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DECOMPOSITION THEOREMS FOR FREDHOLM OPERATORS

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This paper is devoted to proving and discussing several consequences of the following decomposition theorem:

Let A and B be closed densely-defined linear operators from the Banach space X to the Banach space Y such that $D(B) \supseteqq D(A)$, $D(B^*) \supseteqq D(A^*)$, the range $R(A)$ of A is closed, and the dimension of the null-space $N(A)$ of A is finite. Then X and Y can be decomposed into direct sums $X = X_0 \oplus X_1$, $Y = Y_0 \oplus Y_1$, where X_1 and Y_1 are finite dimensional, $X_1 \subseteqq D(A)$, $X_0 \cap D(A)$ is dense in X, and (X_0, Y_0) and (X_1, Y_1) are invariant pairs of subspaces for both A and B. Let A_i and B_i be the restrictions of A and B respectively to X_i. For all integers k, $(B_0A_0^{-1})^k(0) \subseteqq R(A_0)$, and

$$\dim (B_0A_0^{-1})^k(0) = k \dim (B_0A_0^{-1})(0) = k \dim N(A_0) .$$

Also, the action of A_1 and B_1 from X_1 to Y_1 can be given a certain canonical description.

The object of this paper is to study the operator equation $Ax - \lambda Bx = y$, where A and B are (unbounded) linear operators from a Banach space X to a Banach space Y. In §1, an integer $\mu(A{:}B)$ is defined, which expresses a certain interrelationship between the null space of A and the null space of B. In §1 and 2, decomposition theorems are proved which refine theorem 4 of [2]. The theorems allow us to split off certain finite dimensional invariant pairs of subspaces of X and Y so that A and B are well-behaved with respect to $\mu(A{:}B)$ on the remainder.

In §4, the stability of these decompositions under perturbation of A by λB is investigated. In §5, relations between the dimensions of certain subspaces of X and Y are given, and a formula for the Fredholm index of $A - \lambda B$ is obtained. These extend results of Kaniel and Schechter [1], who consider the case $X = Y$ and B the indentity operator.

It should be noted that the results of Kaniel and Schechter referred to here follow from theorems 3 and 4 of [2]. The results of this paper properly refine Kato's results only when the null space of B is not $\{0\}$.

1. We will be considering linear operators T defined on a dense linear subset $D(A)$ of a Banach space X, and with values in a Banach space Y. $N(T)$ and $R(T)$ will denote the null space and range of T respectively, while $\alpha(T)$ is the dimension of $N(T)$, and $\beta(T)$ is the

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codimension of $\overline{R(T)}$ in Y. T is a Fredholm operator if T is closed, $R(T)$ is closed, and both $\alpha(T)$ and $\beta(T)$ are finite. The index of a Fredholm operator is the integer.

$$\kappa(T) = \alpha(T) - \beta(T) .$$

Let P be a subspace of X, Q a subspace of Y. (P, Q) is an *invariant pair of subspaces* for T if $T(P \cap D(T)) \subseteqq Q$.

Standing assumptions: In the remainder of the paper, A and B are closed linear operators from X to Y, $D(A)$ is dense in X, $D(B) \supseteqq D(A)$, and $D(B^*) \supseteqq D(A^*)$; A is semi-Fredholm, in the sense that $R(A)$ is closed and $\alpha(A) < \infty$.

The assumption $D(B^*) \supseteqq D(A^*)$ seems necessary for the proof of the decomposition theorems. It is often met when A and B are differential operators on some domain in Euclidean space, and the order of B is less than the order of A. It is always met when B is bounded.

The linear manifolds $N_k = N_k(A{:}B)$ and $M_k = M_k(A{:}B)$ are defined by induction as follows:

$$\begin{aligned} N_1 &= N(A) \\ N_k &= A^{-1}(BN_{k-1}) , \qquad k > 1 \\ M_k &= BN_k . \end{aligned}$$

N_k and M_k are increasing sequences of linear manifolds in X and Y respectively.

