# QUASI-POSITIVE OPERATORS

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**1. Introduction.** The classical results of Perron and Frobenius ([6], [7], [12]) assert that a finite dimensional, nonnegative, non-nilpotent matrix has a positive eigenvalue which is not exceeded in absolute value by any other eigenvalue and the matrix has a nonnegative eigenvector corresponding to this positive eigenvalue. If the matrix has strictly positive entries, then there is a positive eigenvalue which exceeds every other eigenvalue in absolute value, and the corresponding space of eigenvectors is one-dimensional and is spanned by a vector with strictly positive coordinates. Numerous generalizations of these results to order-preserving linear operators acting in ordered linear spaces have appeared in recent years; a short bibliography is included at the end of this paper. In this paper a generalization in a different direction is obtained which reduces, in the finite dimensional case, to the assertion that the Perron-Frobenius theorems hold if it is only required that all but a finite number of the powers of the matrix satisfy the given conditions. The principal results are theorems of the Perron-Frobenius type which are applicable to any compact linear operator (the compactness condition is weakened somewhat), acting in an ordered real Banach space $B$, which satisfies a condition weaker than order-preserving. In addition, the results apply to the case when the "cone" of positive elements in $B$ has no interior.

**2. Preliminaries.** Throughout the sequel, $B$ will denote a real Banach space with norm $\|\cdot\|$. The complex extension of $B$, $\tilde{B}$, is the complex Banach space $\tilde{B} = \{x + iy \mid x, y \in B\}$ with the obvious definitions of addition and complex scalar multiplication and the norm in $\tilde{B}$ is $\| x + iy \| = \sup_\theta \| \cos\theta \cdot x + \sin\theta \cdot y \|$. If $T$ is a (real) linear operator on $B$ into $B$, the (complex) linear operator $\tilde{T}$ on $\tilde{B}$ into $\tilde{B}$ is defined by $\tilde{T}(x + iy) = Tx + iTy$. $T$ is bounded if and only if $\tilde{T}$ is bounded, in which case $\| T \| = \| \tilde{T} \|$. The spectrum, $\sigma(T)$, and the resolvent, $\rho(T)$, are defined to be the corresponding sets associated with the operator $\tilde{T}$. We denote the spectral radius of $T$ by $r_T$, $r_T = \lim_{n\to\infty} \| T^n \|^{1/n} = \sup_{\lambda \in \sigma(T)} |\lambda|$ (provided $\| T \| < \infty$).

In all of our results there will be a basic assumption that the linear operator under consideration is quasi-compact, a notion which we will now define. A bounded linear operator $T$ is compact (also called completely continuous) if each sequence $Tx_1$, $Tx_2$, $\cdots$, with

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$\| x_i \| \leqq 1$, $i = 1, 2, \cdots$, has a convergent subsequence. $T$ is quasi-compact if there exists a positive integer $n$ and a bounded linear operator $V$ such that $T^n - V$ is compact and $r_V < r_T^n$.[1] There are a number of properties possessed by quasi-compact operators some of which we state now without proof.[2] If $\lambda_0 \in \sigma(T)$ and $|\lambda_0| = r_T$, then $\lambda_0$ is an isolated point in $\sigma(T)$ and is in the point spectrum, i.e., $(\lambda_0 I - \tilde{T})$ is not one-to-one. The resolvent operator, $R(\lambda, T) \equiv (\lambda I - \tilde{T})^{-1}$, exists in a neighborhood of $\lambda_0$ (excluding $\lambda_0$) and, in this neighborhood, $R(\lambda, T)$ has a Laurent series expansion of the form

$$R(\lambda, T) = \sum_{k=1}^{n(\lambda_0)} \frac{(\lambda_0 I - \tilde{T})^{k-1}}{(\lambda - \lambda_0)^k} P(\lambda_0, T) + \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k A_k(\lambda_0, T)$$

where $A_k(\lambda_0, T)$ is a bounded linear operator and the series on the right is convergent in the uniform operator topology. The integer $n(\lambda_0)$ is the index of $\lambda_0$, i.e., $n(\lambda_0)$ is the smallest integer $n$ such that $\{x \mid (\lambda_0 I - \tilde{T})^{n+1} x = 0\} = \{x \mid (\lambda_0 I - \tilde{T})^n x = 0\}$. $P(\lambda_0, T)$ is a projection onto the finite dimensional space $\{x \mid (\lambda_0 I - \tilde{T})^{n(\lambda_0)} x = 0\}$. The minimal property of $n(\lambda_0)$ implies that $(\lambda_0 I - \tilde{T})^{n(\lambda_0)-1} P(\lambda_0, T) \neq 0$.

