PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 3, 1965

NORMS AND NONCOMMUTATIVE JORDAN ALGEBRAS

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Roughly speaking, a norm on a nonassociative algebra is a nondegenerate form Q satisfying $Q(M_x y) = m(x)Q(y)$ for all x, y in the algebra where M_x is a linear transformation having something to do with multiplication by x and where m is a rational function; taking $M_x = L_x$ or $M_x = U_x = 2L_x^2 - L_{x^2}$ we get the forms Q satisfying $Q(xy) = Q(x)Q(y)$ or $Q(U_x y) = Q(x)^2 Q(y)$ investigated by R. D. Schafer. This paper extends the known results by proving that any normed algebra $\mathfrak{A}$ is a separable noncommutative Jordan algebra whose symmetrized algebra $\mathfrak{A}^+$ is a separable Jordan algebra, and that the norm is a product of irreducible factors of the generic norm. As a consequence we get simple proofs of Schafer's results on forms admitting associative composition and can extend his results on forms admitting Jordan composition to forms of arbitrary degree q rather than just $q = 2$ or 3. We also obtain some results of M. Koecher on algebras associated with ω-domains. In the process, simple proofs are obtained of N. Jacobson's theory of inverses and some of his results on generic norms. The basic tool is the differential calculus for rational mappings of one vector space into another. This affords a concise way of linearizing identities, and through the chain rule and its corollaries furnishes methods not easily expressed "algebraically".

Algebras having some sort of "norm" have appeared in various investigations. R. D. Schafer proved in [12] that any algebra $\mathfrak{A}$ with a nondegenerate form Q admitting associative composition $Q(xy) = Q(x)Q(y)$ is a separable alternative algebra. In [11] he proved that if $\mathfrak{A}$ is commutative and has a form Q of degree 2 or 3 admitting Jordan composition $Q(U_x y) = Q(x)^2 Q(y)$, where $U_x = 2L_x^2 - L_{x^2}$, then it is a separable Jordan algebra. In the applications of Jordan algebras to several complex variables [9] M. Koecher considered domains in a real vector space on which a positive homogeneous real-analytic function ω was defined satisfying $\omega(H_x y) = \det H_x \cdot \omega(y)$, where H_x was essentially the Hessian of $\log \omega$ at x. He associated with such an ω-domain a real semisimple Jordan algebra $\mathfrak{A}$ in which $H_x = U_x^{-1}$. In all these cases the algebra was a separable noncommutative Jordan algebra and the norm Q (or ω) was essentially a product of the irreducible factors of the generic norm of $\mathfrak{A}$.

Received April 7, 1964. This reserch was supported by a fellowship from the National Aeronautics and Space Administration and consists of part of the author's doctoral dissertation written at Yale University under Professor Nathan Jacobson.

The present paper originated in an attempt to prove algebraically that the algebra of an ω-domain is a Jordan algebra. Professor Nathan Jacobson suggested that the resulting proof (Lemma 1.3) could be extended to an arbitrary field and might yield at the same time a uniform derivation of Schafer's results, and this led to a general investigation of normed algebras. Speaking very roughly, a norm on a nonassociative algebra $\mathfrak{A}$ is a form Q satisfying $Q(M_x y) = m(x)Q(y)$ where M_x is a linear transformation on $\mathfrak{A}$ having something to do with multiplication by x, and where m is a rational function.

In this paper we will make a systematic study of normed algebras. The basic tool is the standard differential calculus for rational mappings of one vector space into another. This affords a concise way of linearizing identities, and through the chain rule and its corollaries furnishes methods not easily expressed "algebraically".

