESCRIBED COEFFICIENTS

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Introduction. The main result of this paper is Theorem 5 which deals with the structure of infrapolynomials with prescribed coefficients. This theorem was quoted (without proof) in a previous paper [Shisha and Walsh, 1961][1], and was used there to prove a few results concerning the geometrical location of the zeros of some infrapolynomials with prescribed coefficients [loc. cit., Theorems 11, 12, 16, 17]. Two similar results are given here in Theorem 6.

We refer the reader to the Introduction of the last mentioned paper for a review of the development of the concept of infrapolynomial. Here we shall just mention two of the underlying definitions.

A. Let n and q be natural numbers ($q \leq n$), $n_1, n_2, \cdots, n_q$ integers such that $0 \leq n_1 < n_2 \cdots < n_q \leq n$, and S a set in the complex plane[2]. An nth *infrapolynomial on* S *with respect to* $(n_1, n_2, \cdots, n_q)$ is a polynomial $A(z) \equiv \sum_{\nu=0}^{n} a_\nu z^\nu$ such that no $B(z) \equiv \sum_{\nu=0}^{n} b_\nu z^\nu$ exists, satisfying the following properties.

(1) $B(z) \not\equiv A(z)$,

(2) $b_{n_\nu} = a_{n_\nu}$ ($\nu = 1, 2, \cdots, q$),

(3) $|B(z)| < |A(z)|$ whenever $z \in S$ and $A(z) \neq 0$, and

(4) $B(z) = 0$ whenever $z \in S$ and $A(z) = 0$.

B. Let n be a natural number. *A simple n-sequence* is a sequence having one of the forms

$$(0, 1, \cdots, k, n-l, n-l+1, \cdots, n)\ [k \geq 0,\ l \geq 0,\ k+l+2 \leq n]\,,$$
$$(0, 1, \cdots, k)\ [0 \leq k < n],\ (n-l, n-l+1, \cdots, n)\ [0 \leq l < n]\,.$$

Theorem 5 may yield information on the location of the zeros of an nth infrapolynomial $A(z)$ on a set S with respect to a simple n-sequence σ. For it allows (under quite general conditions) to set $A(z) \equiv B(z)\ D(z)$ where $D(z)$ is a polynomial all of whose zeros lie in S, whereas $B(z)$ is a divisor of a polynomial $Q(z)$ whose structure is given by the theorem. By studying the location of the zeros of $Q(z)$, one may get information on the location of the zeros of $A(z)$. By this method, Theorems 11, 12, 16, 17 [loc. cit.] were proved. (Compare

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[1] Dates in square brackets refer to the bibliography.

[2] We deal throughout this paper with the *open* plane of complex numbers.

also the proof of Theorem 6 below.)

Theorem 5 is a generalization of Fekete's structure theorem [1951], and we use his method of proof [cf. also Fekete 1955]. The concept of a "juxtafunction" (Definition 1) is a generalization of Fekete's "nearest polynomial" [1955], later termed "juxtapolynomial" [Walsh and Motzkin 1957]. Theorems 1–4 and Lemmas 1–4 are contained in the author's Ph. D. thesis [1958]; they are needed for the proof of Theorem 5, and they generalize previous results of Fekete [1951, 1955]. The principal results of the present paper were published by the author (without proof) in abstracts (1958a, 1959, 1961].

1. DEFINITION 1. *Let S be a set in the complex plane and let Π be a set of complex functions defined on*[3] *S such that whenever $f_1 \in \Pi$, $f_2 \in \Pi$ and c_1, c_2 are complex numbers, then*[4] *$c_1 f_1 + c_2 f_2 \in \Pi$. Let f be a complex function defined on S. A juxtafunction to f on S with respect to Π is an element p of Π having the property: there does not exist a $q \in \Pi$ satisfying*
(a) $q(z) \neq f(z)$ *for at least one* $z \in S$,
(b) $|f(z) - q(z)| < |f(z) - p(z)|$ *whenver* $z \in S$ and $p(z) \neq f(z)$,
(c) $q(z) = f(z)$ *whenever* $z \in S$ and $p(z) = f(z)$.

EXAMPLES A. Let $S(\neq \varnothing)$ be[5] a closed and bounded set in the complex plane. Let $f, p_1, p_2, \cdots, p_n, \mu$ be complex functions with domain S which are continuous on S, and assume, furthermore, that $\mu(z) \neq 0$ throughout S. For every complex function ψ with domain S which is continuous on S, let $\|\psi\| = \max[|\mu(z)\psi(z)|, z \text{ on } S]$. It is known that there exist complex numbers $\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*$ such that for every complex $\lambda_1 \lambda_2, \cdots, \lambda_n$,

$$\left\| f - \sum_{\nu=1}^{n} \lambda_\nu^* f_\nu \right\| \leq \left\| f - \sum_{\nu=1}^{n} \lambda_\nu f_\nu \right\| . \tag{1}$$

Consider the linear space Π of all linear combinations (with complex coefficients) of $p_1, p_2, \cdots, p_n$. Then $p = \sum_{\nu=1}^{n} \lambda_\nu^* p_\nu$ is a juxtafunction to f on S with respect to Π. Indeed, suppose that some $q = \sum_{\nu=1}^{n} \lambda_\nu' p_\nu$ satisfies (a), (b) and (c) of Definition 1. Let ζ be a point of S such that

$$\|f - q\| = |\mu(\zeta)(f(\zeta) - q(\zeta))| .$$

Then by (a) $q(\zeta) \neq f(\zeta)$, and therefore, by (c), $p(\zeta) \neq f(\zeta)$. From (b) we get $\|f - \sum_{\nu=1}^{n} \lambda_\nu' f_\nu\| = \|f - q\| = |\mu(\zeta)(f(\zeta) - q(\zeta))| < |\mu(\zeta)(f(\zeta)$

[3] i.e. their domains include S.
[4] The domain of $c_1 f_1 + c_2 f_2$ is the intersection of those of f_1 and f_2.
[5] $\varnothing$ denotes the empty set.