We recall that for an arbitrary bounded linear operator, the resolvent $R(\lambda, T) = (\lambda I - \tilde{T})^{-1}$ is an analytic function of $\lambda$ for $\lambda \in \rho(T)$ and the expansion $R(\lambda, T) = \sum_{k=0}^{\infty} (1/\lambda)^{k+1} \tilde{T}^k$ is valid for $|\lambda| > r_T$.

**3. Quasi-positive operators.** A *cone* in $B$ is a convex set $K$ which contains $\lambda x$ for all $\lambda \geqq 0$ if it contains $x$. $K$ is a *proper* cone if $x \in K$ and $-x \in K$ imply $x = 0$. A cone $K$ induces an ordering $\geqq$ in $B$ with $x \geqq y$ if and only if $x - y \in K$. This transitive ordering satisfies

(1) if $x \geqq y$, $u \geqq v$, then $x + u \geqq y + v$,

(2) if $x \geqq y$ and $\lambda \geqq 0$, then $\lambda x \geqq \lambda y$, and

(3) $x \geqq y$ if and only if $-y \geqq -x$.

If the cone is proper, then the ordering satisfies, in addition,

(4) if $x \geqq y$ and $y \geqq x$, then $x = y$.

We will use the notation $x > y$ to denote $x \geqq y, x \neq y$. Associated with a cone $K$ is a closed cone $K^+$ in the conjugate space $B^*$ of continuous, real-valued, linear functions on $B$, consisting of those $x^* \in B^*$ with the property that $x^*(x) \geqq 0$ for all $x \in K$. $K^+$ is a proper cone if and only if the linear space spanned by $K$ is dense in $B$ (a set with this property is called *fundamental*). This is an easy consequence of the Hahn-Banach theorem on the extension of linear functionals. We will use the notations $x^* \geqq y^*$ and $x^* > y^*$ to denote $x^* - y^* \in K^+$

[1] Note that a compact operator is quasi-compact if and only if it has a positive spectral radius.

[2] For details, see Yu. L. Smvl'yan, *Completely continuous perturbations of operators*, Amer. Math. Soc. Translations **10**, 341-344.

and $x^* - y^* \in K^+$, $x^* \neq y^*$, respectively. An element $x > 0$ ($x^* > 0$) will be called *strictly positive* if $x^*(x) > 0$ for all $x^* > 0$ ($x^*(x) > 0$ for all $x > 0$).

The following theorem is a characterization of a closed cone and its interior (when the latter is nonvoid) in terms of $K^+$. The proof may be found, for example, in [11] (Theorem 1.3 and its corollaries, pg. 16).

THEOREM 1. *Let $K$ be a closed cone in $B$. Then $x \in K$ if and only if $x^*(x) \geqq 0$ for all $x^* \geqq 0$. If $K$ has a nonvoid interior, then*

(1) *$x$ is in the interior of $K$ if and only if $x$ is strictly positive and*

(2) *for each $x$ on the boundary of $K$ there exists an $x^* > 0$ such that $x^*(x) = 0$.*

COROLLARY. *If $K$ is a closed proper cone, $K^+$ is a total set of functionals,* i.e., *for each $x \neq 0$, $x \in B$, there exists $x^* > 0$ such that $x^*(x) \neq 0$.*

*Proof.* Since either $x \notin K$ or $-x \notin K$ if $x \neq 0$, this follows immediately from Theorem 1.