The paper is divided into three parts, the first of which is devoted to proving that all normed algebras are separable noncommutative Jordan algebras. After recalling the fundamental results of the differential calculus we make precise the definition of a form Q admitting composition on an algebra. With such a Q we associate an associative symmetric bilinear form called the trace of Q. We define Q to be nondegenerate if its trace form is nondegenerate, and show that this agrees with Schafer's definition of nondegeneracy in the cases of interest. A norm on an algebra is then a nondegenerate form admitting composition. The results of Schafer and Koecher follow easily from the main theorem that every normed algebra is a separable noncommutative Jordan algebra. Our definition of nondegeneracy has the advantage that bilinear forms are easier to work than q-linear forms, hence we can extend Schafer's results on forms admitting Jordan composition to forms of arbitrary degree q rather than just $q = 2, 3$. It also allows us to obtain his results when the base field has more than q elements, which is a weaker hypothesis than his condition that the characteristic is 0 or is greater than q.

The second part of the paper is devoted to characterizing the norm of a normed algebra. We first extend N. Jacobson's theory of inverses in Jordan algebras to noncommutative Jordan algebras; the proof of the main properties of these is simpler than his. We next prove a lemma to the effect that under fairly general conditions if Q admits some kind of composition then so do all its irreducible factors. Using this and a technique of N. Jacobson's we can easily derive the basic properties of the generic norm. Applying these results to normed algebras we show that the norm of any normed algebra is a product of irreducible factors of the generic norm.

The last part of the paper is devoted to the work of Koecher [9] and N. Jacobson [6] on isotopes and the group of norm-preserving transformations of a commutative Jordan algebra.

CHAPTER I

1. Some conventions. In this paper we will always work over a field Φ of characteristic $p \neq 2$ (p may be zero); $|\Phi|$ will denote the cardinality of Φ. *In Chapter I we will always assume that Φ is infinite* (except in § 5), *and that $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ are finite-dimensional vector spaces over Φ.* "Algebra" will always mean nonassociative algebra with identity.

We briefly recall the well-known facts about the differential calculus for rational mappings over an infinite field [2, pp. 21–37]. Relative to bases $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_m\}$ for the vector spaces $\mathfrak{X}$ and $\mathfrak{Y}$ a *rational mapping* $F: \mathfrak{X} \rightarrow \mathfrak{Y}$ has the form

$$x = \Sigma \xi_i x_i \xrightarrow{F} y = \Sigma \eta_j y_j$$

where $\eta_j = F_j(\xi_1, \cdots, \xi_n)$ are rational expressions in $\Phi(\xi_1, \cdots, \xi_n)$. The value $F(x)$ is defined only for those x in the Zariski-open subset of $\mathfrak{X}$ where the denominators of the components F_j of F don't vanish. A rational mapping $F: \mathfrak{X} \rightarrow \Phi$ is called a *rational function.* We will always use x, y, z to denote the "independent variables" of rational mappings on $\mathfrak{X}$, while u, v, w and a, b, c will denote fixed vectors in $\mathfrak{X}$.

If F is a rational mapping of $\mathfrak{X}$ into $\mathfrak{Y}$ then ∂F denotes the differential of F and $\partial F|_x$ the differential of F at a point $x \in \mathfrak{X}$; the latter is a linear map from $\mathfrak{X}$ into $\mathfrak{Y}$, and we denote by $\partial_u F|_x$ its value $\partial F|_x(u)$ at a vector $u \in \mathfrak{X}$. Relative to bases $\{x_i\}$ for $\mathfrak{X}$ and $\{y_j\}$ for $\mathfrak{Y}$ the matrix of $\partial F|_x$ is the Jacobian $(\partial_j F_i|_x)$ where $\partial_j F_i|_x$ is the formal partial derivative of $F_i(\xi_1, \cdots, \xi_n)$ with respect to the indeterminate ξ_j evaluated at x. $x \rightarrow \partial_u F|_x$ is a rational mapping $\partial_u F$ of $\mathfrak{X}$ into $\mathfrak{Y}$, and the map $F \rightarrow \partial_u F$ is just partial derivation ∂_u in the direction u.

