PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 1, 1965

APPROXIMATION BY CONVOLUTIONS

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This paper is concerned mainly with approximating functions on closed subsets P of a locally compact Abelian group G by absolute-convex combinations of convolutions $f * g$, with f and g extracted from bounded subsets of conjugate Lebesgue spaces $L^p(G)$ and $L^{p'}(G)$. It is shown that the Helson subsets of G can be characterised in terms of this approximation problem, and that the solubility of this problem for P is closely related to questions concerning certain multipliers of $L^p(G)$. The final theorem shows in particular that the P. J. Cohen factorisation theorem for $L^1(G)$ fails badly for $L^p(G)$ whenever G is infinite compact Abelian and $p > 1$.

1. The Approximation Problem.

(1.1) Throughout this note, G denotes a locally compact Abelian group and X its character group. For the most part we shall be concerned with the possibility of approximating functions on closed subsets P of G by absolute-convex combinations

$$\sum_{r=1}^{n} \alpha_r(f_r * g_r) , \tag{1}$$

of convolutions $f * g$, where f and g are selected freely from bounded subsets of conjugate Lebesgue spaces $L^p(G)$ and $L^{p'}(G)$ $(1/p + 1/p' = 1)$. In the sums (1), the number n of terms is variable, whilst the complex coefficients α_r are subject to the condition

$$\sum_{r=1}^{n} | \alpha_r | \leqq 1 . \tag{2}$$

Accordingly, if the f_r and g_r are respectively free to range over subsets A and B of $L^p(G)$ and $L^{p'}(G)$, the allowed sums (1) compose precisely the convex, balanced envelope of

$$A * B = \{ f * g : f \in A, g \in B \} .$$

We denote by $C_0(G)$ the Banach space of continuous, complex-valued functions on G which tend to zero at infinity, the norm being $\| u \| = \sup \{ | u(x) | : x \in G \}$. The space $C_0(P)$ is defined similarly, P replacing G throughout. If G (or P) is compact, the restriction that the functions tend to zero at infinity becomes void; we then write $C(G)$ (or $C(P)$) in place of $C_0(G)$ (or $C_0(P)$).

It is well-known that if $1 < p < \infty$ then $f * g \in C_0(G)$ whenever

Received February 5, 1964.

$f \in L^p(G)$ and $g \in L^{p'}(G)$, so that restriction from G to P results in a member of $C_0(P)$.

(1.2) Given an exponent p satisfying $1 < p < \infty$ and a closed subset P of G, we shall consider the following assertion:—

(A_P^p) To each member u of a second category subset of $C_0(P)$ corresponds a number $K = K(P,p,u) < \infty$ such that u is the uniform limit on P of absolute-convex combinations (1), the f_r and g_r being subject to the restrictions

$$\| f_r \|_p \leqq \sqrt{K}, \ \| g_r \|_{p'} \leqq \sqrt{K} .$$

It is evident that (A_P^p) and ($A_P^{p'}$) are equivalent assertions. Furthermore, only a little reflection is required to see that (A_P^1) is true for every P, so that the restriction $1 < p < \infty$ is reasonable. With this restriction on p, (A_P^p) signifies that each u belonging to the said second category set belongs to the closed, convex, balanced envelope in $C_0(P)$ of $A * B$, where A and B are respectively the closed balls in $L^p(G)$ and $L^{p'}(G)$ of radius $\sqrt{K}$ (which a priori may depend upon u).

(1.3) As well shall see, the truth or falsity of (A_P^p) is equivalent to an assertion about bounded measures supported by P which may conveniently be expressed by regarding such a measure as a multiplier (or centraliser) of ($L^p(G)$.

We denote by $M(G)$ the space of bounded, complex (Radon) measures on G; it may be regarded as the dual of $C_0(G)$. Furthermore, $M(P)$ may be thought of as the subset of $M(G)$ composed of measures $\mu \in M(G)$ whose supports are contained in P.