A linear operator $T$ on $B$ into $B$ will be called *positive with respect to a cone $K$* if $TK \subseteqq K$. In the absence of ambiguity we will simply say $T$ is positive. In our applications $K$ will be a closed cone and in this case, in view of Theorem 1, $T$ is positive if and only if $x^*(Tx) \geqq 0$ for all $x \geqq 0$, $x^* \geqq 0$. Since $Tx \geqq 0$ if $x \geqq 0$, we have $x^*(T^2x) \geqq 0$ and, in general, $x^*(T^nx) \geqq 0$ for all $n$ and all $x \geqq 0$, $x^* \geqq 0$. We define $T$ to be *quasi-positive* if for each pair $x \geqq 0$, $x^* \geqq 0$, there exists an integer $n(x, x^*) \geqq 1$ such that $x^*(T^nx) \geqq 0$ if $n \geqq n(x, x^*)$. We define $T$ to be *strictly quasi-positive* if for each pair $x > 0$, $x^* > 0$, there exists an integer $n(x, x^*) \geqq 1$ such that $x^*(T^nx) > 0$ if $n \geqq n(x, x^*)$. Finally we define $T$ to be *strongly quasi-positive* if it is not nilpotent[3] and for each pair $x > 0$, $x^* > 0$, $\liminf_{n\to\infty} x^*(T^nx)/\| T^n \| > 0$.

**4. Spectral properties.** Throughout this section, $K$ will denote a closed proper cone in $B$ and $K$ will be assumed to be fundamental. $T$ will denote a quasi-compact bounded linear operator with spectral radius 1. This restriction on the spectral radius is for convenience only and the results given may be interpreted for a general (quasi-compact) bounded linear operator $S$ with spectral radius $r_S > 0$ by considering the operator $T = (1/r_S)\, S$ which has spectral radius 1.

[3] An operator $T$ is nilpotent if $T^n = 0$ for some $n$.

THEOREM 2. *If $T$ is quasi-positive and quasi-compact with spectral radius* 1, *then* $1 \in \sigma(T)$ *and the index of* 1 *is not exceeded by the index of any other point* $\lambda \in \sigma(T)$, $|\lambda| = 1$.

*Proof.* Assume that $1 \in \rho(T)$. Since $\rho(T)$ is open and $R(\lambda, T)$ is analytic in $\lambda$ for $\lambda \in \rho(T)$, it follows that the function $g(\lambda) = x^*(R(1/\lambda, T)x)$, $x > 0$, $x^* > 0$, is analytic for $1/\lambda \in \rho(T)$, in particular for $\lambda$ in some neighborhood of 1. Moreover, $R(\lambda, T) = \sum_{k=0}^{\infty}(1/\lambda)^{k+1}\tilde{T}^k$ if $|\lambda| > 1$, hence $g(\lambda) = \sum_{k=0}^{\infty}\lambda^{k+1}x^*(T^k x)$ if $|\lambda| < 1$. A theorem of Pringsheim states that if a power series has nonnegative coefficients and converges in the open unit disk, either 1 is a singularity of the series or the series has radius of convergence greater than 1.[4] Clearly it is sufficient to assume that all but a finite number of the coefficients are nonnegative. Since $x^*(T^n x) \geqq 0$ if $n \geqq n(x, x^*)$, and $g(\lambda)$ is analytic in a neighborhood of 1, we conclude that the series $\sum_{k=0}^{\infty}\lambda^{k+1}x^*(T^k x)$ converges in $|\lambda| < 1 + \delta$ for some $\delta > 0$. By assumption $r_T = 1$, hence $R(\lambda, T)$ has a singularity somewhere on $|\lambda| = 1$, say at $\lambda_0$. Since $T$ is quasi-compact, the expansion

$$R(\lambda, T) = \sum_{k=1}^{n} \frac{(\lambda_0 I - \tilde{T})^{k-1}}{(\lambda - \lambda_0)^k} P(\lambda_0, T) + \sum_{k=0}^{n} (\lambda - \lambda_0)^k A_k(\lambda_0, T)$$

is valid for $0 < |\lambda - \lambda_0| < \delta'$, where $n = n(\lambda_0)$ is the index of $\lambda_0$ and $(\lambda_0 I - \tilde{T})^{n-1}P(\lambda_0, T) \neq 0$. We may choose $x > 0$ such that $(\lambda_0 I - \tilde{T})^{n-1}P(\lambda_0, T)x = y \neq 0$ since $K$ is fundamental and by Theorem 1 we may choose $x^* > 0$ such that $x^*(y) \neq 0$. It follows easily that