$-p(\zeta))| \leq \|f - q\| = \|f - \sum_{\nu=1}^{n}\lambda_\nu^* f_\nu\|$, contradicting (1).

B. Let $f, p_1, p_2, \cdots, p_n$ be real functions with domain $S = [0, 1]$, continuous there, and assume furthermore that $p_1, p_2, \cdots, p_n$ are orthonormal on $[0, 1]$. Let Π be again the set of all linear combinations (with complex coefficients) of $p_1, p_2, \cdots p_n$. Let $\lambda_\nu^* = \int_0^1 f(x)p_\nu(x)dx$ $(\nu = 1, 2, \cdots, n)$. Then $p = \sum_{\nu=1}^{n}\lambda_\nu^* p_\nu$ is a juxtafunction to f on S with respect to Π. Indeed, if $p = f$, then the last assertion follows from Lemma 1 below. We thus assume that $p(x_0) \neq f(x_0)$ for some $x_0 \in [0, 1]$. Suppose there exists a $q = \sum_{\nu=1}^{n}\lambda_\nu p_\nu$ satisfying (a), (b) and (c) of Definition 1. Then $|f(x) - q(x)| \leq |f(x) - p(x)|$ throughout $[0, 1]$, and $|f(x_0) - q(x_0)| < |f(x_0) - p(x_0)|$. Thus

$$\int_0^1\left[f(x) - \sum_{\nu=1}^{n} Re(\lambda_\nu)p_\nu(x)\right]^2 dx < \int_0^1\left[f(x) - \sum_{\nu=1}^{n}\lambda_\nu^* p_\nu(x)\right]^2 dx ,$$

contradicting the least squares property of the Fourier coefficients λ_ν^*.

LEMMA 1. *Let S and Π be as in Definition* 1 *and let f be an element of Π with domain S. Then f is the unique function with domain S which is a juxtafunction to f on S with respect to Π.*

Proof. f is such a juxtafunction, since (a) and (c) of Definition 1 are mutually contradictory when p is f. If p (with domain S) belongs to Π and $p \neq f$, then $q = \frac{1}{2}(p + f)$ belongs to Π and satisfies (a), (b) and (c), so that p is not a juxtafunction to f on S with respect to Π.

THEOREM 1.

Hypotheses.

1. $S(\neq \emptyset)$ is a closed and bounded set in the complex plane, $f, p_1, p_2, \cdots, p_n$ are complex functions defined and continuous on[6] S.

2. *Π is the set of all complex functions defined on S which can be represented throughout S as linear combinations* (*with complex coefficients*) *of the p_ν's.*

3. *p is a juxtafunction to f on S with respect to Π, and $p(z) \neq f(z)$ throughout S.*

[6] As the domain of f may properly include S, its continuity on S means that if $a \in S$, and if $(a_j)_{j=1}^{\infty}$ is a sequence of points of S converging to a, them $\lim_{j\to\infty} f(a_j) = f(a)$. Similarly for $p_1, p_2, \cdots, p_n$ and in Lemma 2.

Conclusion. There exist distinct points $z_1, z_2, \cdots, z_m$ of S $(1 \leq m \leq 2n+1)$ and positive $\lambda_1, \lambda_2, \cdots, \lambda_m$ such that:

(I). $p(z)$ is a juxtafunction to f on $s = \{z_1, z_2, \cdots, z_m\}$ with respect to Π,

(II). *No* complex $b_1, b_2, \cdots, b_n$ exist such that $|f(z) - \sum_{\nu=1}^{n} b_\nu p_\nu(z)| < |f(z) - p(z)|$ throughout s,

(III). $\sum_{\mu=1}^{m} \lambda_\mu p_\nu(z_\mu)/\{f(z_\mu) - p(z_\mu)\} = 0$, $\nu = 1, 2, \cdots, n$.

REMARK 1. Observe that (I) is implied by (II).

For the proof of Theorem 1 we shall need two lemmas.

LEMMA 2. *Let $S(\neq \varnothing)$ be a closed and bounded set in the complex plane, and Π a set of complex functions, defined and continuous on S such that whenever $f_1 \in \Pi$, $f_2 \in \Pi$, and c_1 and c_2 are complex numbers, then $c_1 f_1 + c_2 f_2 \in \Pi$. Let f be a complex function defined and continuous on S, and let p be an element of Π such that $p(z) \neq f(z)$ throughout S. A necessary and sufficient condition for the existence of a $q \in \Pi$ satisfying throughout S*

$$|f(z) - q(z)| < |f(z) - p(z)| \tag{2}$$

is the existence of an $r \in \Pi$, satisfying throughout S

$$|f(z) - p(z) - r(z)| < |f(z) - p(z) + r(z)| \, . \tag{3}$$

Proof of Lemma 2.

