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SOME INEQUALITIES FOR SYMMETRIC MEANS

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This paper was received before the synoptic introduction became a requirement.

1. In two recent papers, [3, 4], Everitt has generalised certain known inequalities, by replacing the known monotonicity of certain set (or sequence) functions by super-additivity; the sequence functions are zero if all the terms of the sequence are equal.

Included in the inequalities generalised is one due to Rado, [5, p. 61]. Bullen and Marcus, [1], recently proved a multiplicative analogue of this inequality and a generalisation to symmetric means. It is one of the intentions of this note to show that the corresponding sequence function, which is 1 when all the terms of the sequence are equal is logarithmically super-additive, (Corollary 5, below). Further properties of these sequence functions are then investigated.

2. $(a) = (a_1, \cdots, a_m)$ will denote an m-tple of positive numbers. $E_r(a)$, $1 \leqq r \leqq m$, is the rth elementary symmetric function of (a),

$$E_r = E_r(a) = \sum \prod_{j=1}^{r} a_{i_j} , \qquad E_0 = 1 , \tag{1}$$

the sum being over all r-tples, $i_1, \cdots, i_r$, such that $1 \leqq i_1 < \cdots < i_r \leqq m$. $P_r(a)$ is the mean of $E_r(a)$,

$$P_r = P_r(a) = \binom{m}{r}^{-1} E_r . \tag{2}$$

If $m = n + q$, $(\bar{a}) = (a_1, \cdots, a_n)$, $(\tilde{a}) = (a_{n+1}, \cdots, a_{n+q})$ and correspondingly $\bar{E}_r = E_r(\bar{a})$, $\tilde{E}_r = E_r(\tilde{a})$, etc., if r has suitable values. When $r = 1$ the symmetric means are arithmetic means and will be written $P_1 = A_{n+q}$, $\bar{P}_1 = \bar{A}_n$, $\tilde{P}_1 = \tilde{A}_q$. Similarly, P_{n+q}, $\bar{P}_n$, $\tilde{P}_q$ are powers of geometric means and will be written G_{n+q}^{n+q}, $\bar{G}_n^n$ and $\tilde{G}_q^q$ respectively.

3. It is known, [5, p. 52] that

(3) $s < t$ implies $P_s^t \geqq P_t^s$, with equality if and only if $a_1 = \cdots = a_m$.

It is easily seen from (1) that

(i) if $s \leqq \min(n, q)$ then $E_s = \sum_{t=0}^{s} \bar{E}_{s-t}\tilde{E}_t$,

(ii) if $s > \max(n, q)$ then $E_s = \sum_{t=0}^{n+q-s} \bar{E}_{n-t}\tilde{E}_{s-n+t}$,

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(iii) if $q < s \leqq n$ then $E_s = \sum_{t=0}^{q} \bar{E}_{s-t}\tilde{E}_t$.

Using these identities and (2) we have

LEMMA 1. (i) *If* $1 \leqq s \leqq n + q$

$$P_s = \sum_{t=u}^{v} \lambda_t^{(s)} \bar{P}_{s-t}\tilde{P}_t \,, \tag{4}$$

where $u = \max(s - n, 0)$, $v = \min(s, q)$, $\lambda_0^{(s)} = \lambda_s^{(s)} = 1$ *and* $t \neq 0, s$,
$\lambda_t^{(s)} = \left[\binom{n}{s-u-t}\binom{q}{u+t}\right] \Big/ \binom{n+q}{s}$.

(ii) *In particular if* $a_{n+1} = \cdots = a_{n+q} = \beta$ *then*

$$P_s = \sum_{t=u}^{v} \lambda_t^{(s)} \bar{P}_{s-t}\beta^t \,, \tag{5}$$

and if in addition $a_1 = \cdots = a_n = \alpha$,

$$P_s = \sum_{t=u}^{v} \lambda_t^{(s)} \alpha^{s-t}\beta^t \,. \tag{6}$$

When $q = 1$ this reduces to formulae (2) and (4) of [1].