The smallest integer n such that N_n is not a subset of $B^{-1}R(A)$ will be denoted by $\nu(A:B)$. If N_n is a subset of $B^{-1}R(A)$ for all n, then we define $\nu(A{:}B) = \infty$. (cf. [2])

The dimension of N_k will be denoted by $\pi_k = \pi_k(A{:}B)$, and the dimension of M_k by $\rho_k = \rho_k(A{:}B)$. Then $\pi_1 = \alpha(A)$, and, in general, $\pi_k \leqq k\alpha(A)$. $\mu(A{:}B)$ will denote the first integer n such that $\pi_n < n\alpha(A)$. It $\pi_n = n\alpha(A)$ for all intergers n, then we define $\mu(A:B) = \infty$.

In general, $\mu(A:B) \geqq \nu(A:B) + 1$. This inequality is trivial if $\nu = \infty$. If $\nu < \infty$, then $M_{\nu-1} \subseteqq R(A)$, while $M_\nu \not\subseteqq R(A)$. Consequently, $\pi_{\nu+1} < \pi_\nu + \alpha(A) \leqq (\nu + 1)\alpha(A)$, and so $\mu(A:B) \leqq \nu + 1$.

We define $\sigma_k(A{:}B) = \pi_k - \pi_{k-1}$. Then σ_k is the dimension of the quotient space N_k/N_{k-1}. $\{\sigma_k\}$ is a decreasing sequence of nonnegative integers, and so the limit

$$\sigma(A:B) = \lim_{k\to\infty} \sigma_k(A:B) \qquad \text{exists.}$$

If $\mu(A:B) = \infty$, then $\sigma(A:B) = \alpha(A)$.

2. Theorem 1. *Assume, in addition to the standing assumptions on A and B, that $\nu(A:B) = \infty$. Then X and Y can be decomposed*

into direct sums

$$X = X_0 \oplus X_1$$
$$Y = Y_0 \oplus Y_1 ,$$

where X_1 *and* Y_1 *are finite dimensional,* $X_1 \subseteqq D(A)$, $X_0 \cap D(A)$ *is dense in* X_0, *and* (X_0, Y_0) *and* (X_1, Y_1) *are invariant pairs for both* A *and* B. *If* A_i *and* B_i *are the restrictions of* A *and* B *respectively to* X_i, *then* $\mu(A_0, B_0) = \infty$, *while* A_1 *and* B_1 *map* X_1 *onto* Y_1.

Furthermore, X_1 and Y_1 can be decomposed as direct sums

$$X_1 = P_1 \oplus \cdots \oplus P_p$$
$$Y_1 = Q_1 \oplus \cdots \oplus Q_p ,$$

where A_1 and B_1 map P_j onto Q_j. Bases $\{x_j^i: 1 < i \leqq \eta(j)\}$ and $\{y_j^i: 1 \leqq i \leqq \eta(j) - 1\}$ can be chosen for P_j and Q_j respectively so that

$$Ax_j^{i+1} = Bx_j^i = y_j^i , \qquad 1 \leqq i \leqq \eta(j) - 1$$
$$Ax_j^1 = 0 = Bx_j^{\eta(j)} .$$

Although the decomposition is not, in general, unique, the integers p and $\eta(j)$, $1 \leqq j \leqq m$, are uniquely determined by A and B. In fact,

$$p = \alpha(A) - \sigma(A : B) .$$

Proof. Let $n = \alpha(A)$, and suppose that $\{z_1^1, \cdots z_n^1\}$ is a basis for $N(A)$. Since $\nu(A : B) = \infty$, z_j^i can be chosen by induction so that $Az_j^i = Bz_j^{i-1}$. $\{z_j^i: 1 \leqq j \leqq n,\ 1 \leqq i \leqq m\}$ is a spanning set for N_m, while $\{Bz_j^i: 1 \leqq j \leqq n,\ 1 \leqq i \leqq m\}$ is a spanning set for M_m. Also, $\{z_i^m: 1 \leqq i \leqq n\}$ span N_m modulo N_{m-1}.

Recall that $\sigma_m = \sigma(m) = \dim (N_m/N_{m-1})$. By induction, the order of the z_j^i can be chosen so that $\{z_{n-\sigma(m)+1}^m, \cdots , z_n^m\}$ span N_m modulo N_{m-1}. Then

$$G_m = \{z_j^i: n - \sigma(i) + 1 \leqq j \leqq n,\ 1 \leqq i \leqq m\}$$

is a basis for N_m.