Necessity. Let $r = q - p$. Then throughout S

$$|f(z) - p(z) - r(z)| < |f(z) - p(z)| < |f(z) - p(z)|\{2 - |f(z) - q(z)| \times |f(z) - p(z)|^{-1}\} \leq |2\{f(z) - p(z)\} - \{f(z) - q(z)\}| = |f(z) - p(z) + r(z)| \, .$$

Sufficiency. We use the fact that if a, b are arbitrary complex numbers, the inequalities $|a - b| < |a + b|$, $Re(b\,\bar{a}) > 0$, are equivalent. Since throughout S

$$Re[r(z)/\{f(z) - p(z)\}] = |f(z) - p(z)|^{-2}\, Re[r(z)\overline{\{f(z) - p(z)\}}] > 0 \, ,$$

we have there $\alpha\,|r(z)/\{f(z) - p(z)\}|^2 < 2Re[r(z)/\{f(z) - p(z)\}]$ where $\alpha = \min\,[|\{f(z) - p(z)\}/r(z)|^2 Re(r(z)/\{f(z) - p(z)\}),\ z \text{ on } S]$. Let $q = p + \alpha r$. Then throughout S,

$$|f(z) - q(z)| = |f(z) - p(z)|\,|1 - \alpha r(z)\{f(z) - p(z)\}^{-1}| = |f(z) - p(z)| \times [1 + \alpha^2 |r(z)/(f(z) - p(z))|^2 - 2\alpha Re\{r(z)(f(z) - p(z))^{-1}\}]^{1/2} < |f(z) - p(z)| \, .$$

LEMMA 3. *Let the Hypotheses* 1, 2 *of Theorem* 1 *hold, and let* p *be an element of* Π *such that* $p(z) \neq f(z)$ *throughout* S. *For every* $z \in S$, *let* $F(z)$ *denote the point* $(x_1(z), y_1(z), x_2(z), y_2(z), \cdots, x_n(z), y_n(z))$ *of the* (*real*) *Euclidean* $2n$-*space* E_{2n}, *where* $x_\nu(z)$ *is the real part and* $y_\nu(z)$ *the imaginary part of* $p_\nu(z)\ \overline{\{f(z) - p(z)\}}$. *A necessary and sufficient condition for the existence of a* $q \in \Pi$ *satisfying* (2) *throughout* S, *is that the point* $\Omega_{2n} = (0, 0, \cdots 0)$ *of* E_{2n} *does not belong to the convex hull* H *of*[7] $F(s)$.

Proof of Lemma 3.

Necessity. By Lemma 2 there exists an $r \in \Pi$ such that (3), i.e. the inequality

$$Re[r(z)\overline{\{f(z) - p(z)\}}] > 0 \tag{3a}$$

holds throughout S. Let $s_1, t_1, s_2, t_2, \cdots, s_n, t_n$ be reals such that throughout S, $r(z) = \sum_{\nu=1}^{n}(s_\nu - it_\nu)p_\nu(z)$. Then throughout S we have

$$\sum_{\nu=1}^{n} s_\nu x_\nu(z) + t_\nu y_\nu(z) > 0 \tag{4}$$

and thus $F(s)$ is a subset of the half-space

$$s_1x_1 + t_1x_2 + \cdots + s_nx_{2n-1} + t_nx_{2n} > 0 . \tag{5}$$

Therefore H is also a subset of this half-space, and consequently $\Omega_{2n} \notin H$.

Sufficiency. Since H is compact and $\Omega_{2n} \notin H$, we can find a half-space (5) containing $F(S)$. Thus (4) holds for every $z \in S$. Setting $r = \sum_{\nu=1}^{n}(s_\nu - it_\nu)p_\nu$, we have throughout S, (3a), and therefore (3). Thus, by Lemma 2, there exists a $q \in \Pi$ satisfying (2) throughout S.

Proof of Theorem 1. f cannot belong to Π, for otherwise, by Lemma 1, the restrictions of f and of p to S would coincide, contradicting Hypothesis 3. By Definition 1, there does not exist a $q \in \Pi$ satisfying (2) throughout S. Using notations of the last lemma, it follows that $\Omega_{2n} \in H$. By a well known theorem of Carathéodory there exist in $F(S)$ distinct points $A_1, A_2, \cdots, A_m$ $(m \leq 2n + 1)$ and there exist positive $\Lambda_1, \Lambda_2, \cdots, \Lambda_m$ such that

$$\Omega_{2n} = \sum_{\mu=1}^{m} \Lambda_\mu A_\mu . \tag{6}$$

Let

[7] $F(s)$ is, as usual, the set of all $F(z)$, $z \in S$.

$$A_\mu = F(z_\mu)\ ,\ z_\mu \in S \qquad (\mu = 1, 2, \cdots, m). \tag{7}$$

Then the z_μ are distinct, and from (6) we get by taking components,

$$\sum_{\mu=1}^{m} \Lambda_\mu p_\nu(z_\mu)\overline{\{f(z_\mu) - p(z_\mu)\}} = 0 \qquad (\nu = 1, 2, \cdots, n)\ . \tag{8}$$

Thus

$$\sum_{\mu=1}^{m} \lambda_\mu p_\nu(z_\mu)/\{f(z_\mu) - p(z_\mu)\} = 0 \qquad (\nu = 1, 2, \cdots, n)$$

where $\lambda_\mu = \Lambda_\mu\,|f(z_\mu) - p(z_\mu)|^2 > 0$ $(\mu = 1, 2, \cdots, m)$. Let $s = \{z_1, z_2, \cdots, z_m\}$, and let π be the set of all functions defined on s which can be represented throughout s as linear combinations (with complex coefficients) of the p_ν. Obviously $p \in \pi$, since $p \in \Pi$. From (6) and (7) it follows that Ω_{2n} belongs to the convex hull of $F(s)$ and therefore, by Lemma 3 (taking there s in place of S and π in place of Π) there does not exist a $q \in \pi$ satisfying (2) throughout s. This concludes the proof.