$$g(\lambda) = (\lambda/\lambda_0)^n(1/\lambda_0 - \lambda)^{-n}h(\lambda)\ , \quad |1/\lambda - \lambda_0| < \delta\ ,$$

where $h(\lambda)$ is analytic and $h(1/\lambda_0) = x^*(y) \neq 0$. Thus $g$ has a pole at $1/\lambda_0$ which contradicts the fact that $g$ has a Taylor's series about the origin with radius of convergence greater than 1. Our assumption that $1 \in \rho(T)$ leads to a contradiction, hence $1 \in \sigma(T)$.

Now let the index of 1 be $n$. It is easy to see that $\lim_{\lambda \to 1}(\lambda - 1)^k R(\lambda, T) = 0$ if $k > n$. It follows that for $|\lambda| > 1$, $\lim_{\lambda \to 1}(\lambda - 1)^k \sum_{m=0}^{\infty}(1/\lambda)^{m+1}x^*(T^m x) = 0$ for every pair $x > 0$, $x^* > 0$ and clearly this implies $\lim_{\lambda \to 1}(\lambda - 1)^k \sum_{m=j}^{\infty}(1/\lambda)^{m+1}x^*(T^m x) = 0$ if $k > n$ and $j \geqq 0$. If $\lambda_0 \in \sigma(T)$, $|\lambda_0| = 1$ and $\lambda_0$ has index $l$, then $\lim_{\lambda \to \lambda_0}(\lambda - \lambda_0)^l R(\lambda, T) \neq 0$. We may choose $x > 0$ and $x^* > 0$ such that $\lim_{\lambda \to \lambda_0}(\lambda - \lambda_0)^l x^*(R(\lambda, T)x) \neq 0$ and it follows that for $|\lambda| > 1$, $\lim_{\lambda \to \lambda_0}(\lambda - \lambda_0)^l \sum_{m=j}^{\infty}(1/\lambda)^{m+1}x^*(T^m x) \neq 0$. Let $\lambda_0 = e^{i\varphi}$, $\lambda = \rho e^{i\varphi}$, $\rho > 1$. If $j \geqq n(x, x^*)$, $|(\lambda - \lambda_0)^l \sum_{m=j}^{\infty}(1/\lambda)^{m+1}x^*(T^m x)| \leqq (\rho - 1)^l \sum_{m=j}^{\infty}(1/\rho)^{m+1}x^*(T^m x)$. The expression on the right in this last inequality tends to zero as

[4] See Titchmarsh, *Theory of Functions*, pg. 214. Acknowledgement is due here to S. Karlin for the essence of the proof in Theorem 2 (see [**10**], Theorem 4).

$\rho$ tends to 1 if $l > n$, hence $l \leqq n$. This completes the proof.

THEOREM 3. *If $T$ is quasi-positive and quasi-compact with spectral radius 1, there exist elements $u > 0$ and $u^* > 0$ such that $Tu = u$, $T^*u^* = u^*$.*[5]

*Proof.* By Theorem 2, $1 \in \sigma(T)$. We have

$$R(\lambda, T) = \sum_{k=1}^{n} \frac{(I - \tilde{T})^{k-1}}{(\lambda - 1)^k} P(1, T) + \sum_{k=0}^{\infty} (\lambda - 1)^k A_k(1, T)$$