Let $\eta(j)$ be the greatest integer k such that $z_j^k \in G_k$. If $z_j^k \in G_k$ for all k, let $\eta(j) = \infty$. Then $1 \leqq \eta(1) \leqq \eta(2) \leqq \cdots \leqq \eta(n)$. Let p be the greatest integer k such that $\eta(k) < \infty$. By definition of σ, it is clear that

$$p = \alpha(A) - \sigma .$$

Suppose $1 \leqq j \leqq p$. $z_j^{\eta(j)+1}$ is linearly dependent on the set $G_{\eta(j)+1}$, and so we can write

$$z_j^{\eta(j)+1} = \sum \alpha_{ik} z_k^i ,$$

where the sum is taken over all pairs of integers (i, k), with the understanding that $z_k^i = 0$ if $i \leqq 0$ and $\alpha_{ik} = 0$ if $z_k^i \notin G_{\eta(j)+1}$. For $-1 \leqq q \leqq \eta(j)$ define

$$x_j^{\eta(j)-q} = z_j^{\eta(j)-q} - \sum \alpha_{ik} z_k^{i-q-1}.$$

For $0 \leqq q \leqq \eta(j)$,

$$\begin{aligned} Bx_j^{\eta(j)-q} &= Bz_j^{\eta(j)-q} - \sum \alpha_{ik} Bz_k^{i-q-1} \\ &= Az_j^{\eta(j)-q+1} - \sum \alpha_{ik} Az_k^{i-q} \\ &= Ax_j^{\eta(j)-q+1} \end{aligned}$$

In particular, $Bx_j^{\eta(j)} = 0$.

Since the sum for $x_j^{\eta(j)-q}$ involves $z_j^{\eta(j)-q}$ only in the first term, the $z^{\eta}_{j(j)-q}$ may be replaced by the $x_j^{\eta(j)-q}$, $0 \leqq q \leqq \eta(j)$, to obtain another basis for $N_{\eta(j)+1}$. Repeating this process for $1 \leqq j \leqq p$, and making other appropriate replacements, we arrive at vectors x_j^i such that.

(1) $$x_1^1, \cdots, x_1^n \text{ are a basis for } N(A)$$

(2) $$Bx_j^i = Ax_j^{i+1} , \qquad 1 \leqq i \leqq \eta(j)$$

(3) $$Bx_j^{\eta(j)} = 0 , \qquad 1 \leqq j \leqq p .$$

For convenience, it is assumed that

(4) $$x_j^i = 0 \qquad \text{if } i > \eta(j) .$$

If $1 \leqq j \leqq p$, let P_j be the subspace of X with basis $\{x_j^1, \cdots, x_j^{\eta(j)}\}$. Let Q_j be the subspace of Y with basis $\{y_j^1, \cdots, y_j^{\eta(j)-1}\}$, where $y_j^i = Bx_j^i = Ax_j^{i+1}$. Let $X_1 = P_1 \oplus \cdots \oplus P_p$ and $Y_1 = Q_1 \oplus \cdots \oplus Q_p$. Then X_1 and Y_1 satisfy all the conclusions of the theorem. To conclude the proof, it suffices to produce complementary subspaces to X_1 and Y_1 which also form an invariant pair.

We will construct functionals

$$\begin{aligned} &\{g_j^i : 1 \leqq i \leqq \eta(j) , \qquad 1 \leqq j \leqq p\} \text{ on } X \text{ and} \\ &\{f_j^i : 1 \leqq i \leqq \eta(j) - 1 , \ 1 \leqq j \leqq p\} \text{ on } Y \text{ such that} \end{aligned}$$

the f_j^i are in the domain of A^* and

(5) $$g_j^{i+1} = A^* f_j^i , \qquad 1 \leqq i \leqq \eta(j) - 1$$

(6) $$g_j^i = B^* f_j^i , \qquad 1 \leqq i \leqq \eta(j) - 1$$

(7) $$f_j^i(y_k^q) = \delta_{iq}\delta_{jk} , \qquad \begin{aligned} &1 \leqq j,\ k \leqq n \\ &1 \leqq q \leqq i \end{aligned}$$

$$(8) \qquad g_j^i(x_k^q) = \delta_{iq}\delta_{jk}\,, \qquad \begin{array}{l} 1 \leqq j,\ k \leqq n \\ 1 \leqq q \leqq i\,. \end{array}$$

Let $g_j^{\eta(j)}$ be any functional on X which satisfies (8). The other g_j^i will be chosen by induction.