As an example, in this notation the *chain rule,* which is fundamental in the sequel, becomes

$$\partial\{F \circ G\}|_x = \partial F|_{G(x)} \circ \partial G|_x$$

or

$$\partial_u\{F \circ G\}|_x = \partial_v F|_{G(x)} \quad \text{for} \quad v = \partial_u G|_x \,.$$

The *logarithmic derivative*

$$\partial_u \log F = F^{-1}\partial_u F$$

of a rational function is well defined even though there is no function

$\log F$, and the usual rules hold:

$$\partial \log \{F \cdot G\} = \partial \log F + \partial \log G$$

$$\partial_u \partial_v \log F = \partial_v \partial_u \log F = F^{-2}\{F \cdot \partial_u \partial_v F - \partial_u F \cdot \partial_v F\} .$$

If F is a homogeneous mapping of degree q, ie. $F(\lambda x) = \lambda^q F(x)$, then the *Euler differential equations* imply.

$$\partial_x F \,|_x = qF(x) .$$

Finally, if F is a rational function such that $\partial_u F = 0$ for all $u \in \mathfrak{X}$ then $F \in \Phi(\xi_1^p, \cdots, \xi_n^p)$ relative to any basis for $\mathfrak{X}$ (p always denotes the characteristic of Φ; if $p = 0$ the condition is that $F \in \Phi$).

2. Forms admitting composition. By a *form* on a vector space $\mathfrak{X}$ we mean a homogeneous polynomial function; *throughout the rest of this chapter Q will denote a form on $\mathfrak{X}$ of degree $q > 0$.* If $\mathfrak{A}$ is an algebra on $\mathfrak{X}$ with identity c we say a form Q *admits composition* on $\mathfrak{A}$ if there are two rational mappings $E: x \to E_x$, $F: x \to F_x$ of $\mathfrak{X}$ into $\mathrm{Hom}(\mathfrak{X}, \mathfrak{X})$ satisfying

(1.1)

(a) $E_c = F_c = I$

(b) $\partial_u E\,|_c = \alpha L_u$, $\partial_u F\,|_c = \beta R_u$ for $0 \neq \alpha, \beta \in \Phi$ where L_u, R_u are left and right multiplications by $u \in \mathfrak{A}$

(c) $Q(E_x y) = e(x)Q(y)$, $Q(F_x y) = f(x)Q(y)$ for some rational functions e, f on $\mathfrak{X}$ wherever all mappings involved are defined.

Note that Q admits composition on any extension algebra $\mathfrak{A}_\Omega$, Ω an extension field of Φ.

For example, if $\mathfrak{A}$ admits associative composition $Q(xy) = Q(x)Q(y)$ we may take $E_x = L_x$, $F_x = R_x$, $\alpha = \beta = 1$, $e = f = Q$. If $\mathfrak{A}$ is commutative and admits Jordan composition $Q(U_x y) = Q(x)^2 Q(y)$, where $U_x = 2L_x^2 - L_{x^2}$, we may take $E_x = F_x = U_x$, $\alpha = \beta = 2$, $e = f = Q^2$. Indeed,

$$\begin{aligned}\partial_u U_x\,|_c &= 2\partial_u L_x^2\,|_c - L(\partial_u x^2\,|_c) \\ &= 2\{L_u L_c + L_c L_u\} - L_{cu+uc} = 2L_u = 2R_u .\end{aligned}$$

For typographical reasons we will often write $E(x)$, $L(x)$ in place of E_x, L_x etc.

A more complicated example is the following. Suppose $\mathfrak{A}$ is *quasi-associative*, that is, there is an extension Ω of Φ such that $\mathfrak{A}_\Omega$ is obtained by defining $xy = \lambda x \cdot y + (1 - \lambda) y \cdot x$ for some $\lambda \in \Omega$ where $\widetilde{\mathfrak{A}} = (\mathfrak{X}_\Omega, \cdot)$ is an associative algebra on $\mathfrak{X}_\Omega$ (if $\mathfrak{A}$ itself is associative we assume $\widetilde{\mathfrak{A}} = \mathfrak{A}$, $\lambda = 1$). Suppose Q is a form on $\mathfrak{X}$ whose extension to $\mathfrak{X}_\Omega$ satisfies $Q(x \cdot y) = Q(x)Q(y)$ identically (such as the generic norm of $\mathfrak{A}$, since