where $P(1, T)$ is a projection onto the finite-dimensional space $\{x \mid (I - \tilde{T})^n x = 0\}$ and $(I - \tilde{T})^{n-1} P(1, T) \neq 0$. Let $\Gamma = (I - \tilde{T})^{n-1} P(1, T)$. It is easy to see that $R(\lambda, T)B \subseteqq B$ for $\lambda$ real. Since $\Gamma = \lim_{\lambda \to 1} (\lambda - 1)^n R(\lambda, T)$, it follows that $\Gamma B \subseteqq B$. Also $\tilde{T}\Gamma = \Gamma\tilde{T} = \Gamma$. Let $x \geqq 0$, $x^* \geqq 0$ be arbitrary and let $N = n(x, x^*)$. If $\lambda > 1$, we have $x^*(T^N R(\lambda, T)x) = \sum_{m=0}^{\infty} (1/\lambda)^{m+1} x^*(T^{N+m})x \geqq 0$. It follows that for $\lambda > 1$, $x^*(T^N \Gamma x) = \lim_{\lambda \to 1} (\lambda - 1)^n \sum_{m=0}^{\infty} (1/\lambda)^{m+1} x^*(T^{N+m}x) \geqq 0$. Since $T^N \Gamma = \Gamma$, $\Gamma$ is a positive operator. We choose $v > 0$ such that $\Gamma v = u \neq 0$. Then $u > 0$ and $Tu = T\Gamma v = \Gamma v = u$. We choose $v^* > 0$ such that $v^*(u) > 0$. Letting $u^* = \Gamma^* v^*$, we see that for $x \geqq 0$, $u^*(x) = (\Gamma^* v^*)(x) = v^*(\Gamma x) \geqq 0$ since $v^* > 0$ and $\Gamma$ is a positive operator. Hence $u^* \geqq 0$, and since $u^*(v) = (\Gamma^* v^*)(v) = v^*(\Gamma v) = v^*(u) > 0$, $u^* > 0$. Finally, we have $\Gamma T = \Gamma$ which implies $T^* \Gamma^* = \Gamma^*$, hence $T^* u^* = T^*(\Gamma^* v^*) = \Gamma^* v^* = u^*$ which completes the proof.

For strictly quasi-positive operators we obtain stronger results in the next two theorems.

THEOREM 4. *If $T$ is strictly quasi-positive and quasi-compact with spectral radius 1, then $1 \in \sigma(T)$, 1 has index one and $\tilde{T}$ has a representation of the form $\tilde{T} = \sum_{j=1}^{m} \lambda_j P_j + S$ where $\lambda_1 = 1$, $|\lambda_j| = 1$, $P_j^2 = P_j$, $SP_j = P_j S = 0$, $j = 1, 2, \cdots, m$, $P_i P_j = 0$ if $i \neq j$, and $r_s < 1$.*

*Proof.* By Theorem 2, $1 \in \sigma(T)$. By Theorem 3, there exists $u^* > 0$ such that $T^* u^* = u^*$ and for $x > 0$, $u^*(x) = u^*(T^n x) > 0$ if $n \geqq n(x, u^*)$, hence $u^*$ is strictly positive. Let the index of 1 be $n$. Then $\Gamma = \lim_{\lambda \to 1} (\lambda - 1)^n R(\lambda, T) \neq 0$. For $\lambda > 1$ and arbitrary $x$ we have

$$\begin{aligned} u^*(\Gamma x) &= \lim_{\lambda \to 1} (\lambda - 1)^n \sum_{k=0}^{\infty} (1/\lambda)^{k+1} u^*(T^k x) = \lim_{\lambda \to 1} u^*(x)(\lambda - 1)^n \sum_{k=0}^{\infty} (1/\lambda)^{k+1} \\ &= u^*(x) \lim_{\lambda \to 1} (\lambda - 1)^{n-1} = 0 \end{aligned}$$

[5] $T^*$ is the adjoint of $T$, defined on $B^*$ by $(T^* x^*)(x) = x^*(Tx)$.