Each $\mu \in M(G)$ generates a multiplier T_μ of $L^p(G)$ defined by $T_\mu f = \mu * f$ for $f \in L^p(G)$. In general, by a multiplier of $L^p(G)$ is meant a continuous endomorphism of $L^p(G)$ which commutes with translations. Each multiplier T of $L^p(G)$ has a norm

$$\| T \| = \sup \{ \| Tf \|_p : \| f \|_p \leqq 1 \} .$$

Accordingly we may define $N_p(\mu)$ for $\mu \in M(G)$ as the norm of T_μ regarded as a multiplier of $L^p(G)$.

It is easily seen that

$$N_p(\mu) \leqq \| \mu \| , \tag{3}$$

equality holding if $p = 1$ (and hence also if $p = \infty$).

Although, as will be seen in (2.3), the norms $N_p(\mu)$ and $\| \mu \|$ are not generally equivalent on $M(G)$ when $1 < p < \infty$, yet equivalence may obtain on $M(P)$ for suitable closed subsets P of G. In fact, as the next theorem shows, the suitable sets P are just those for which the assertion (A_P^p) is true. When $p = 2$ one obtains in this way a new characterisation of the so-called Helson subsets of G; see (1.6) *infra.* A further link between (A_P^p) and properties of certain sets of multipliers of $L^p(G)$ is expressed in Theorem (2.1).

(1.4) THEOREM. *Let P be a closed subset of G, and let $1 < p < \infty$. Then (A_P^p) is true if and only if there exists a number $k = (P, p) < \infty$ such that*

$$\| \mu \| \leqq k . N_p(\mu) \,, \tag{4}$$

for each $\mu \in M(P)$.

Proof. Suppose first that (4) holds for $\mu \in M(P)$. This signifies that

$$\| \mu \| \leqq k. \ \mathrm{Sup} \left| \int_G (\mu * f)\, g dx \right| \,,$$

the supremum being taken over those f and g lying respectively in the unit balls in $L^p(G)$ and $L^{p'}(G)$. Since

$$\int_G (\mu * f)\, g dx = \int_G (\check{f} * g)\, d\mu \,,$$

where $\check{f}(x) = f(-x)$, it follows that

$$\| \mu \| \leqq \mathrm{Sup} \left\{ \left| \int_G (f * g)\, d\mu \right| : \| f \|_p \leqq \sqrt{K}, \| g \|_{p'} \leqq \sqrt{K} \right\} .$$

From this it follows that for each $u \in C_0(P)$ one has

$$\left| \int_G u d\mu \right| \leqq \mathrm{Sup} \left|_G (f * g)\, d\mu \right| \,, \tag{5}$$

where now f and g vary subject to the conditions

$$\| f \|_p \leqq \sqrt{K.} \sqrt{\| u \|} \,, \| g \|_{p'} \leqq \sqrt{K.} \sqrt{\| u \|} \,. \tag{6}$$

Now (5), combined with the Bipolar Theorem, shows that u belongs to the closed, convex, balanced envelope in $C_0(P)$ of the functions $f * g$ (or, more precisely, their restrictions to P), where f and g are subject to (6). Thus the assertion (A_P^p) is true for each $u \in C_0(P)$, with

$$K(P,p,u) \leqq k(P,p). \| u \| \,.$$

Conversely, suppose that (A_P^p) is satisfied. Let Σ denote the set of $u \in C_0(P)$ for which $K(P,p,u)$ exists finitely, so that Σ is a second category subset of $C_0(P)$. For a given $u \in \Sigma$, the set of admissible numbers $K(P,p,u)$ is easily seen to be closed. Denote by S the set of $u \in \Sigma$ for which the infimum of this set of admissible values of $K(P,p,u)$ is at most unity. Thus S consists precisely of those $u \in C_0(P)$ which are limits in $C_0(P)$ of sums (1), wherein