$\mathfrak{A}_\Omega$ and $\tilde{\mathfrak{A}}$ have the same generic norm; see the Corollary to Theorem 2.5). Set $\varphi = \lambda(1-\lambda)$, $E_x = L_x + \varphi U_{x-c}$, $F_x = R_x + \varphi U_{x-c}$, $\alpha = \beta = 1$, $e(x) = f(x) = Q(\varphi(x-c)^2 + x)$ where $U_x = \frac{1}{2}(L_x + R_x)^2 - \frac{1}{2}(L_{x^2} + R_{x^2})$. It is known [1, p. 583] that $\varphi \in \Phi$,[1] so E, F, e, f are all rational mappings on $\mathfrak{X}$ (not just $\mathfrak{X}_\Omega$). Clearly $E_c = F_c = I$. It is also easily checked that $U_x y = x \cdot y \cdot x$ in terms of the multiplication in $\tilde{\mathfrak{A}}$. Then

$$\partial_u\{U_{x-c}\}\,|_c\,(y) = u \cdot y \cdot (c-c) + (c-c) \cdot y \cdot u = 0\ ,$$

so $\partial_u E_x\,|_c = L_u$, $\partial_u F_x\,|_c = R_u$. Finally,

$$\begin{aligned} Q(E_x y) &= Q(xy + \varphi U_{x-c} y) \\ &= Q(\lambda x \cdot y + (1-\lambda) y \cdot x + \lambda(1-\lambda)(x-c) \cdot y \cdot (x-c)) \\ &= Q(\{\lambda(x-c) + c\} \cdot y \cdot \{(1-\lambda)(x-c) + c\}) \\ &= Q(\lambda(x-c) + c) Q((1-\lambda)(x-c) + c) Q(y) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by assumption on } Q) \\ &= Q(\lambda(1-\lambda)(x-c)^2 + (x-c) + c) Q(y) \\ &= e(x) Q(y)\ . \end{aligned}$$

Similarly $Q(F_x y) = f(x) Q(y)$, so Q admits composition on $\mathfrak{A}$.

The following weak form of the open mapping theorem is well known.

LEMMA 1.1. *If $F: \mathfrak{X} \to \mathfrak{Y}$ is a rational mapping whose differential ∂F is surjective at a point $x \in \mathfrak{X}$ then there is no nonzero rational function $G: \mathfrak{Y} \to \Phi$ which vanishes on the range $F(\mathfrak{X})$ of F.*

Proof. If there were such a G there would be one which was a polynomial; since Φ is infinite, the hypotheses would remain valid over a perfect extension Ω of Φ. But there the nonexistence of such a polynomial follows from [7, p. 268].

Suppose Q admits composition on an algebra $\mathfrak{A}$ with identity c. The mapping $\tilde{E}: x \to E_x c$ has differential αI at c since

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[1] In fact, if $L_x, R_x, \tilde{L}_x, \tilde{R}_x$ denote the multiplications in $\mathfrak{A}, \tilde{\mathfrak{A}}$ then

$$L_x = \lambda \tilde{L}_x + (1-\lambda)\tilde{R}_x\ , \quad R_x = \lambda \tilde{R}_x + (1-\lambda)\tilde{L}_x\ , \quad L_x + R_x = \tilde{L}_x + \tilde{R}_x\ .$$

Associativity of $\tilde{\mathfrak{A}}$ means $[\tilde{L}_u, \tilde{R}_v] = 0$ for all u, v. Hence

$$\begin{aligned} [L_x, R_y] &= \lambda(1-\lambda)\{[\tilde{L}_x, \tilde{L}_y] + [\tilde{R}_x, \tilde{R}_y]\} \\ &= \lambda(1-\lambda)[\tilde{L}_x + \tilde{R}_x, \tilde{L}_y + \tilde{R}_y] \\ &= \lambda(1-\lambda)[L_x + R_x, L + R_y]\ . \end{aligned}$$