REMARK 2. Suppose that one of the p_ν in Theorem 1 equals throughout S a constant $c(\neq 0)$. Then from (8) we obtain $\sum_{\mu=1}^{m} \Lambda_\mu\{f(z_\mu) - p(z_\mu)\} = 0$. Thus 0 belongs to the convex hull of the image of s (and a fortiori of S) under $f - p$. [Compare Motzkin and Walsh 1953, §2, and Fekete 1955, §18].

REMARK 3. Let $s' = \{z_1, z_2, \cdots, z_M\}$ be a finite set in the complex plane and suppose that $f, p_1, p_2, \cdots, p_n$ are complex functions defined on s'. Let π' be the set of all complex functions representable throughout s' as a linear combination with complex coefficients of $p_1, p_2, \cdots, p_n$. Let p be an element of π' such that $p(z) \neq f(z)$ throughout s', and suppose there exist nonnegative reals $\lambda'_1, \cdots, \lambda'_M$ (not all zero) such that

$$\sum_{\mu=1}^{M} \lambda'_\mu p_\nu(z_\mu)/\{f(z_\mu) - p(z_\mu)\} = 0 \ \ (\nu = 1, 2, \cdots, n)\ .$$

Then there does not exist a $q \in \pi'$ such that (2) holds throughout s'. Indeed, we have

$$\sum_{\mu=1}^{M} \Lambda'_\mu p_\nu(z_\mu)\overline{\{f(z_\mu) - p(z_\mu)\}} = 0 \ \ (\nu = 1, 2, \cdots, n)$$

where Λ'_μ are nonnegative reals, not all zero. Therefore (using notations of Lemma 3) Ω_{2n} belongs to the convex hull of $F(s')$. By Lemma 3, there does not exist a $q \in \pi'$ satisfying (2) throughout s'. Consequently, p is a juxtafunction to f on s' with respect to π'.

THEOREM 2. *Let the hypotheses of Theorem* 1 *hold and suppose furthermore that* $f - p, p_1, p_2, \cdots ; p_n$ *are real valued throughout* S. *Then the inequality* $1 \leq m \leq 2n + 1$ *in the conclusion of Theorem* 1 *can be replaced by* $1 \leq m \leq n + 1$.

Theorem 2 is proved with the aid of the following lemma, in the same way that Theorem 1 was proved with the aid of Lemma 3.

LEMMA 4. *Let the hypotheses* 1, 2 *of Theorem* 1 *hold, let* p *be an element of* Π *such that* $f(z) \neq p(z)$ *throughout* S, *and suppose that* $f - p, p_1, p_2, \cdots, p_n$ *are real throughout* S. *For every* $z \in S$, *let* $F_1(z)$ *denote the point* $(p_1)(z)\{f(z) - p(z)\}, \ p_2(z)\{f(z) - p(z)\}, \cdots, p_n(z)\{f(z) - p(z)\})$ *of the (real) Euclidean* n-*space* E_n. *A necessary and sufficient condition for the existence of a* $q \in \Pi$ *satisfying* (2) *throughout* S, *is that the point* $\Omega_n = (0, 0, \cdots, 0)$ *of* E_n *does not belong to the convex hull of* $F_1(S)$.

The proof of the last lemma is analogous to that of Lemma 3.

We shall make frequent use of the concept of unisolvence. We mention therefore the following

DEFINITION 2. *Let* S *be a set in the complex plane, and* $(p_\nu(z))_{\nu=1}^n$ *a finite sequence of complex functions defined on* S. *The sequence will be called unisolvent on* S *if and only if for every complex* $c_1, c_2, \cdots, c_n$ *(not all zero) the set of all* $z \in S$ *for which* $\sum_{\nu=1}^n c_\nu p_\nu(z) = 0$, *contains less than* n *points.*

REMARK 4. Thus $(p_\nu(z))_{\nu=1}^n$ is unisolvent on S if and only if this sequence is linearly independent on every n-point subset of S. A simple example is the sequence $(z^{\nu-1})_{\nu=1}^n$, which is unisolvent on every subset of the complex plane. A unisolvent sequence has been termed also (for an important particular case) a "Tchebycheff system". Other terms used in this connection are "Haar system" and "interpolational system".

THEOREM 3. *Let the hypotheses of Theorem* 1 *hold and suppose that each of the sequences* $(p_\nu(z))_{\nu=1}^j (j = 1, 2, \cdots, n)$ *is unisolvent on* S. *Then the inequalities*

$$1 \leq m \leq 2n + 1 \tag{9}$$

in Theorem 1, *can be replaced by the sharper estimate* $n + 1 \leq m \leq 2n + 1$. *Furthermore, if the additional hypothesis of Theorem* 2 *is* *made too,* (9) *can be replaced by* $m = n + 1$.