$$(7) \qquad \| f_r \|_p \leqq 1, \ \| g_r \|_{p'} \leqq 1 .$$

It is almost evident that S is closed, convex, and balanced in $C_0(P)$. Moreover, Σ is the union of the sets nS $(n = 1, 2, \cdots)$. Since Σ is second category in $C_0(P)$, it follows that S must be a neighbourhood of zero in $C_0(P)$. Consequently, $\Sigma = C_0(P)$ and, for some $r > 0$, each $u \in C_0(P)$ satisfying $\| u \| \leqq r$ is the limit in $C_0(P)$ of sums (1) with the f_r and g_r subject to (7). Then, however, each $u \in C_0(P)$ belongs to the closed, convex, balanced envelope in $C_0(P)$ of the set of convolutions $f * g$ with

$$\| f \|_p \leqq r^{-1/2} \sqrt{\| u \|} , \ \| g \|_{p'} \leqq r^{-1/2} \sqrt{\| u \|} .$$

For $\mu \in M(P)$ it is therefore the case that

$$\begin{aligned} \| \mu \| &= \operatorname{Sup}\left\{ \left| \int_G u d\mu \right| : \| u \| \leqq 1 \right\} \\ &\leqq \operatorname{Sup}\left\{ \left| \int_G (f * g) d\mu \right| : \| f \|_p \leqq r^{-1/2}, \| g \|_{p'}, \leqq r^{-1/2} \right\} . \end{aligned}$$

Using again the relation

$$\int_G (f * g)\, d\mu = \int_G (\mu * \check{f})\, g dx ,$$

it appears that

$$\begin{aligned} \| \mu \| &\leqq r^{-1/2} . \operatorname{Sup}\left\{ \| \mu * f \|_p : \| f \|_p \leqq r^{-1/2} \right\} \\ &= r^{-1} . N_p(\mu) , \end{aligned}$$

which is (4), with $k = r^{-1}$. The proof is thus complete.

(1.5) REMARK. It has appeared in the course of the preceding proof that, if the approximation specified in (A_P^p) is possible for each member of a second category subset of $C_0(P)$, then it is indeed possible for each $u \in C_0(P)$, and this with a value of $K(P,p,u)$ not exceeding $K_0(P,p)$. $\| u \|$.

(1.6) *The case $p = 2$: relation with Helson sets.* When $p = 2$ it is a simple consequence of the Parseval formula and Plancherel's theorem that

$$\begin{aligned} N_2(\mu) &= \| \hat{\mu} \| \\ &= \operatorname{Sup}\left\{ | \hat{\mu}(\xi) | : \xi \in X \right\} , \end{aligned}$$

where

$$\hat{\mu}(\xi) = \int_G \overline{\xi(x)}\, d\mu(x) ,$$

is the Fourier-Stieltjes transform of μ. Reference to Rudin [4], p.115, Theorem 5.6.3 shows then that as a Corollary to Theorem (1.4) one obtains the fact that (A_P^2) is true for a closed set $P \subset G$ if and only if P is a Helson subset of G. (Rudin assumes his Helson sets to be compact, but this restriction is unnecessary in the present connection.)

From the case $p = 2$ of Theorem (1.4) we may also derive a known property of Helson subsets of discrete groups G. (For historical reasons, Helson subsets of discrete groups are often termed Sidon sets; see [4], Section 5.7.)

(1.7) COROLLARY. *Suppose that G is discrete and that P is a Helson (or Sidon) subset of G. Then each bounded, complex-valued function on P is the restriction to P of the Fourier-Stieltjes transform of some measure on the (compact) character group X.* (Cf. [4], p. 121, *Theorem* 5.7.3(d).)

Proof. Let $B(P)$ be the superspace of $C_0(P)$ formed of all bounded, complex-valued functions on P. On $B(P)$ take the topology of pointwise convergence on P. Let T denote the linear mapping of $M(X)$ into $B(P)$ which assigns to $\lambda \in M(X)$ the function $T\lambda$ defined by

$$T\lambda(x) = \int_X \xi(x) d\lambda(\xi) .$$

It is evident that T is continuous for the weak topology $t = \sigma(M(X), C(X))$ on $M(X)$. For any $k > 0$, the set

$$S_k = \{\lambda \in M(X) : \| \lambda \| \leqq k\} ,$$

is compact for t, so that its image $T(S_k)$ is compact, and therefore closed, in $B(P)$. It will therefore suffice to show that, for some $k > 0$, $T(S_k)$ is dense in

$$V = \{ v \in B(P) : \| v \| \leqq 1 \} ;$$

and this will certainly be the case if $T(S_k)$ is shown to be dense in the closed unit ball $V_0 = V \cap C_0(P)$ in $C_0(P)$.