If we can find $x, y \in \mathfrak{A} \subset \mathfrak{A}_\Omega$ with $[L_x, R_y] \neq 0$ we can conclude that $\varphi = \lambda(1-\lambda) \in \Phi$; otherwise, $[L_x, R_y] = 0$ identically and $\mathfrak{A}$ is associative, so by assumption $\lambda = 1$, $\varphi = 0 \in \Phi$.

$$\partial\widetilde{E}\,|_c\,(u) = \partial_u\widetilde{E}\,|_c = (\partial_u E_x\,|_c)c = \alpha L_u c = \alpha u\ .$$

If $Q(c) = 0$ then $Q(\widetilde{E}(x)) = Q(E_x c) = e(x)Q(c) = 0$ would imply $Q = 0$ by Lemma 1.1, contradicting $\deg Q > 0$. Thus if Q admits composition we necessarily have $Q(c) \neq 0$; in particular, we can always normalize Q so that $Q(c) = 1$ if we wish.

We define a rational mapping $x \to \tau_x$ of $\mathfrak{X}$ into the space of symmetric bilinear forms on $\mathfrak{X}$ by

$$\tau_x(u, v) = -\partial_u\partial_v \log Q\,|_x\ .$$

The form τ_x is defined whenever $Q(x) \neq 0$. We just saw $Q(c) \neq 0$, so $\tau = \tau_c$ is defined; τ is the *trace form* of Q. If Q is the generic norm, τ is the generic trace, and if $Q(x) = \det L_x$ or $\det U_x$ for $U_x = 2L_x^2 - L_{x^2}$ then $\tau = tr$ or $2tr$ where tr is the usual trace form $tr(u, v) = \operatorname{trace} L_{uv}$.

LEMMA 1.2. *If Q admits composition on $\mathfrak{A}$ then for those x, y where all functions involved are defined we have*

$$\partial_{E(x)v} \log Q\,|_{E(x)y} = \partial_v \log Q\,|_y \tag{1.2}$$

$$\tau_{E(x)y}(E_x u, E_x v) = \tau_y(u, v) \tag{1.3}$$

$$\tau_y(L_u y, v) = \partial_{L(u)v} \log Q\,|_y \tag{1.4}$$

$$\tau(L_u w, v) + \tau(w, L_u v) = \partial_w\partial_u\partial_v \log Q\,|_c \tag{1.5}$$

and dually with E, L replaced by F, R. Also, τ is an associative form:

$$\tau(uv, w) = \tau(u, vw)\ . \tag{1.6}$$

Proof. For (1.2), first observe that $E_x\colon y \to E_x y$ is linear in y, so $\partial E_x\,|_y = E_x$ where ∂ is applied to functions of y. Then using the chain rule we have

$$\begin{aligned}\partial_{E(x)v} \log Q\,|_{E(x)y} &= \partial \log Q\,|_{E(x)y}\,(E_x v)\\ &= \partial \log Q\,|_{E(x)y} \circ \partial E_x\,|_y\,(v)\\ &= \partial \log \{Q \circ E_x\}\,|_y\,(v) = \partial_v\{\log e(x)Q\}\,|_y\\ &= \partial_v \log e(x)\,|_y + \partial_v \log Q\,|_y\\ &= \partial_v \log Q|_y\ .\end{aligned}$$

Regarding (1.2) as a function of y and applying $\partial_u\,|_y$, the right side becomes $\partial_u\partial_v \log Q\,|_y = -\tau_y(u, v)$, while the left side yields

$$\partial_u\{\{\partial_{E(x)v} \log Q\} \circ E_x\}\,|_y = \partial_w\{\partial_{E(x)v} \log Q\}\,|_{E(x)y}$$

by the chain rule where $w = \partial_u E_x\,|_y = E_x u$; thus the left side finally reduces to $\partial_{E(x)u}\partial_{E(x)v}\log Q\,|_{E(x)y} = -\tau_{E(x)y}(E_x u, E_x v)$. Equating gives (1.3).