Suppose that f_k^q and g_k^q are chosen, for $q > i \geqq 1$, to satisfy (5) through (8). By (8), g_k^{i+1} is orthogonal to $N(A)$, and so g_k^{i+1} is in the closure of $R(A^*)$. Since $R(A)$ is closed, $R(A^*)$ is closed, and there is an $f_k^i \in D(A^*)$ for which $A^*f_k^i = g_k^{i+1}$. Let $g_k^i = B^*f_k^i$. Then (5) and (6) hold by definition.

To verify (7), we have for $q \leqq i$,

$$\begin{aligned} f_j^i(y_k^q) &= f_j^i(Ax_k^{q+1}) \\ &= (A^*f_j^i)(x_k^{q+1}) \\ &= g_j^{i+1}(x_k^{q+1}) = \delta_{iq}\delta_{jk}\,. \end{aligned}$$

(8) is an immediate consequence of (7).

$$\begin{aligned} \text{Let } X_0 &= \cap\,\{N(g_j^i) \colon 1 \leqq i \leqq \eta(j)\,, \qquad 1 \leqq j \leqq p\} \\ Y_0 &= \cap\,\{N(f_j^i) \colon 1 \leqq i \leqq \eta(j) - 1\,, \qquad 1 \leqq j \leqq p\}\,. \end{aligned}$$

From (7) and (8), it is clear that $X_0 \cap X_1 = \{0\}$ and $Y_0 \cap Y_1 = \{0\}$. Since the codimension of X_0 in X is no greater than the number of functionals g_j^i defining it, and since this number is the dimension of X_1, we must have $X = X_0 \oplus X_1$. Similarly, $Y = Y_0 \oplus Y_1$.

Suppose $x \in D(A) \cap X_0$. Then $f_j^i(Ax) = (A^*f_j^i)(x) = g_j^{i+1}(x) = 0$, and so $Ax \in Y_0$. Similarly, $Bx \in Y_0$, and $(X_0,\ Y_0)$ is an invariant pair for both A and B.

Since $(X_0,\ Y_0)$ and $(X_1,\ Y_1)$ are invariant pairs, $N_k(A:B) \cap X_0 = N_k(A_0:B_0)$. For k sufficiently large, $X_1 \subseteqq N_k(A:B)$, and so

$$\begin{aligned} \dim\{N_{k+1}(A_0:B_0)/N_k(A_0:B_0)\} &= \dim\{N_{k+1}(A:B)/N_k(A:B)\} \\ &= \sigma \\ &= \alpha(A) - p \\ &= \alpha(A_0)\,. \end{aligned}$$

This can occur only if $\dim N_k(A_0:B_0) = k\alpha(A_0)$ for all integers k. Hence $\mu(A_0:B_0) = \infty$.

3. Let $(P,\ Q)$ be an invariant pair of finite dimensional subspaces for A and B. $(P,\ Q)$ is an *irreducible invariant pair of type* ν if there are bases $\{x_i\}_{i=1}^n$ for P and $\{y_i\}_{i=1}^n$ for Q such that $Bx_i = y_i$, $Ax_1 = 0$, and $Ax_i = y_{i-1}$, $2 \leqq i \leqq n$.

$(P,\ Q)$ is an *irreducible invariant pair of type* μ if there are bases $\{x_i\}_{i=1}^n$ for P and $\{y_i\}_{i=1}^{n-1}$ for Q such that

$$\begin{aligned} Ax_1 &= 0 = Bx_n \\ Ax_{i+1} &= y_i = Bx_i\ , \qquad 1 \leqq i \leqq n-1\ . \end{aligned}$$

(P, Q) is an *irreducible invariant pair of type* μ^* if there are bases $\{x_i\}_{i=1}^{n-1}$ for P and $\{y_i\}_{i=1}^{n}$ for Q such that

$$\begin{aligned} Bx_i &= y_i\ , \qquad 1 \leqq i \leqq n-1 \\ Ax_i &= y_{i+1}\ , \qquad 1 \leqq i \leqq n-1\ . \end{aligned}$$

(P, Q) is an *invariant pair of type* ν if $P = P_1 \oplus \cdots \oplus P_k$ and $Q = Q_1 \oplus \cdots \oplus Q_k$, where (P_j, Q_j) is an irreducible invariant pair of type ν, $1 \leqq j \leqq k$. *Invariant pairs of type* μ or *type* μ^* are defined similarly.