Suppose then that $u \in V_0$. Since P is a Helson set, (1.5) affirms the existence of a number $k = K_0(P,2)$ such that u is the limit, uniformly on P, and so a fortiori in the sense of the pointwise topology, of functions (1) with $\| f_r \|_2 \leqq \sqrt{K}$ and $\| g_r \|_2 \leqq \sqrt{K}$. By the Plancherel theory, these approximating functions form a sequence $(u_s)_{s=1}^{\infty}$, each term of which is expressible in the form

$$u_s(x) = \int_X \xi(x)\, F_s(\xi) d\xi = T\lambda_s\,(x)\,,$$

where $\lambda_s \in M(X)$ is defined by $d\lambda_s(\xi) = F_s(\xi)d\xi$, and where

$$F_s = \sum_{r=1}^{n} \alpha_r^{(s)}\, \hat{f}_r^{(s)}.\, \hat{g}_r^{(s)}\,,$$

so that

$$\begin{aligned} \|\,\lambda_s\,\| = \int_X |\,F_s(\xi)\,|\, d\xi &\leqq \sum_{r=1}^{n_s} |\,\alpha_r^{(s)}\,|\cdot\|\,\hat{f}_r^{(s)}\,\|_2\cdot\|\,\hat{g}_r^{(s)}\,\|_2 \\ &\leqq \sum_{r=1}^{n_s} |\,\alpha_r^{(s)}\,|\cdot\sqrt{k}\cdot\sqrt{k} \leqq k\,. \end{aligned}$$

Thus $u_s \in T(S_k)$ for each s, which shows that each $u \in V_0$ belongs to the closure in $B(P)$ of $T(S_k)$, as we wished to show.

2. *Falsity of* (A_G^p). It is not altogether trivial to decide whether or not (A_G^p) is true. By expressing this assertion in terms of multipliers of $L^p(G)$, we shall show that (A_G^p) is false at any rate whenever $1 < p < \infty$ and G is infinite compact Abelian. The same conclusion is derivable without explicit mention of multipliers; see Remark (3.2) *infra*.

Let us denote by $m^p(G)$ the set of all multipliers of $L^p(G)$. As observed in (1.3), we may regard $M(G)$ as a subset of $m^p(G)$. The next theorem makes reference to the so-called weak and uniform operator topologies on $m^p(G)$, and for brevity we shall label these "W.O.T." and "U.O.T." respectively.

(2.1) THEOREM. *If P is a closed subset of G, the following four statements are equivalent:—*

(i) *$M(P)$ is closed in $m^p(G)$ for the* U.O.T.;

(ii) *$M(P)$ is sequentially closed in $m^p(G)$ for the* W.O.T.;

(ii′) *$M(P)$ contains the closure in $m^p(G)$, relative to the* W.O.T., *of any N_p-bounded subset of $M(P)$;*

(iii) *there exists a number $k = k(P,p) < \infty$ such that*

$$\|\,\mu\,\| \leqq k.N_p(\mu)\,,$$

for $\mu \in M(P)$, i.e., by Theorem (1.4), *(A_P^p) is true.*

Proof. Since P is closed, $M(P)$ is in any case complete for the norm $\|\,\mu\,\|$. Since $m^p(G)$ is complete for the U.O.T., $M(P)$ is complete for N_p if and only if (i) holds. In any case, $N_p(\mu) \leqq \|\,\mu\,\|$. These remarks, combined with the Inversion Theorem for Banach spaces, show that (i) and (iii) are equivalent.