Now regard (1.2) as a function of x and apply $\partial_u\,|_c$. The differentiation is routine, but we will go through it in detail this once. The left side of (1.2) is $F(E_x v, E_x y)$ if $F(z, y) = \partial_z \log Q\,|_y$, and

$$\partial_u F(E_x v, E_x y)\,|_c = \partial_u F(E_x v, E_c y)\,|_c + \partial_u F(E_c v, E_x y)\,|_c$$

by the usual rule for differentiating a function of two variables. F is linear in the first variable so we can move the partial inside; hence the first term is $F(\partial_u\{E_x v\}\,|_c, y) = F(\alpha L_u v, y) = \alpha\partial_{L(u)v}\log Q\,|_y$. By the chain rule the second term reduces to $\partial_w F(v, x)\,|_{E(c)y} = \partial_w\partial_v \log Q\,|_y = -\tau_y(w, v)$ for $w = \partial_u\{E_x y\}\,|_c = \alpha L_u y$. Now the right side of (1.2) is independent of x, so applying $\partial_u\,|_c$ gives 0. Thus

$$\alpha\partial_{L(u)v}\log Q\,|_y - \tau_y(w, v) = 0\ ,$$
$$\alpha\partial_{L(u)v}\log Q\,|_y = \alpha\tau_y(L_u y, v)\ .$$

Canceling α gives (1.4).

Similarly, applying $\partial_w\,|_c$ to (1.4) as a function of y gives

$$\tau_c(L_u w, v) + \partial_w\{\tau_y(u, v)\}\,|_c = \tau(L_u w, v) - \partial_w\partial_u\partial_v \log Q\,|_c$$

for the left side and $\partial_w\partial_{L(u)v}\log Q\,|_c = -\tau(w, L_u v)$ for the right side. Equating gives (1.5).

Interchanging u and v in the dual of (1.5) we get

$$\tau(R_w u, v) + \tau(u, R_w v) = \partial_u\partial_v\partial_w \log Q\,|_c\ .$$

The latter is symmetric in u, v, w so comparing with (1.5) we get $\tau(u, vw) = \tau(w, uv)$, proving (1.6).

3. Nondegenerate forms. We call a form Q on an algebra $\mathfrak{A}$ with identity c *nondegenerate* if the trace form $\tau = \tau_c$ is a nondegenerate bilinear form.

Assume for the moment that the characteristic of Φ is 0 or $p > q$, and let Q be an arbitrary form of degree q on a vector space $\mathfrak{X}$. Then $q!$ is not zero in Φ; we claim

$$[x_1, \cdots, x_q] = 1/q!\,\partial_{x_1}\cdots\partial_{x_q}Q\,|_c$$

is a symmetric q-linear form with $Q(x) = [x, \cdots, x]$. Clearly it is a symmetric and multilinear. Note that since Q is of degree q, $\partial_{x_1}\cdots\partial_{x_q}Q$ is of degree 0, ie. constant, so $[x_1, \cdots, x_q] = 1/q!\,\partial_{x_1}\cdots\partial_{x_q}Q\,|_x$ for any x. From the Euler equations we have

$$q!\,[x, \cdots, x] = \overbrace{\partial_x \cdots \partial_x}^{q} Q\,|_x = 1!\,\overbrace{\partial_x \cdots \partial_x}^{q-1} Q\,|_x = \cdots \tag{1.7}$$
$$= (q-1)!\,\overbrace{\partial_x}^{1} Q\,|_x = q!\,Q(x)\,.$$