Proof. Choose distinct points $z_1, z_2, \cdots, z_m$ of S and positive $\lambda_1, \lambda_2, \cdots, \lambda_m$ such that (I), (II) and (III) of Theorem 1 hold, where $1 \leq m \leq 2n + 1$ and where, furthermore, $1 \leq m \leq n + 1$ in case the additional hypothesis of Theorem 2 holds. We shall prove that $n + 1 \leq m$. Indeed: suppose $m \leq n$. Then since $(p_\nu(z))_{\nu=1}^m$ is unisolvent on S, the determinant whose jth row is $p_1(z_j)\ p_2(z_j) \cdots p_m(z_j)$ is different from zero. Therefore there exist constants $c_1, \cdots, c_m$ such that $f(z) = \sum_{\nu=1}^n c_\nu p_\nu(z)$ throughout s. Let π have the same meaning as in the proof of Theorem 1; then $f \in \pi$. By Theorem 1, (II), p is a juxtafunction to f on s with respect to π. By Lemma 1 (with S replaced by s, Π by π, and f by the restriction of our f to s) we have $f(z) = p(z)$ throughout s, contradicting hypothesis 3 of Theorem 1.

2. We apply now Theorems 1, 2 and 3 to nth infrapolynomials (cf. the Introduction).

THEOREM 4. *Let n and q be natural numbers ($q \leq n$), $n_1, n_2, \cdots, n_q$ integers such that $0 \leq n_1 < n_2 \cdots < n_q \leq n$, and S a closed and bounded set in the complex plane. Let $A(z)$ ($\neq 0$ throughout S) be an nth infrapolynomial on S with respect to $(n_1, \cdots, n_q)$. Then*[8] *there exist distinct points $z_1, z_2, \cdots, z_m$ of S,*

$$1 \leq m \leq 2(n - q) + 3 \tag{10}$$

and positive $\lambda_1, \lambda_2, \cdots, \lambda_m$ such that $A(z)$ is an nth infrapolynomial on $s = \{z_1, z_2, \cdots, z_m\}$ with respect to $(n_1, n_2, \cdots, n_q)$ and such that

$$\sum_{\mu=1}^{m} \lambda_\mu z_\mu^{l_\nu}/A(z_\mu) = 0 \qquad (\nu = 1, 2, \cdots, n + 1 - q) \tag{11}$$

where $l_1, l_2, \cdots, l_{n+1-q}$ ($l_1 < l_2 \cdots < l_{n+1-q}$) are the elements of $\{0, 1, \cdots, n\} - \{n_1, n_2, \cdots, n_q\}$. If the polynomials $A(z), z^{l_1}, \cdots, z^{l_{n+1-q}}$ are real valued throughout S, then (10) *can be replaced by $1 \leq m \leq n + 2 - q$. If each of the sequences $(z^{l_\nu})_{\nu=1}^j$ ($j = 1, 2, \cdots, n + 1 - q$) is unisolvent on S, then* (10) *can be replaced by*

$$n - q + 2 \leq m \leq 2(n - q) + 3\ . \tag{12}$$

If the polynomials $A(z)$, $z^{l_1}, \cdots, z^{l_{n+1-q}}$ are real valued throughout S and each of the sequences $(z^{l_\nu})_{\nu=1}^j$ ($j = 1, 2, \cdots, n + 1 - q$) is unisolvent on S, then (10) *can be replaced by $m = n - q + 2$.*

REMARK 5. If $(n_1, n_2, \cdots, n_q)$ of Theorem 4 is a simple n-sequ-

[8] As is easily seen, S cannot be empty. [Cf. Shisha and Walsh, 1961, footnote 7 on p. 117].

ence (cf. the Introduction) and if, in case $n_1 = 0$, $0 \notin S$, then as is easily seen, the sequences $(z^{l_\nu})_{\nu=1}^{j}$ $(j = 1, 2, \cdots, n+1-q)$ are unisolvent on S.

Proof of Theorem 4. Let Π be the set of all complex functions defined on S which are expressible throughout S as linear combinations of $z^{l_1}, z^{l_2}, \cdots, z^{l_{n+1-q}}$ with complex coefficients, and let $f(z) \equiv \sum_{\nu=1}^{q} a_{n_\nu} z^{n_\nu}$, $p(z) \equiv -\sum_{\nu=1}^{n+1-q} a_{l_\nu} z^{l_\nu}$. It is easily seen that $p(z)$ is a juxtafunction to f on S with respect to Π. Therefore, by Theorem 1 there exist distinct points $z_1, \cdots, z_m$ $(m \leq 2(n+1-q)+1 = 2(n-q)+3)$ of S and positive $\lambda_1, \lambda_2, \cdots, \lambda_m$ such that (11) holds, and such that no complex $b_1, b_2, \cdots, b_{n+1-q}$ exist satisfying

$$\left| \sum_{\nu=1}^{q} a_{n_\nu} z^{n_\nu} - \sum_{\nu=1}^{n+1-q} b_\nu z^{l_\nu} \right| < \left| A(z) \right|$$

throughout $s = \{z_1, z_2, \cdots, z_m\}$. Thus $A(z)$ is an nth infrapolymial on s with respect to $(n_1, n_2, \cdots, n_q)$. The rest of Theorem 4 follows from Theorems 2 and 3.