It is evident that (ii) implies (i). Also, since any sequence in $M(P)$ which is convergent for the W.O.T. is N_p-bounded (a direct application of the uniform boundedness principle), (ii′) implies (ii). It therefore remains only to show that (iii) implies (ii′).

Suppose then that (μ_i) is an N_p-bounded net in $M(P)$ such that $\lim_i T_{\mu_i} = T$ in the W.O.T.: we have to show that $\mathrm{T} = T_\mu$ for some $\mu \in M(P)$. Now, since (iii) is true by hypothesis, $\mathrm{Sup}_i \| \mu_i \| < \infty$. Hence the net (μ_i) has a weak limiting point $\mu \in M(G)$. Since P is closed, μ necessarily belongs to $M(P)$. The definition of the weak topology on $M(G)$ ensures that, for each $f \in L^p(G)$ and each $g \in L^{p'}(G)$, the number $\int_G (\mu * f) g dx$ is a limiting point of the numerical net

$$\left(\int_G (\mu_i * f) g dx\right) = \left(\int_G (T_{\mu_i} f)\, g dx\right).$$

But this last net is convergent to $\int (Tf) g dx$. It follows that $Tf = \mu * f$ for each $f \in L^p(G)$, i.e., $T = T_\mu \in M(P)$, which is what we wished to prove.

(2.2) REMARK. It is simple to verify that if $\mu \in M(P)$, then the multiplier T_μ has the property that $T_\mu f$ is, for each $f \in L^p(G)$, the limit of linear combinations of translates $f(x\text{-}a)$ of f with $a \in P$. Problem: Is it true that conversely any $T \in m^p(G)$, which is so approximable, is the limit in the W.O.T. of multipliers T_μ with μ ranging over some N_p-bounded subset of $M(P)$? The answer is affirmative if $P = G$ is compact, as will appear in the proof immediately below.

(2.3) COROLLARY. *Suppose that G is infinite compact Abelian. Then (A_G^p) is false for every p satisfying $1 < p < \infty$.*

Proof. Let us show first that any $T \in m_p(G)$ is the limit in the W.O.T. of an N_p-bounded net (μ_i) in $M(G)$. Take any base (U_i) of compact neighbourhoods of zero in G, and choose for each i a non-negative, continuous function h_i on G with support contained in U_i and such that $\int h_i dx = 1$. Then $\lim_i h_i * f = f$ in $L^p(G)$ for each $f \in L^p(G)$, so that

$$T_x = \lim_i T(h_i * f) = \lim_i Th_i * f = \lim_i k_i * f,$$

where $k_i = Th_i \in L^p(G)$ and

$$\begin{aligned} \| k_i * f \|_p &= \| T(h_i * f) \|_p \leqq \| T \| \cdot \| h_i * f \|_p \\ &\leqq \| T \| \cdot \| f \|_p . \end{aligned}$$

Let $\mu_i \in M(G)$ be defined by $d\mu_i(x) = k_i(x) dx$. Then $N_p(\mu_i) \leqq \| T \|$, and $\lim_i T_{\mu_i} f = \lim_i k_i * f = f$ in $L^p(G)$. Thus $\lim_i T_{\mu_i} = T$ in the

W.O.T. (even in the strong operator topology), and the net (μ_i) is N_p-bounded. This verifies our claim.

This being so, Theorem (2.1) shows that it is now sufficient to show that $M(G) \neq m^p(G)$, when G and P satisfy the stated conditions. To this end, we choose and fix any infinite Sidon subset S of X, and aim to show that corresponding to any bounded-complex-valued function b on X which vanishes on $X \cap S'$ there is a multiplier $T \in m^p(G)$ for which

$$(8) \qquad (Tf)^{\wedge}(\xi) = b(\xi)\hat{f}(\xi) \qquad (\xi \in X) .$$

Indeed, if $1 < p \leqq 2$, this follows from the substance of p. 130 of [4]. If, on the other hand, $2 < p < \infty$ there is by that same token a multiplier T_1 of $L^{p'}(G)$ such that (8) is true with T_1 in place of T, and it then suffices to take for the desired T the adjoint of T_1.