Thus $Q(x) = [x, \cdots, x]$, and hence $[x_1, \cdots, x_q]$ is the unique symmetric q-linear form obtained by polarizing Q. R. D. Schafer has called Q nondegenerate if there is no $u \neq 0$ such that $[x, \cdots, x, u] = 0$ for all x (or equivalently, by linearizing, $[x_1, \cdots, x_{q-1}, u] = 0$ for all x_k). As in (1.7) we see $[x, \cdots, x, u] = 0$ for all x if and only if $1/q\,\partial_u Q\,|_x = 0$, so Schafer's condition of nondegeneracy is that $\partial_u Q = 0 \Rightarrow u = 0$. Since the characteristic is 0 or $p > q$, this means that Q is not independent of any variables in the sense that there is no basis $\{x_1, \cdots, x_n\}$ for $\mathfrak{X}$ relative to which $Q \in \Phi[\xi_1, \cdots, \xi_{n-1}]$. Certainly this is a reasonable restriction.

If Q is nondegenerate in our sense it is always nondegenerate in Schafer's, since $\partial_u Q = 0$ would imply

$$\tau(u, v) = -\partial_v \partial_u \log Q\,|_c = -\partial_v\{Q^{-1}\partial_u Q\}\,|_c = 0$$

for all v. We claim the converse is true if Q is a form admitting composition on an algebra $\mathfrak{A}$ (still assuming the characteristic is 0 or $p > q$). Suppose such a Q is degenerate in our sense, so τ has a nonzero radical $\mathfrak{R}$. τ is associative by (1.6), so $\mathfrak{R}$ is an ideal. As we remarked after Lemma 1.1, we may assume $Q(c) = 1$. Then $\partial_u Q\,|_c = Q(c)^{-1}\partial_u Q\,|_c = \partial_u \log Q\,|_c = \tau(c, u)$ (taking $y = v = c$ in (1.4)) so as in (1.7) $[c, \cdots, c, u] = 1/q\,\partial_u Q\,|_c = 1/q\,\tau(c, u) = 0$ if $u \in \mathfrak{R}$. Thus we have proven the relation $[c, \cdots, c, y, \cdots, y, u] = 0$ for all $u \in \mathfrak{R}, y \in \mathfrak{A}$ when the number of y's is 0. Assume it proven when there are m y's. Applying $\partial_{x_1} \cdots \partial_{x_q}$ to $Q(E_x y) = e(x)Q(y)$ as a function of y gives

$$[E_x x_1, \cdots, E_x x_q] = e(x)[x_1, \cdots, x_q]\,.$$

By our induction hypothesis we conclude

$$[E_x c, \cdots, E_x c, E_x y, \cdots, E_x y, E_x u] = 0$$

for $u \in \mathfrak{R}$ if there are m y's. If we apply $\partial_y\,|_c$ to this as a function of x we get three groups of terms. Applying $\partial_y\,|_c$ to $E_x u$ gives the term $[c, \cdots, c, y, \cdots, y, u^*]$ where there are m y's and where

$$u^* = \partial_y\{E_x u\}\,|_c = \alpha L_y u = \alpha y u\,;$$

but $u^* \in \mathfrak{R}$ since $\mathfrak{R}$ is an ideal, so the term vanishes by the induction hypothesis. Applying $\partial_y\,|_c$ to the $E_x y$'s gives the term $m[c, \cdots, c, y, \cdots, y, y^*, u]$ where there are $m - 1$ y's and where $y^* = \alpha y y$. But this is just the result of applying $\partial_{y^*}|_y$ to the induction hypothesis as a function of y, so it vanishes too. Thus the final term $(q-1-m)$ $[c, \cdots, c, c^*, y, \cdots, y, u]$ obtained from the $E_x c$'s must vanish. But $c^* = \alpha y$, so $[c, \cdots, c, y, \cdots, y, u] = 0$ when there are $m+1$ y's, and the induction is complete. Hence $[y, \cdots, y, u] = 0$ for all $u \in \mathfrak{R}$, $y \in \mathfrak{A}$ and Q is degenerate in Schafer's sense.