It is straightforward to verify that if (P, Q) is an (irreducible) invariant pair of type $\mu(A:B)$ (resp. $\mu^*(A:B)$), then (P, Q) is an (irreducible) invariant pair of type $\mu(A-\lambda B:B)$ (resp. $\mu^*(A-\lambda B:B)$), for all complex numbers λ. If (P, Q) is an invariant pair of type μ, then $\nu(A \mid P, B \mid P) = \infty$ and $\mu((A \mid P)^*, (B \mid P)^*) = \infty$. If (P, Q) is of type μ^*, then $\nu(A \mid P, B \mid P) = \infty$ and $\mu(A \mid P, B \mid P) = \infty$.

THEOREM 2. *Suppose A and B satisfy the standing hypothesis. Then there exist decompositions*

$$\begin{aligned} X &= X_0 \oplus X_1 \oplus X_2 \\ Y &= Y_0 \oplus Y_1 \oplus Y_2 \end{aligned}$$

Where (X_0, Y_0) is an invariant pair, (X_1, Y_1) is an invariant pair of type μ, and (X_2, Y_2) is an invariant pair of type ν. If A_0 and B_0 are the restrictions of A and B respectively to X_0, then $\nu(A_0, B_0) = \infty$ and $\mu(A_0, B_0) = \infty$.

Proof. Theorem 2 follows from Theorem 1 and Kato's Theorem 4 [1], after it is noted that the latter theorem, although stated only for bounded operators B, is valid under the less restrictive assumption that $D(B^*) \supseteqq D(A^*)$.

THEOREM 3. *In addition to the standing hypotheses, suppose that A is a Fredholm operator. Then there exist decompositions*

$$\begin{aligned} X &= X_0 \oplus X_1 \oplus X_2 \oplus X_3 \\ Y &= Y_0 \oplus Y_1 \oplus Y_2 \oplus Y_3\ , \end{aligned}$$

where each (X_i, Y_i) is an invariant pair, (X_1, Y_1) is of type μ, (X_2, Y_2) is of type ν, and (X_3, Y_3) is of type μ^*. If A_0 and B_0 are the restrictions of A and B to X_0, then $\nu(A_0 : B_0) = \infty$, $\mu(A_0 : B_0) = \infty$, $\mu(A_0^* : B_0^*) = \infty$, and *$\nu(A_0^* : B_0^*) = \infty$.*

If $X^* = X_0^* \oplus X_1^* \oplus X_2^* \oplus X_3^*$ and $Y^* = Y_0^* \oplus Y_1^* \oplus Y_2^* \oplus Y_3^*$ are the corresponding decompositions of the adjoint spaces, then (Y_1^*, X_1^*) is an invariant pair of type $\mu_*(A^* : B^*)$, (Y_2^*, X_2^*) is an invariant pair of type $\nu(A^* : B^*)$, and (Y_3^*, X_3^*) is an invariant pair of type $\mu(A^* : B^*)$.

Proof. In view of Theorem 2, we may assume that $\mu(A : B) = \infty$ and $\nu(A : B) = \infty$. Then $\nu(A^* : B^*) = \infty$, and we can proceed to decompose X^* and Y^*, as in the proof of Theorem 1. The only difficulty encountered is to produce vectors x_j^i to span X_3 which actually lie in $D(A)$. An induction argument similar to that used in Theorem 1 to produce the f_j^i and g_j^i can also be employed in this case.

4. Let $\Phi^+(A : B)$ be the set of complex numbers λ such that $A - \lambda B$ is a closed operator from $D(A)$ to Y, and such that $R(A - \lambda B)$ is closed and $\alpha(A - \lambda B) < \infty$. $\Phi^+(A : B)$ is an open subset of the complex plane which, by assumption, contains the point $\lambda = 0$.