4. We are now in a position to state and prove

THEOREM 2. *Let* $1 \leqq r \leqq k \leqq n + q$ *and* $u = \max(r - n, 0)$, $v = \min(r, q)$, $w = \max(k - n, 0)$, $x = \min(k, q)$. *Then*

(i) *if* $v \leqq w$ *and* $r - u \leqq k - x$

$$\frac{P_r^{k/r}}{P_r} \geqq \frac{\bar{P}_{r-u}^{(k-x)/(r-u)}}{\bar{P}_{k-x}} \frac{\tilde{P}_v^{w/v}}{\tilde{P}_w} \,, \tag{7}$$

(ii) *if* $v \leqq w$

$$\frac{P_r^{k/r}}{P_k} \geqq \frac{\tilde{P}_v^{w/v}}{\tilde{P}_w} \,, \tag{8}$$

with equality in each case if and only if either $r = k$ *or* $a_1 = \cdots = a_{n+q}$.

Before proceeding with the proof it should be noted that the condition $v \leqq w$ becomes $r - u \leqq k - x$ if n and q are interchanged. So if $r - u \leqq k - x$ inequality (8) holds, with the role of n and q interchanged; or equivalently $P_r^{k/r}/P_k \geqq \bar{P}_{r-u}^{(k-x)/(r-u)}/\bar{P}_{k-x}$. If neither $v \leqq w$ nor $r - u \leqq k - x$ then, from (3), nothing is true. The condition $v \leqq w$ is equivalent to $\min(r, q) \leqq \max(k - r, 0)$ and for this either $r < q$ and $k \geqq n + r$ or $r \geqq q$ and $k = n + q$; that is $k \geqq n + v$. For both $v \leqq w$ and $r - n \leqq k - x$ either $r < \min(n, q)$ and $k \geqq r + \max(n, q)$ or $r \geqq \min(n, q)$ and $k = n + q$.

Proof of Theorem 2. If $r = k$ the results are trivial so assume $r < k$. Rewrite (7) as

$$L = \frac{P_k^r}{\bar{P}_{k-x}^r \tilde{P}_w^r} \leqq \frac{P_r^k}{\bar{P}_{r-u}^{[(k-x)/(r-u)]r} \tilde{P}_v^{wr/v}} = R\ .$$

By (4) with $s = r$

$$P_r^k = \left(\sum_{t=u}^{v} \lambda_t^{(r)} \bar{P}_{r-t} \tilde{P}_t\right)^k . \tag{9}$$

Using (3) on each term of this sum

$$P_r^k \geqq \left(\sum_{t=u}^{v} \lambda_t^{(r)} \bar{P}_{r-u}^{(r-t)/(r-u)} \tilde{P}_v^{t/v}\right)^k .$$

By (6) the right hand side of this inequality is the kth power of the rth symmetric mean of $b_1, \cdots, b_{n+q}$ where $b_1 = \cdots = b_n = P_{r-u}^{1/(r-u)}$ and $b_{n+1} = \cdots = b_{n+q} = P_v^{1/v}$. Using (3), (6) and $r < k$ this gives

$$\begin{aligned} P_r^k &\geqq \left(\sum_{t=w}^{x} \lambda_t^{(k)} \bar{P}_{r-u}^{(k-t)/(r-u)} \tilde{P}^{t/v}\right)^r \\ &= \bar{P}_{r-u}^{[(k-x)/(r-u)]r} \tilde{P}^{wr/v} \left(\sum_{t=w}^{x} \lambda_t^{(k)} \bar{P}_{r-u}^{(x-t)/(r-u)} \tilde{P}_v^{(t-w)/v}\right)^r . \end{aligned}$$

On rewriting we get,

$$R \geqq \left(\sum_{t=w}^{x} \lambda_t^{(k)} \bar{P}_{r-u}^{(x-t)/(r-u)} \tilde{P}_v^{(t-w)/v}\right)^r = S\ , \quad \text{say}\ .$$

Similarly by (4)

$$P_k^r = \left(\sum_{t=w}^{x} \lambda_t^{(k)} \bar{P}_{k-t} \tilde{P}_t\right)^r . \tag{10}$$

Using (3) on each term of this sum gives

$$\begin{aligned} P_k^r &\leqq \left(\sum_{t=w}^{x} \lambda_t^{(k)} \bar{P}_{k-x}^{(k-t)/(k-x)} \tilde{P}_w^{t/w}\right)^r \\ &= \bar{P}_{k-x}^r P_w^r \left(\sum_{t=w}^{x} \lambda_t^{(k)} \bar{P}_{k-x}^{(x-t)/(r-x)} \tilde{P}_w^{(t-w)/w}\right)^r . \end{aligned}$$

Rewriting we have that

$$L \leqq \left(\sum_{t=w}^{x} \lambda_t^{(k)} \bar{P}_{k-x}^{(x-t)/(k-x)} \tilde{P}_w^{(t-w)/w}\right)^r = T\ , \quad \text{say}\ .$$

By the condition in (i) and (3), $T \leqq S$, which proves (7). Some terms in the above proof become undefined in certain limiting cases. If they are defined to be 1 the proof is then correct. Finally, since $r < k$, the inequality is clearly strict when (3) is. This completes the proof of (i).