If the multiplier T defined by (8) were of the form T_μ with $\mu \in M(G)$, then (8) would entail that

$$(9) \qquad \hat{\mu}(\xi) = b(\xi) \qquad (\xi \in X) .$$

Since therefore $\hat{\mu}$ vanishes off S, the lemma immediately below would combine with (9) to show that

$$(10) \qquad \Sigma_{\xi \in S} \mid b(\xi) \mid^2 < \infty .$$

However, S being infinite, we are at liberty to suppose that (10) is false, in which case T is not of the form T_μ. Thus $M(G)$ is a proper subset of $m^p(G)$, and the proof is complete.

(2.4) *Let G be a compact Abelian group and S a Sidon subset of X. If $\mu \in M(G)$ is such that*

$$(11) \qquad \hat{\mu}(\xi) = 0 \qquad (\xi \in X \cap S') ,$$

then μ is absolutely continuous (relative to Haar measure on G) and its Radon-Nikodym derivative h belongs to $L^q(G)$ for every finite q. In particular,

$$\Sigma_{\xi \in S} \mid \hat{\mu}(\xi) \mid^2 < \infty .$$

Proof. It is known ([4], p. 128, Theorem 5.7.7) that

$$(12) \qquad \| t \|_q \leqq B_q \| t \|_1 ,$$

for every $q < \infty$ and every trigonometric polynomial t on G for which $\hat{t}(\xi) = 0$ for $\xi \in X \cap S'$, the number B_q being independent of t. On the other hand one may select in many ways a net (t_i) of trigonometric polynomials on G such that $\lim_i t_i * \mu = \mu$ weakly in $M(G)$ and $C \equiv$

$\sup_i \| t_i \|_1 < \infty$. The inequality (12) applies to $t_i * \mu$ and gives

$$\| t_i * \mu \|_q \leqq B_q \| t_i * \mu \|_1 \leqq B_q C \| \mu \| .$$

Supposing that $q > 1$, it follows that the net $(t_i * \mu)$ has a weak limiting point h_q in $L_q(G)$ and, since $t_i * \mu \to \mu$ weakly in $M(G)$, μ can be none other than the measure defined by $d\mu(x) = h_q(x)dx$. Putting $h = h_2 \in L_2(G)$, it is seen that $h_q = h$ a.e. for each $q > 1$, so that $h \in L^q(G)$ for every finite q. This h is, modulo negligible functions, the Radon-Nikodym derivative of μ, and the lemma is established.

3. Impossibility of factorisation in $L^p(G)$, $p > 1$. It was shown by P.J. Cohen [1] that each $h \in L^1(G)$ can be factorised as $f * g$ with f and g in $L^1(G)$. Now, if $p > 1$, $L^p(G)$ is an algebra under convolution if G is compact (and, if Abelian as we assume throughout, in no other cases). The next theorem, still concerned with approximation by sums of the type (1), though now with different restrictions on the f_r and g_r, shows that Cohen's result is very far from being extendible to $L^p(G)$ with $p > 1$.

(3.1) THEOREM. *Let G be infinite compact Abelian, and let $1 < p \leqq \infty$. Let Σ denote the set of functions h in $L^p(G)$ with the following property:— There exists a number $R = R(p, h) < \infty$ such that h is the weak limit in $M(G)$ of finite sums*

$$\sum_{r=1}^{n} f_r * g_r , \tag{13}$$

subject to the condition

$$\sum_{r=1}^{n} \| f_r \|_p \cdot \| g_r \|_p \leqq R . \tag{14}$$

Then Σ is a first category subset of $L^p(G)$.

Note. In the statement of Theorem (3.1) we are regarding $L^p(G)$ as a subset of $M(G)$, identifying a function $f \in L^p(G)$ with the measure μ defined by $d\mu(x) = f(x)dx$.