unless $n = 1$. In proving Theorem 3 we showed that $\Gamma$ is a positive operator, hence there exists $x > 0$ such that $\Gamma x > 0$ and therfore $u^*(\Gamma x) > 0$. It follows that $n = 1$. By Theorem 2, every $\lambda_0 \in \sigma(T)$, $|\lambda_0| = 1$, has index 1 and hence $P(\lambda_0, T) = \lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)R(\lambda, T)$ exists and is a projection onto the finite dimensional space $\{x \mid (\lambda_0 I - \tilde{T})x = 0\}$. Let $\lambda_1 = 1, \lambda_2 \cdots, \lambda_m$ be an enumeration of the points in $\sigma(T)$ with absolute value 1 and let $P_j = P(\lambda_j, T)$. Since $\tilde{T}$ commutes with $R(\lambda, T)$ and $P_j = \lim_{\lambda \to \lambda_j}(\lambda - \lambda_j)R(\lambda, T)$, it follows that $\tilde{T}$ commutes with $P_j$. For $i \neq j$ we have $\lambda_i P_i P_j = \tilde{T} P_i P_j = P_i \tilde{T} P_j = \lambda_j P_i P_j$, hence $P_i P_j = 0$. Define the bounded linear operator $S$ by the equation $\tilde{T} = \sum_{j=1}^{m} \lambda_j P_j + S$. Since $\tilde{T} P_j = P_j \tilde{T} = \lambda_j P_j$, $P_j^2 = P_j$ and $P_i P_j = 0$ if $i \neq j$, it follows that $P_j S = S P_j = 0$. This implies $\tilde{T}^n = \sum_{j=1}^{m} \lambda_j^n P_j + S^n$. Suppose $r_S \geqq 1$. $T$ is quasi-compact, hence $\tilde{T}^n = U + V$ for some $n$ where $U$ is compact and $r_V < 1$. The operator $U'$ defined by $U'x = Ux - \sum_{j=1}^{m} \lambda_j^n P_j x$ is compact[6] and $S^n = U' + V$. Therefore $S$ is quasi-compact. Let $\lambda \in \sigma(S)$, $|\lambda| = r_S \geqq 1$. Then $Sx = \lambda x$ for some $x \in \tilde{B}$, $x \neq 0$. Since $P_j S = S P_j = 0$, it follows that $\tilde{T} x = \lambda x$ and therefore for some $j$, $\lambda = \lambda_j$ and $P_j x = x$. This implies $Sx = SP_j x = 0$, a contradiction. Therefore $r_S < 1$ and the proof is complete.

Before stating our next result, we state the following lemma which is easily proved.

LEMMA 1. *If $E$ is a finite dimensional real Banach space, $K$ is a cone in $E$ and $K$ is fundamental, then $K$ contains an open set.*

THEOREM 5. *If $T$ is strictly quasi-positive and quasi-compact with spectral radius 1, the eigenspace for $T$ corresponding to the eigenvalue 1 is one-dimensional.*

*Proof.* By Theorem 4 we have $\tilde{T} = \sum_{j=1}^{m} \lambda_j P_j + S$ where $P_j$ is a projection onto the eigenspace corresponding to $\lambda_j$, $\lambda_1 = 1$, $|\lambda_j| = 1$, $P_j S = S P_j = 0$, $j = 1, 2, \cdots, m$ and $P_i P_j = 0$ if $i \neq j$. By a theorem of Kronecker, there exists a sequence $n_1, n_2 \cdots$ of positive integers such that $\lim_{k \to \infty} \lambda_j^{n_k} = 1$, $j = 1, 2, \cdots, m$.[7] Since $r_S < 1$, it follows that $\lim_{n \to \infty} \| S^n \| = 0$. This implies $\lim_{k \to \infty} \tilde{T}^{n_k} = \sum_{j=1}^{m} P_j$. Let $P = \sum_{j=1}^{m} P_j$. For $x \in B$ we have $Px = \lim_{k \to \infty} T^{n_k} x$, hence $PB \subseteqq B$. For $x \geqq 0$ and $x^* \geqq 0$, $x^*(Px) = \lim_{k \to \infty} x^*(T^{n_k} x) \geqq 0$, hence $P$ is a positive operator. Consider the finite dimensional real Banach space $PB$ with closed proper cone $PK$. Since $K$ is fundamental in $B$, it is clear that $PK$ is fundamental in $PB$. Therefore, by Lemma 1, $PK$ contains an open set (open relative to $PB$). Since $T$ is strictly quasi-positive, every

[6] The compact operators from an ideal in the algebra of bounded linear operators and any bounded operator with a finite dimensional range is compact.