REMARK 6. Let $n, n_1, n_2, \cdots, n_q$ be integers $(q \leq n,\ 0 \leq n_1 < n_2 \cdots < n_q \leq n)$, $A(z) \equiv \sum_{\nu=0}^{n} a_\nu z^\nu$ a polynomial, $z_1, z_2, \cdots, z_M$ points of the complex plane, and $\lambda'_1, \lambda'_2, \cdots, \lambda'_M$ $(\sum_{\mu=1}^{M} \lambda'_\mu > 0)$ nonnegative reals such that $A(z_\mu) \neq 0$ $(\mu = 1, 2, \cdots, M)$, and such that $\sum_{\mu=1}^{M} \lambda'_\mu z_\mu^{l_\nu} / A(z_\mu) = 0$ $(\nu = 1, 2, \cdots, n+1-q)$, where the l_ν have the same meaning as in Theorem 4. Then $A(z)$ is an nth infrapolynomial on $s' = \{z_1, z_2, \cdots, z_M\}$ with respect to $(n_1, n_2, \cdots, n_q)$. Indeed: let f and p be as in the last proof, and let π' be the set of all complex functions representable throughout s' as a linear combination (with complex coefficients) of $z^{l_1}, z^{l_2}, \cdots, z^{l_{n+1-q}}$. The asserted conclusion follows from Remark 3.

We give now the following structure theorem which is the main result of this paper.

THEOREM 5. *Let n and q $(1 \leq q \leq n)$ be integers, and σ a simple n-sequence of q elements. Let S be a closed and bounded set in the complex plane, and in case $0 \in \sigma$, assume that $0 \notin S$. Let $A(z)$ $(\not\equiv 0)$ be an nth infrapolynomial on S with respect to σ, and let $B(z)$ $(\neq 0$ throughout $S)$ be a divisor of $A(z)$. Assume also that the degree*[9] *r of $B(z)$ is $\geq q$. Then $B(z)$ is a divisor of some*

$$Q(z) \equiv P(z) g(z) + z^K \sum_{\mu=1}^{M-q+2} \lambda_\mu g(z)/(z - z_\mu) . \tag{13}$$

[9] By degree of a polynomial $(\not\equiv 0)$ we mean its exact degree. The polynomial 0 is assigned the degree -1.

Here M is an integer satisfying $r \leq M \leq 2r - q + 1$, the z_ν are distinct points of S, $g(z) \equiv \prod_{\mu=1}^{M-q+2}(z - z_\mu)$, the λ_μ are positive reals with $\sum_{\mu=1}^{M-q+2}\lambda_\mu = 1$, $P(z)$ is a polynomial of degree $\leq q - 1$ such that $P(z)g(z) + z^{K+M-q+1}$ is of degree $\leq M$, and K is $\min[\nu, \nu \notin \sigma, \nu = 0, 1, 2, \cdots]$.

REMARK 7. As will be seen from the proof of Theorem 5, if S and the coefficients of $B(z)$ are real, the inequality $r \leq M \leq 2r - q + 1$ of the theorem can be replaced by the equality $M = r$.

In the proof of Theorem 5 use will be made of the following

LEMMA 5. *Let n, q, σ and K be as in the last theorem, let S be a set in the complex plane, and let $A(z)$ $(\not\equiv 0)$ be an* nth *infrapolynomial on S with respect to σ. Let $B(z)$ be a polynomial of degree $r(\geq q)$ dividing $A(z)$. Then $B(z)$ is an* rth *infrapolynomial on S with respect to σ_0, where σ_0 is that simple r-sequence of q elements for which* $K = \min[\nu, \nu \notin \sigma_0, \nu = 0, 1, 2, \cdots]$.

The proof of Lemma 5 is straightforward and may be omitted.

Proof of Theorem 5. By Lemma 5, $B(z)$ in an rth infrapolynomial on S with respect to the sequence σ_0 defined there. We choose (cf. Theorem 4 and Remark 5) distinct points $z_1, z_2, \cdots, z_m$ of S and positive $\lambda_1, \lambda_2, \cdots, \lambda_m$ such that $\sum_{\mu=1}^{m} \lambda_\mu = 1$ and

$$\sum_{\mu=1}^{m} \lambda_\mu z_\mu^\rho / B(z_\mu) = 0 \tag{14}$$

for every integer ρ satisfying $0 \leq \rho \leq r$, $\rho \notin \sigma_0$. Here m is an integer satisfying $r - q + 2 \leq 2(r - q) + 3$, and in case S and the coefficients of $B(z)$ are real we may take $m = r - q + 2$. Set

$$g(z) \equiv \prod_{\mu=1}^{m} (z - z_\mu), \quad N(z) \equiv \sum_{\mu=1}^{m} \lambda_\mu z_\mu^{r-q+K+1} g(z)/\{B(z_\mu)(z - z_\mu)\} . \tag{15}$$

If μ and ν are integers, $1 \leq \mu \leq m$, $0 \leq \nu \leq r - q + K$, then

$$[\lambda_\mu z_\mu^{r-q+K+1} g(z)/\{B(z_\mu)(z - z_\mu)\}]_{z=0}^{(\nu)} = - \sum_{j=0}^{\nu} \lambda_\mu z_\mu^{r-q+K-j} \tbinom{\nu}{j}\, j\,!\; g^{(\nu-j)}(0)/B(z_\mu)$$