For all $\lambda \in \Phi^+(A : B)$, Theorems 1 and 2 are applicable to the operators $A - \lambda B$ and B. Also, for $\lambda \in \Phi^+(A : B)$ we define

$$\begin{aligned} \sigma_k(\lambda) &= \sigma_k(A - \lambda B : B) \\ \pi_k(\lambda) &= \pi_k(A - \lambda B : B) \\ \rho_k(\lambda) &= \rho_k(A - \lambda B : B) \\ \sigma(\lambda) &= \sigma(A - \lambda B : B) . \end{aligned}$$

THEOREM 4. *Let A and B satisfy the standing hypotheses. There exists a decomposition*

$$\begin{aligned} X &= X_0 \oplus X_1 \\ Y &= Y_0 \oplus Y_1 \end{aligned}$$

such that (X_0, Y_0) is an invariant pair, and (X_1, Y_1) is an invariant pair of type $\mu(A - \lambda B : B)$ for all complex numbers λ. If A_0 and B_0 are the restrictions of A and B to X_0, then $\mu(A_0 - \lambda B_0 : B_0) = \infty$ for all $\lambda \in \Phi^+(A : B)$ satisfying $\nu(A - \lambda B : B) = \infty$.

Proof. The points $\lambda \in \Phi^+(A : B)$ for which $\nu(A - \lambda B : B) < \infty$ form a discrete subset of $\Phi^+(A : B)$, and so there is a $\lambda' \in \Phi^+$ such that $\nu(A - \lambda' B : B) = \infty$. Let $X = X_0 \oplus X_1$ be the decomposition of Theorem 1 with respect to $A - \lambda' B$ and B. Then (X_1, Y_1) is an invariant pair of type $\mu(A - \lambda B : B)$ for all complex numbers λ, as remarked earlier.

If $\lambda \in \Phi^+(A : B)$ and $\nu(A - \lambda B : B) = \infty$, then X_0 and Y_0 cannot be decomposed further as in Theorem 1, for such a decomposition would violate the fact that $\mu(A_0 - \lambda' B_0 : B) = \infty$. Hence $\nu(A - \lambda B : B) =$

∞ implies $\mu(A_0 - \lambda B_0 : B_0) = \infty$.

Let D be the subset of $\Phi^+(A : B)$ of complex numbers λ for which $\nu(A - \lambda B : B) < \infty$. D is a discrete subset of $\Phi^+(A : B)$ with no limit points in $\Phi^+(A : B)$(cf [1]).

THEOREM 5. *$\mu(A - \lambda B : B)$ is a constant, either finite or infinite, for $\lambda \in \Phi^+(A : B) - D$.*

Proof. In view of Theorem 4, it suffices to prove the theorem when A and B are operators in an invariant pair of type μ. For this, it suffices to look at an irreducible invariant pair of type μ. This case is easy to verify.

THEOREM 6. *$\sigma(\lambda)$ is constant on each component of $\Phi^+(A : B)$.*

Proof. It suffices to show that $\sigma(\lambda)$ is constant in a neighborhood of an arbitrary point $\lambda' \in \Phi^+(A : B)$. Let $X = X_0 \oplus X_1 \oplus X_2$ and $Y = Y_0 \oplus Y_1 \oplus Y_2$ be the decomposition of Theorem 2 with respect to $A - \lambda' B$ and B. Then $\nu(A_0 - \lambda B_0 : B_0) = \infty$ for λ near λ', and so $\sigma(\lambda) = \alpha(A_0 - \lambda B_0)$ for λ near λ'. By Theorem 3, [2], $\alpha(A_0 - \lambda B_0) = \alpha(A_0 - \lambda' B_0)$ for λ near λ'.

5. Let $X = X_0 \oplus X_1 \oplus X_2$ and $Y = Y_0 \oplus Y_1 \oplus Y_2$ be the decompositions of Theorem 2 with respect to A and B. Let $\pi_k = \pi_k^0 + \pi_k^1 + \pi_k^2$ and $\rho_k = \rho_k^0 + \rho_k^1 + \rho_k^2$ be the corresponding decompositions of π_k and ρ_k. Assume that r is chosen small that $0 < |\lambda| < r$ implies $\lambda \in \Phi^+(A : B)$ and $\nu(A - \lambda B : B) = \infty$. Then $\pi_k^0(\lambda) = k\sigma(\lambda)$ for $|\lambda| < r$. If k is sufficiently large,

$$\pi_k^1(\lambda) = \dim X_1 , \qquad |\lambda| < r$$

$$\pi_k^2(\lambda) = \begin{cases} \dim X_2, & \lambda = 0 \\ 0 \quad , & 0 < |\lambda| < r . \end{cases}$$