To prove (ii) the procedure is similar except that when (3) is applied to the right hand sides of (9) and (10) it is applied to the second part of each term only, that is to $\tilde{P}_t$. The analysis is then the same with (5) being used instead of (6).

COROLLARY 3.

$$\left(\frac{A_{n+q}}{G_{n+q}}\right)^{n+q} \geqq \left(\frac{\bar{A}_n}{\bar{G}_n}\right)^n \left(\frac{\tilde{A}_q}{\tilde{G}_q}\right)^q \tag{11}$$

with equality if and only if $a_1 = \cdots = a_{n+q}$.

Proof. From Theorem 2(i) with $r = 1$, $k = n + q$.

COROLLARY 4. *If* $1 \leqq r \leqq s \leqq n$ *then*

$$\frac{P_r^{s+1}}{P_{s+1}^r} \geqq \frac{\bar{P}_r^s}{\bar{P}_s^r}, \tag{12}$$

and in particular

$$\left(\frac{A_{n+1}}{G_{n+1}}\right)^{n+1} \geqq \left(\frac{\bar{A}_n}{\bar{G}_n}\right)^n, \tag{13}$$

with equality if and only if $a_1 = \cdots = a_{n+1}$.

Proof. From Theorem 2 (ii) with $k = s + 1$, $n = 1$. These results are those in [1].

Finally if $r\{(a)\} = r(a) = (A_m/G_m)^m$ then we have

COROLLARY 5. $\log r\{(\bar{a}) \cup (\tilde{a})\} \geqq \log r(\bar{a}) + \log r(\tilde{a})$.

5. The above inequalities (11), (13) and that due to Rado, [5, p. 61] can be further generalised by the use of weighted means. Let $(w) = (w_1, \cdots, w_n)$ be an m-tple of nonnegative numbers, not all zero. Define

$$W_r = \sum_{i=1}^{n} w_r, \qquad W_r > 0,$$

$$A_r = A_r^{(w)} = \frac{1}{W_r}\sum_{i=1}^{r} w_i a_i, \qquad G_r = G_r^{(w)} = \left(\prod_{i=1}^{r} a_i^{w_i}\right)^{1/W_r}.$$

It is known that

$$G_r \leqq A_r, \quad \text{with equality only when} \quad a_1 = \cdots = a_n. \tag{14}$$

A generalisation of Rado's inequality and (13) is given by

THEOREM 6.

$$W_n(A_n - G_r) \leqq W_{n+1}(A_{n+1} - G_{n+1}), \tag{15}$$

$$\left(\frac{A_n}{G_n}\right)^{W_n} \leqq \left(\frac{A_{n+1}}{G_{n+1}}\right)^{W_{n+1}}, \tag{16}$$

with equality if and only if $a_1 = \cdots = a_{n+1}$.

Proof. The proofs are exactly those of the special cases. As direct proofs were not given in [1] they will be given here. In particular the proof of (15) is simpler than that suggested in [5].

(15) is equivalent to $G_{n+1} \leqq (W_n/W_{n+1})G_n + (w_{n+1}/W_{n+1})a_{n+1} = U$, say.

$$\begin{aligned} G_{n+1} &= G_n^{(W_n/W_{n+1})} a_{n+1}^{(w_{n+1})/(W_{n+1})} \\ &\leqq U \end{aligned} \tag{17}$$

by an application of (14).

Similarly (16) is equivalent to

$$A_{n+1} \geqq A_n^{(W_n/W_{n+1})} a_{n+1}^{(w_{n+1})/(W_{n+1})} = V, \quad \text{say}$$

but

$$\begin{aligned} A_{n+1} &= \frac{W_n}{W_{n+1}} A_n + \frac{w_{n+1}}{W_{n+1}} a_{n+1} \\ &\geqq V, \end{aligned} \tag{18}$$

by an application of (14).