(the equality is obvious if $z_\mu = 0$, and otherwise it is obtained by Leibnitz's rule for differentiating a product). Therefore, from (15) we get

$$N^{(\nu)}(0) = - \sum_{j=0}^{\nu} \tbinom{\nu}{j}\, j\,!\; g^{(\nu-j)}(0) \sum_{\mu=1}^{m} \lambda_\mu z_\mu^{r-q+K-j}/B(z_\mu) \tag{16}$$
$$(\nu = 0, 1, \cdots, r - q + K) .$$

Since $\{0, 1, \cdots, r\} - \{\sigma_0\} = \{r - q + K - j\}_{j=0}^{r-q}$, therefore (16) and (14) yield $N^{(\nu)}(0) = 0$, $\nu = 0, 1, \cdots, r - q$. Hence we can write $N(z) \equiv z^{r-q+1}M_1(z)$ where $M_1(z)$ is a polynomial (of degree $\leq m - 2$). Let

$$M_2(z) \equiv \sum_{\mu=1}^{m} \lambda_\mu z_\mu^K g(z)/\{B(z_\mu)(z - z_\mu)\} .$$

By (14),

$$\sum_{\mu=1}^{m} \lambda_\mu z_\mu^K / B(z_\mu) = 0$$

and therefore the degree of $M_2(z)$ is $\leq m - 2$. For every z_j different from zero we have by (15), $M_1(z_j) = z_j^{-r+q-1}N(z_j) = \lambda_j z_j^K g'(z_j)/B(z_j) = M_2(z_j)$. Since there are at least $m - 1$ such z_j, we have $M_1(z) \equiv M_2(z)$. Consider now the polynomial

$$R(z) \equiv B(z)M_2(z) - \sum_{\mu=1}^{m} \lambda_\mu z_\mu^K g(z)/(z - z_\mu) .$$

For $j = 1, 2, \cdots, m$ we have $R(z_j) = B(z_j)M_2(z_j) - \lambda_j z_j^K g'(z_j) = 0$. Therefore we can write $R(z) \equiv g(z)U(z)$, where $U(z)$ is some polynomial. Also, the relation $N(z) \equiv z^{r-q+1}M_2(z)$ and the definition of $R(z)$ imply that the degree of the latter is $\leq m + q - 2$. Therefore the degree of $U(z)$ is at most $q - 2$. If $K \geq 1$, then the relation

$$B(z)M_2(z) \equiv g(z)U(z) + \sum_{\mu=1}^{m} \lambda_\mu z_\mu^K g(z)/(z - z_\mu)$$

yields, upon putting $z_\mu^K = [z + (z_\mu - z)]^K$ and developing the last right member,

$$B(z)M_2(z) \equiv g(z)[U(z) + A_{K-1}(z)] + z^K \sum_{\mu=1}^{m} \lambda_\mu g(z)/(z - z_\mu) ,$$

where $A_{K-1}(z)$ is a polynomial of degree $K - 1$. The last relation (with $A_{K-1}(z) \equiv 0$) holds also when $K = 0$. We set now $P(z) \equiv U(z) + A_{K-1}(z)$, and get that $B(z)$ is a divisor of

$$Q(z) \equiv P(z)g(z) + z^K \sum_{\mu=1}^{m} \lambda_\mu g(z)/(z - z_\mu) .$$

The degree of $Q(z)$, i. e. of $B(z)M_2(z)$, is $\leq m + q - 2$. Thus the degree of $P(z)$ is $\leq q - 1$, and that of $P(z)g(z) + z^{K+m-1}$ is $\leq m + q - 2$. We set now $M = m + q - 2$, and observe that the conclusions of the theorem are all satisfied.

REMARK 8. The polynomial $Q(z)$ of (13) is an Mth infrapolynomial on $\{z_1, z_2, \cdots, z_{M-q+2}\}$ with respect to σ_1, where σ_1 is that simple

M-sequence of q elements for which $\min[\nu, \nu \notin \sigma_1, \nu = 0, 1, 2, \cdots] = K$. This follows from Theorem 1 of Shisha and Walsh [1961].

THEOREM 6. *Let S be a closed and bounded set in the complex plane, $A(z) \equiv \sum_{\nu=0}^{n} a_\nu z^\nu$ $(n \geq 1,\ a_n \neq 0)$ an nth infrapolynomial on S with respect to $(n-1)$, and suppose that $A(z) \neq 0$ throughout S. Then:*

(a) *Every zero ζ of $A(z)$ is of the form*

$$c(\zeta) - \lambda(\zeta)[a_{n-1}/a_n] \tag{17}$$

where $c(\zeta)$ belongs to the convex hull of S and where $0 \leq \lambda(\zeta) \leq 1$.[10]

(b) *Suppose that S lies in a closed disc $C\colon |z-a| \leq r$ (≥ 0). Then all zeros of $A(z)$ belong to $C \cup C_1$, where C_1 is the closed disc $|z - [a - (a_{n-1}/a_n)]| \leq r$. If C and C_1 are disjoint then $A(z)$ has at least $n-1$ zeros belonging to C. [Multlplicities are always being counted].*