Also, $\rho_k^0(\lambda) = k\sigma(\lambda)$ for $|\lambda| < r$. For k sufficiently large,

$$\rho_k^1(\lambda) = \dim Y_1$$

$$\rho_k^2(\lambda) = \begin{cases} \dim Y_2 , & \lambda = 0 \\ 0 \quad , & 0 < |\lambda| < r . \end{cases}$$

We define, for any $\lambda \in \Phi^+(A : B)$,

$$\pi(\lambda) = \lim_{k \to \infty} [\pi_k(\lambda) - k\sigma(\lambda)] \tag{1}$$

$$\rho(\lambda) = \lim_{k = \infty} [\rho_k(\lambda) - k\sigma(\lambda)] \tag{2}$$

$\pi(\lambda)$ and $\rho(\lambda)$ correspond to $\tau(\lambda)$ defined in [1]. From the preceding, we deduce that

$$\pi(\lambda) = \begin{cases} \dim X_1 , & 0 < |\lambda| < r \\ \dim (X_1 \oplus X_2) , & \lambda = 0 \end{cases} \tag{3}$$

$$\rho(\lambda) = \begin{cases} \dim Y_1 , & 0 < |\lambda| < r \\ \dim (Y_1 \oplus Y_2) , & \lambda = 0 . \end{cases} \tag{4}$$

From these formulae, it follows that

$$\alpha(A - \lambda B) = \sigma(\lambda) + \pi(\lambda) - \rho(\lambda) , \qquad 0 < |\lambda| < r , \tag{5}$$

for both sides of this expression are equal to

$$\alpha(A_0 - \lambda B_0) + \dim X_1 - \dim Y_1 .$$

We will assume in the remainder of the discussion that A is a Fredholm operator. The set of complex numbers λ such that $A - \lambda B$ is a Fredholm operator will be denoted by $\Phi(A : B)$. $\Phi(A : B)$ is an open subset of the complex plane, and consists of the union of those components of $\Phi^+(A : B)$ for which $R(A - \lambda B)$ is of finite codimension in Y, i.e., for which $\alpha(A^* - \lambda B^*) < \infty$.

The quantities $\pi_k^*(\lambda) = \pi_k(A^* - \lambda B^* : B^*)$, $\rho_k^*(\lambda)$, $\sigma^*(\lambda)$, $\pi^*(\lambda)$ and $\rho^*(\lambda)$ are then well-defined for $\lambda \in \Phi(A : B)$. The formula for the adjoint operators corresponding to (5) is

$$\alpha(A^* - \lambda B^*) = \sigma^*(\lambda) + \pi^*(\lambda) - \rho^*(\lambda) , \qquad 0 < |\lambda| < r . \tag{6}$$

Since $\alpha(A^* - \lambda B^*) = \beta(A - \lambda B)$, we have

$$\begin{aligned} \kappa(A - \lambda B) = {} & (\sigma(\lambda) - \sigma^*(\lambda)) \\ & + (\pi(\lambda) - \pi^*(\lambda)) - (\rho(\lambda) - \rho^*(\lambda)) \qquad 0 < |\lambda| < r . \end{aligned} \tag{7}$$

In view of the decomposition of Theorem 3, the jump discontinuity of π^* at $\lambda = 0$ is equal to that of π at $\lambda = 0$, i.e., they are both equal to $\dim X_2 = \dim Y_2$. Hence (7) holds also for $\lambda = 0$, and we arrive at the following theorem.

THEOREM 7. *For all* $\lambda \in \Phi(A : B)$,

$$\kappa(A - \lambda B) = (\sigma(\lambda) - \sigma^*(\lambda)) + (\pi(\lambda) - \pi^*(\lambda)) - (\rho(\lambda) - \rho^*(\lambda)).$$

Analogous formulae can be written down if it is assumed, further, that B is a Fredholm operator. If $M(B) = \{0\}$ and $R(B)$ is dense in Y_1 then $\rho(\lambda) = \rho^*(\lambda) = \pi(\lambda) = \pi^*(\lambda) = 0$, and Theorem 7 reduces to

$$\kappa(A - \lambda B) = \sigma(\lambda) - \sigma^*(\lambda), \qquad \lambda \in \Phi(A : B) . \tag{8}$$

This latter formula is due to Kaniel and Schechter [1], when $X = Y$ and B is the identity operator.

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1. Kaniel and Schechter, *Spectral theory for Fredholm operators*, Comm. on Pure and Applied Math., vol. 16, no. 4 (1963), 423-448.

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