In a similar way we can prove

THEOREM 7.

$$W_{n+p}(A_{n+q} - G_{n+q}) \geqq \bar{W}_n(\bar{A}_n - \bar{G}_n) + \tilde{W}_q(\tilde{A}_q - \tilde{G}_q),$$

$$\left(\frac{A_{n+q}}{G_{n+q}}\right)^{W_{n+q}} \geqq \left(\frac{\bar{A}_n}{\bar{G}_n}\right)^{\bar{W}_n} \left(\frac{\tilde{A}_q}{\tilde{G}_q}\right)^{\tilde{W}_q},$$

with equality if and only if $a_1 = \cdots = a_{n+q}$.

Generalisations along the same lines are possible for the inequalities (7), (8) and (12). Suppose $(wa) = (w_1a_1, \cdots, w_ma_m)$; then define

$$F_r(a) = \frac{E_r(wa)}{E_r(w)},$$

a generalisation of $P_r(a)$, to which it reduces if $w_1 = \cdots = w_m \neq 0$. The two m-tples (a), (w) will be said to be similarly ordered if for all i, j, $a_i \leqq a_j$ $(a_i \geqq a_j)$ implies $w_i \leqq w_j$ $(w_i \geqq w_j)$.

THEOREM 8. *If* (a) *and* (w) *are similarly ordered then*

(i) $s<t$ *implies* $F_s^t > F_t^s$, *with equality if and only if* $a_1 = \cdots = a_m$.

(ii) *inequalities* (7), (8) *and* (12) *hold, subject to the relevant conditions, with* P *replaced by* F.

Proof. The proof of (i) is exactly that of (3), [5, p. 53]. Then the inequalities follow as before.

The requirement that (a) and (w) be similarly ordered is essential as the following example shows. If $(a) = (1, 1, 2)$ and $(w) = (2, 1, 1)$ then $F_1 < F_2^{1/2}$ but $F_2^{1/2} > F_3^{1/3}$. The extreme case $s = 1$, $t = m$ of (i) is a weaker form of (14) since $F_m^{1/m}$ is the unweighted geometric mean whereas F_1 is the weighted arithmetic mean with the larger numbers having the larger weights.

6. In recent papers Diananda, [2] and Kober [6], have investigated further properties of $A_n - G_n$. We will now prove multiplicative analogues of their results. Let $(w) = (w_1, \cdots, w_n)$, $w_i > 0$, $W_n = 1$ and define

$$R_n = R_n^{(w)} = \frac{A_n^{(w)}}{G_n^{(w)}} = \frac{A_n}{G_n}$$

$$L_n = \prod_{i,j=1}^{n} \left(\frac{a_i^{1/2}a_j^{-1/2} + a_i^{-1/2}a_j^{1/2}}{2} \right),$$

$$\Lambda_n = \prod_{i,j=1}^{n} \left(\frac{a_i^{1/2}a_j^{-1/2} + a_i^{-1/2}a_j^{1/2}}{2} \right)^{w_i w_j},$$

$w = \min (w_1, \cdots, w_n)$, $W = \max (w_1, \cdots, w_n)$.

THEOREM 9.

$$L_n^{w/(n-1)} \leqq R_n \leqq E_n^{W}, \tag{19}$$

$$\Lambda_n^{1/(1-w)} \leqq R_n \leqq \Lambda_n^{1/w}, \tag{20}$$

with equality if and only if $a_1 = \cdots = a_n$.

Proof. The proofs of (19) and (20) are similar so only that of (20) will be given. Writing $\alpha = 1/(1 - w)$ the left hand inequality in (20) can be rewritten as

$$G_n^{1-\alpha} \Pi_n^{\alpha} \leqq A_n, \tag{21}$$

where

$$\Pi_n = \prod_{i,j=1}^{n} \left(\frac{a_i + a_j}{2} \right)^{w_i w_j}.$$

The left hand side of (21) is equal to

$$\prod_{1 \leqq i < j \leqq n} \left(\frac{a_i + a_j}{2} \right)^{2\alpha w_i w_j} \prod_{i=1}^{n} a^{\{\alpha w_i^2 + (1-\alpha) w_i\}}.$$

Since $\alpha w_i^2 + (1 - \alpha) w_i \geqq 0$ and $\sum_{1 \leqq i < j \leqq n} 2\alpha w_i w_j + \sum_{i=1}^{n} \{\alpha w_i^2 + (1 - \alpha) w_i\} =$

1, an application of (14) gives (21).