Proof. We choose distinct points $z_1, z_2, \cdots, z_m$ of S and positive $\lambda_1, \lambda_2, \cdots, \lambda_m$ $(m \leq 2n+1)$ such that $\sum_{\mu=1}^{m} \lambda_\mu = 1$ and $\sum_{\mu=1}^{m} \lambda_\mu z_\mu^\rho / A(z_\mu) = 0$ for all integers ρ with $0 \leq \rho \leq n$, $\rho \neq n-1$. Then $1 = \sum_{\mu=1}^{m} \lambda_\mu A(z_\mu)/A(z_\mu) = \sum_{\mu=1}^{m} \lambda_\mu a_{n-1} z_\mu^{n-1}/A(z_\mu)$, and so

$$\sum_{\mu=1}^{m} \lambda_\mu z_\mu^{n-1}/A(z_\mu) = 1/a_{n-1}\,.$$

We set $g(z) \equiv \prod_{\mu=1}^{m}(z - z_\mu)$, $N(z) \equiv \sum_{\mu=1}^{m} \lambda_\mu z_\mu^{n-1} g(z)/\{A(z_\mu)(z - z_\mu)\} \equiv a_{n-1}^{-1} z^{m-1} + \cdots$. We follow the proof of Theorem 5 from the sentence following (15). Again we have $N^{(\nu)}(0) = 0$ for every ν satisfying $0 \leq \nu \leq n-2$. Thus we may set $N(z) \equiv z^{n-1} M_1(z)$, where $M_1(z) \equiv a_{n-1}^{-1} z^{m-n} + \cdots$ is some polynomial. Let $M_2(z) \equiv \sum_{\mu=1}^{m} \lambda_\mu g(z)/\{A(z_\mu)(z - z_\mu)\}$. If $n = 1$, then $M_2(z) \equiv N(z) \equiv M_1(z)$. If $n > 1$ then for each z_j different from zero, $M_1(z_j) = \lambda_\mu g'(z_j)/A(z_j) = M_2(z_j)$, and since there are at least $m-1$ such z_j and $M_1(z)$ and $M_2(z)$ are of degrees $\leq m-2$, we have again $M_2(z) \equiv M_1(z)$. Consider now the polynomial $R(z) \equiv A(z)M_2(z) - \sum_{\mu=1}^{m} \lambda_\mu g(z)/(z - z_\mu) \equiv (a_n/a_{n-1}) z^m + \cdots$ For $j = 1, 2, \cdots, m$, $R(z_j) = 0$, and therefore $R(z) \equiv (a_n/a_{n-1}) g(z)$. Thus, $A(z)$ is a divisor of $Q(z) \equiv (a_n/a_{n-1}) g(z) + \sum_{\mu=1}^{m} \lambda_\mu g(z)/(z - z_\mu)$. Let ζ be a zero of $A(z)$. Then $g(\zeta) \neq 0$, and thus $a_n/a_{n-1} + \sum_{\mu=1}^{m} \lambda_\mu/(\zeta - z_\mu) = 0$. Since $\sum_{\mu=1}^{m} \lambda_\mu/(\zeta - z_\mu)$ can be written [Shisha and Walsh 1961, Lemma on p. 127] as $\lambda(\zeta)/(\zeta - c(\zeta))$ where $c(\zeta)$ and $\lambda(\zeta)$ are as required in (a) of our theorem, ζ is of the form (17). Suppose now that S lies in a closed disc $C\colon |z - a| \leq r(\geq 0)$. Then by a theorem due to J. L. Walsh [cf. 1922, Theorem VI; see also Shisha and Walsh 1961, p. 147] all zeros of $Q(z)$ lie in

[10] Thus ζ belongs to the set swept by the convex hull of S while being displaced, the displacement being given by the vector $-a_{n-1}/a_n$.

$C \cup C_1$, and if C and C_1 are disjoint, the number of zeros of $Q(z)$ in them is, respectively, $m - 1$ and 1. From this follow the conclusions of part (b) of our theorem.

BIBLIOGRAPHY

1. M. Fekete 1951. *On the structure of extremal polynomials,* Proc. Natl. Acad. Sci. U.S. **37**, pp. 95-103. 1955. *On the structure of polynomials of least deviation,* Bull. Research Council Israel **5**A, pp. 11-19.

2. T. S. Motzkin and J. L. Walsh 1953. *On the derivative of a polynomial and Chebychev approximation,* Proc. Amer. Math. Soc. **4**, pp. 76-87.

3. O. Shisha 1958. *Properties of extremal systems of points* (mimeographed), Ph. D. thesis. 1958a. *Nearest functions (abstract)*, Notices Amer. Math. Soc. **5**, No. 7, p. 853. 1959. *On generalized infrapolynomials of a certain type* (abstract), Ibid. **6**, No. 3, p. 283. 1961. *A structure theorem for infrapolynomials with prescribed coefficients* (abstract), Ibid. **8**, No. 1, p. 66.

4. O. Shisha and J. L. Walsh 1961. *The zeros of infrapolynomials with some prescribed coefficients,* J. d'analyse Math. **9**, pp. 111-160.

5. J. L. Walsh 1922. *On the location of the roots of certain types of polynomials,* Trans. Amer. Math. Soc. **24**, pp. 163-180. 1958. *On infrapolynomials with prescribed constant term,* J. Math. Pures Appl. **37**, pp. 295-316.

6. J. L. Walsh and T. S. Motzkin 1957. *Polynomials of best approximation on a real finite point set,* Proc. Natl. Acad. Sci. U. S. **43**, pp. 845-846.

7. J. L. Walsh and M. Zedek 1956. *On generalized Tchebycheff polynomials,* Ibid. **42**, pp. 99-104.

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