[7] See, for example, Hardy & Wright, *The Theory of Numbers*, Oxford Univ. Press.

non-trivial fixed vector of $T$ in $K$ is strictly positive. By Theorem 3, there exists $u > 0$ such that $Tu = u$. Let $Tx = x, x \neq 0$. We wish to show $u$ and $x$ are linearly dependent and for this purpose we may assume $x \notin K$ (otherwise replace $x$ by $-x$). It is clear that $u \in PK$ and $x \in PB$. Let $t_0 = \sup\{t \mid u + tx \in PK\}$. Since $u$ is in the interior of $PK$ and $x \notin PK$, it is easy to see that $0 < t_0 < \infty$ and that $u + t_0x$ is on the boundary of $PK$. Hence, by Theorem 1, there exists $x^* \in (PK)^+$ such that $x^*(u + t_0x) = 0$. We extend $x^*$ to $y^* \in B^*$ by defining $y^*(y) = x^*(Py)$. Since $PK \subseteqq K$, it follows that $y^* \in K^+$. We have $P(u+t_0x) = u + t_0x$, hence $y^*(u + t_0x) = x^*(u_0 + t_0x) = 0$. Now $u + t_0x$ is a fixed vector of $T$ which is not strictly positive, hence $u + t_0x = 0$, which completes the proof.

Our next result is a characterization of strongly quasi-positive operators.

THEOREM 6. *If $T$ is quasi-compact with spectral radius* 1, *then $T$ is strongly quasi-positive if and only if the following conditions are satisfied*:

(1) $1 \in \sigma(T)$ *and* 1 *is the only point in* $\sigma(T)$ *with absolute value one*,

(2) *the eigenspace for $T$ corresponding to the eigenvalue* 1 *is one-demensional and is spanned by a strictly positive element $u$*,

(3) *there exists a strictly positive element $u^*$ such that $T^*u^* = u^*$.*

*Proof.* In Theorems 3, 4, 5 we have seen that if $T$ is strictly quasi-positive (in particular, if it is strongly quasi-positive), then $1 \in \sigma(T)$ and (2) and (3) hold. There remains to show 1 is the only point in $\sigma(T)$ with absolute value one. We define the operator $P = \sum_{j=1}^m P_j$ as in Theorem 5 and recall that $PB$ is a finite dimensional real Banach space with closed proper cone $PK$ containing interior elements. Let $\lambda = e^{i\theta}$ be a point in $\sigma(T)$ and let $\tilde{T}(x + iy) = e^{i\theta}(x + iy)$ for some $x, y$ in $B$, not both zero. It is easy to see that $Px = x$ and $Py = y$, hence $x \in PB$ and $y \in PB$. At least one of the four elements $x + y$, $x - y$, $y - x$, $-x - y$ must be not in $PK$ since otherwise $x + y = 0$, $x - y = 0$, hence $x = y = 0$. Therefore $ax + by \notin PK$ for some choice of $a = \pm 1$ and $b = \pm 1$. Now choose $t > 0$ such that $u + t(ax + by) = v$ is on the boundary of $PK$. By Theorem 1, there exists $x^* \in (PK)^+$, $x^* \neq 0$, such that $x^*(v) = 0$. We extend $x^*$ to $y^* \in K^+ : y^*(y) = x^*(Py)$. Now choose a sequence of positive integers $n_1, n_2, \cdots$ such that $\lim_{k \to \infty} e^{in_k\theta} = 1$. It follows that $\lim_{k\to\infty} T^{n_k}v = v$. Since $r_T = 1$, we have $\| T^n \| \geqq 1$ for all $n$ and hence if $v > 0$,

$$\liminf_{n\to\infty} y^*(T^nv) \geqq \liminf_{n\to\infty} y^*(T^nv)/\| T^n \| > 0 \ .$$

This is impossible since $\lim_{k\to\infty} y^*(T^{n_k}v) = y^*(v) = 0$. Therefore $v = 0$, i.e., $ax + by = -(1/t)u$. Since $\tilde{T}(x + iy) = e^{i\theta}(x + iy)$, it follows that $u^*(x) + iu^*(y) = e^{i\theta}(u^*(x) + iu^*(y))$. This implies either $e^{i\theta} = 1$ or $u^*(x) = u^*(y) = 0$. The second alternative is incompatible with $ax + by = -(1/t)u$ since $u^*(u) > 0$. Therefore $e^{i\theta} = 1$ and the necessity of (1), (2), (3) is proved.