Proof. Take again an infinite Sidon subset S of X. Since $p > 1$ there exists ([4], p. 130) a number $c = c(p, S)$ such that

$$\| \hat{f} \|_{2,S} \equiv [\Sigma_{\xi \in S} | \hat{f}(\xi) |^2]^{1/2} \leqq c . \| f \|_p ,$$

for each $f \in L^p(G)$. If k is a sum of the type (13), then $\hat{k} = \sum_{r=1}^{n} \hat{f}_r \cdot \hat{g}_r$ and so, by the Cauchy-Schwarz inequality,

$$\Sigma_{\xi \in S} | \hat{k}(\xi) | \leqq \sum_{r=1}^{n} || \hat{f}_r ||_{2,S} \cdot || \hat{g}_r ||_{2,S} ,$$
$$\leqq c^2 \sum_{r=1}^{n} || f_r ||_p \cdot || g_r ||_p ,$$
$$\leqq c^2 R ,$$

the last step by virtue of (14). Consequently, the inequality

$$\Sigma_{\xi \in S} | \hat{h}(\xi) | < \infty , \tag{15}$$

is satisfied by each $h \in \Sigma$.

If $\sum$ were second category in $L^p(G)$, an argument similar to that used in the proof of Theorem (1.4) would show that

$$\Sigma_{\xi \in S} | \hat{h}(\xi) | \leqq c' || h ||_p , \tag{16}$$

for each $h \in L^p(G)$, c' being independent of h. This in turn would entail the existence of a measure $\mu \in M(G)$ (actually a function in $L^{p'}(G)$ if $p < \infty$) such that

$$\hat{\mu}(\xi) = \begin{cases} 1 & \text{if } \xi \in S , \\ 0 & \text{if } \xi \in X \cap S' . \end{cases}$$

But this would contradict Lemma (2.4). Thus Σ must be a first category subset of $L^p(G)$, as asserted.

(3.2) REMARK. The preceding proof can be modified slightly to show that $\Sigma \cap C(G)$ is a first category subset of $C(G)$, thus providing an alternative proof of Corollary (2.3).

(3.3) REMARK. The final phase of the preceding proof, leading from (16) to the contradiction, may be completed without reference to Lemma (2.4), and is in fact quite independent of the notion of Sidon sets and their properties. This is shown by the following lemma.

(3.4) LEMMA. *Let G be compact Abelian. If S is a subset of X such that*

$$\Sigma_{\xi \in S} | \hat{u}(\xi) | < \infty , \tag{17}$$

holds for each u in a second category subset of $C(G)$, then S is necessarily finite.

Proof. The hypothesis entails (cf. the proof of Theorem (1.4)) the existence of a number c'' such

$$\Sigma_{\xi \in S} | \hat{u}(\xi) | \leqq c'' || u || ,$$

for each $u \in C(G)$. This and the Riesz theorem combine to show that to each bounded, complex-valued function b on S corresponds a measure $\mu \in M(G)$ for which

$$\hat{\mu}(\xi) = b(\xi) \text{ for } \xi \in S, \ \hat{\mu}(\xi) = 0 \text{ for } \xi \in X \cap S'.$$

This μ is uniquely determined by b and the mapping T which carries b into μ is an algebraic isomorphism of the algebra $B(S)$ of all bounded, complex-valued functions on S (with the sup norm and pointwise product) into the convolution algebra $M(G)$. By Theorem 1 of [2], this entails that $B(S)$ is of finite dimension, so that S must be finite.

(3.5) REMARK. Yet another way of deriving a contradiction from (16), or from the apparently weaker variant (17), is to invoke a known theorem which says that if S is a Sidon subset of the character group of a compact Abelian group G, then for any given $v \in l^2(S)$ there exists $u \in C(G)$ such that $\hat{u}(\xi) = v(\xi)$ for $\xi \in S$. For the circle group this is established by Rudin ([5], 5.1 and 5.3), though the result for Hadamard sets S of integers is much older; and for general G it follows from Theorem 5.7.7 of [4] together with a result due to Hewitt and Zuckerman ([3], Theorem 8.6) which applies even to non-Abelian compact G.

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