The proof of the right hand inequality in (20) is slightly longer. The proof is by induction on n and the result is trivial when $n = 1$. By rewriting, the inequality is equivalent to

$$\beta_n(a) = \frac{A_n^w G_n^{1-w}}{\Pi_n} \leqq 1 \,. \tag{22}$$

Using (17) and (18) it is easy to show that

$$\beta_n(a) = \frac{\{(1 - w_n)A_{n-1} + w_n a_n\}^w G_{n-1}^{(1-w_n)(1-w)} a_n^{w_n(1-w_n-w)}}{\Pi_{n-1}\cdot \prod_{i=1}^{n-1}\left(\frac{a_i + a_n}{2}\right)^{2w_i w_n}} \,.$$

In particular therefore, if $a_1 = \cdots = a_{n-1} = \alpha$,

$$\beta_n(a) = \frac{\{(1 - w_n)\alpha + w_n a_n\}^w \alpha^{(1-w_n)(w_n-w)} a_n^{w_n(1-w_n-w)}}{\left(\frac{a_n + \alpha}{2}\right)^{2w_n(1-w_n)}} \,. \tag{23}$$

Further if $v = \min(w_1, \cdots, w_{n-1})$ then $v \geqq w$ and

$$\beta_{n-1}(a_1, \cdots, a_{n-1}) = \frac{A_{n-1}^v G_{n-1}^{1-v}}{\Pi_{n-1}^{1/(1-w_n)^2}} \,.$$

Now, since $1 - w_n - w \geqq 0$ and $w + (1-w_n)(1-w) + w_n(1-w_n-w) = 2w_n(1 - w_n)$, an application of (14) to (23) demonstrates (22) in this special case.

If we now assume $\beta_{n-1} \leqq 1$ then

$$\begin{aligned}\beta_n \leqq \frac{\beta_n}{\beta_{n-1}^{(1-w_n)^2}} &= \frac{\{(1 - w_n)A_{n-1} + w_n a_n\}^w G_{n-1}^{(1-w_n)(1-w)-(1-w_n)^2(1-v)} a_n^{w_n(1-w_n-w)}}{A_{n-1}^{v(1-w_n)^2} \prod_{i=1}^{n}\left(\frac{a_i + a_n}{2}\right)^{2w_i w_n}} \\ &\leqq \frac{\{(1 - w_n)A_{n-1} + w_n a_n\}^w A_{n-1}^{(1-w_n)(w_n-w)} a_n^{w_n(1-w_n-w)}}{\prod_{i=1}^{n-1}\left(\frac{a_i + a_n}{2}\right)^{2w_i w_n}}\end{aligned}$$

using (14). Without any loss of generality we can assume that $a_n = \max(a_1, \cdots, a_n)$, when in particular $a_n \geqq A_{n-1}$. Then

$$\begin{aligned}\beta_n(a) &\leqq \frac{\{(1 - w_n)A_{n-1} + w_n a_n\}^w A_{n-1}^{(1-w_n)(w_n-w)} a_n^{w_n(1-w_n-w)}}{\left(\frac{a_n + A_{n-1}}{2}\right)^{2w_n(1-w_n)}} \\ &\leqq 1 \,,\end{aligned}$$

by the particular case (23) with $\alpha = A_{n-1}$.

The cases of equality are immediate.

It might be remarked that if W' is second largest and w' the second smallest of (w) tnen

$$1 \leqq L_n^{ww'} \leqq \Lambda_n \leqq L_n^{WW'} .$$

It is possible to generalise Hölder's inequality using Theorem 9.

THEOREM 10. *Let* $a_{ij} \geqq 0$ $(i = 1, \cdots, m,\ j = 1, \cdots, n)$ *and*

$$\sum_{i=1}^{m} a_{ij} = s_j > 0 \qquad (j = 1, \cdots, n) .$$

Then

$$D \prod_{j=1}^{n} s_j^{w_j} \geqq \sum_{i=1}^{m} \prod_{j=1}^{u} a_{ij}^{w_j} \geqq d \prod_{j=1}^{n} s_j^{w_j}$$

where

$$D = \min(1, L, \Lambda)$$
$$d = \max(l, \lambda)$$

and

$$L = \max_i L_n^{-(w/n-1)}(a_{i1}, \cdots, a_{in}) = \max_i L_{n\,i}^{-(w/n-1)}$$
$$= \max_i \Lambda_n^{-(1/1-w)}(a_{i1}, \cdots, a_{in}) = \max_i \Lambda_{n\,i}^{-(1/1-w)}$$
$$l = \min_i L_{n,i}^{-W} ,$$
$$\lambda = \min_i \Lambda_{n,i}^{-1/w} .$$

Proof. A simple modification of the usual proof [5, p. 23] using Theorem 9 instead of (14).

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