Now let $T$ satisfy conditions (1), (2), (3). We assume without loss of generality that $u^*$ is normalized so that $u^*(u) = 1$. Define the bounded linear operator $S$ by $Tx = u^*(x)u + Sx$. As in Theorem 4, it can be shown that $r_S < 1$. We have $Su = Tu - u^*(u)u = u - u = 0$ and it follows that $T^n x = u^*(x)u + S^n x$. Since $r_S < 1$, $\| S^n \| \leqq M$ for all $n$ and hence $\| T^n \| \leqq \| u^* \| \| u \| + \| S^n \| \leqq M'$ for all $n$. Moreover, $S^n x \to 0$ as $n \to \infty$ for all $x$. Hence if $x > 0$ and $x^* > 0$,

$$\begin{aligned}\liminf_{n\to\infty} x^*(T^n x)/\| T^n \| &\geqq \liminf_{n\to\infty} (u^*(x)x^*(u) + x^*(S^n x))/M' \\ &\geqq u^*(x)x^*(u)/M' > 0 .\end{aligned}$$

Therefore $T$ is strongly quasi-positive and the theorem is proved.

THEOREM 7. *Assume that $B$ is a lattice*[8] *with respect to the ordering given by $K$. Then Theorem 6 is true if "strongly quasi-positive" is replaced by "strictly quasi-positive."*

*Proof.* Conditions (1), (2) and (3) in Theorem 6 imply $T$ is strongly quasi-positive, hence, a fortiori, $T$ is strictly quasi-positive. Now suppose $T$ is strictly quasi-positive. Then $1 \in \sigma(T)$ and (2), (3) hold. It is easy to see from the representation of Theorem 4, $\tilde{T} = \sum_{j=1}^{m} \lambda_j P_j + S$, that $\| T^n \|$ is bounded independently of $n$. Hence, by a theorem of Krein-Rutman ([11], Theorem 8.1 and corollary), every $\lambda \in \sigma(T)$, $|\lambda| = 1$, is a root of unity. It is easily verified that every power of $T$ is quasi-compact and strictly quasi-positive, hence the eigenspace for $T^n$ corresponding to the eigenvalue 1 is one-dimensional for all $n$. If $\tilde{T}x = \lambda x$, $|\lambda| = 1$, $\lambda^n = 1$, then $\tilde{T}^n x = \lambda^n x = x$ and it follows that $\lambda = 1$ which completes the proof.

An immediate consequence is the following corollary.

COROLLARY. *If $B$ is a lattice, every strictly quasi-positive and quasi-compact operator is strongly quasi-positive.*

The conclusion of this corollary is not true in general as we will illustrate by an example. Let $B$ be three-dimensional (real) Euclidean

[8] I.e., each pair of elements in $B$ has a greatest lower bound and a least upper bound.

space, $B = \{(x_1, x_2, x_3)\}$, and let $K = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leqq x_3^2, x_3 \geqq 0\}$. If we interpret "to the right" to mean any direction in which the $x_3$ coordinate is increasing, each non-trivial element $x^* \in K^+$ is represented by a plane through the origin whose unit normal at the origin directed to the right lies in $K$. Let $T$ be a rotation about the $x_3$ axis through $\theta$ radians where $\theta$ and $2\pi$ are incommensurable. It is clear that $\| T^n \| = 1$ for all $n$ and that $TK \subseteqq K$. To show that $T$ is strictly quasi-positive it suffices to consider $x^* \in K^+$ which is represented by a plane tangent to $K$. If $p$ is in the interior of $K$, $T^n p$ is in the interior for all $n$, hence $x^*(T^n p) > 0$. Now let $p$ be on the boundary of $K$. There exists exactly one point $q$ which has the same $x_3$ coordinate as $p$ and such that $x^*(q) = 0$. Since $\theta$ and $2\pi$ are incommensurable, there is at most one value of $n$ such that $T^n p = q$. Therefore, $x^*(T^m p) > 0$ for all $m$ sufficiently large and, hence, $T$ is strictly quasi-positive. If $p$ is on the boundary of $K$, so is $T^n p$ for all $n$. We can pick a sequence $n_1, n_2, \cdots$ such that $T^{n_k} p$ converges to a point $q$ on the boundary of $K$ and there exists $x^* \in K^+$ such that $x^*(q) = 0$, $x^* \neq 0$. This shows $T$ is not strongly quasi-positive.

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