4. Normed algebras. A *norm* on an algebra $\mathfrak{A}$ is a nondegenerate form Q admitting composition on $\mathfrak{A}$. If $\mathfrak{A}$ has a norm it is called a *normed algebra*. Since a nondegenerate form remains nondegenerate on any extension of the base field, $\mathfrak{A}_\Omega$ is normed for all extensions Ω of Φ.

We saw in § 2 that the algebras studied by Schafer [11, 12] were normed algebras. Another important example of a normed algebra is the following. Suppose $\mathfrak{X}$ is just a vector space, with no algebra in sight. Let Q be a form on $\mathfrak{X}$, $c \in \mathfrak{X}$ a point where the trace form $\tau_c(u, v) = -\partial_u \partial_v \log Q|_c$ is nondegenerate. Then each bilinear form τ_x is obtained from $\tau = \tau_c$ as

$$\tau_x(u, v) = \tau(H_x u, v)$$

where $H: x \to H_x$ is a rational mapping of $\mathfrak{X}$ into $\mathrm{Hom}(\mathfrak{X}, \mathfrak{X})$ defined for all x with $Q(x) \neq 0$. H depends on the choice of basepoint c. $H_c = I$, and the H_x's are self-adjoint relative to τ since each τ_x is symmetric. Consider the symmetric trilinear form

$$\sigma(u, v, w) = -\tfrac{1}{2}\partial_u\{\tau_x(v, w)\}|_c = \tfrac{1}{2}\partial_u \partial_v \partial_w \log Q|_c ;$$

since τ is nondegenerate $\sigma(u, v, w) = \tau(L_u v, w)$ for some $L_u \in \mathrm{Hom}(\mathfrak{X}, \mathfrak{X})$ where $u \to L_u$ is linear. Then $u \cdot v = L_u v$ defines a bilinear pairing on $\mathfrak{X}$. From the definition of σ we see $L_u = -\tfrac{1}{2}\partial_u H_x|_c$; H is homogeneous of degree -2 in x since τ_x is, so the Euler equations imply $\partial_c H_x|_c = -2H_c$, and $L_c = H_c = I$. Since τ is symmetric the pairing is commutative, so $u \cdot c = c \cdot u = u$. If Q admits composition in the sense that $Q(H_x y) = h(x)Q(y)$ for some rational function h whenever both sides are defined we define the *Koecher algebra* $\mathfrak{A}(Q, c)$ to be $\mathfrak{A} = (\mathfrak{X}, \cdot)$. Then $\mathfrak{A}$ is a commutative algebra with c as identity. Taking $E = F = H$, $\alpha = \beta = -2$, $e = f = h$ we see that $\mathfrak{A}$ is a normed algebra. Such algebras were first studied by M. Koecher [9, pp. 39–43] in connection with ω-domains. We will see later there is a close connection between normed algebras in general and Koecher algebras (Theorem 2.9).

From now on let $\tau = \tau_c$ be the trace form of a norm Q on an algebra $\mathfrak{A}$. We define H_x for $Q(x) \neq 0$ as above by $\tau(H_x u, v) = \tau_x(u, v)$. Since τ is nondegenerate, each linear functional $\partial \log Q|_x$ can be obtained from τ as

$$\partial_v \log Q\,|_x = \tau(x\#, v)$$

for some vector $x\#$. $\#$ is a rational mapping of $\mathfrak{X}$ into itself defined whenever $Q(x) \neq 0$. We have $\partial_u \tau(x\#, v)\,|_x = \partial_u \partial_v \log Q\,|_x = -\tau_x(u, v) = -\tau(H_x u, v)$, so $\partial_u \#\,|_x = -H_x u$.

LEMMA 1.3. *If $\mathfrak{A}$ is a normed algebra then for those x for which all mappings involved are defined we have*

$$L_x^* = R_x\,, \quad R_x^* = L_x \tag{1.8}$$

(* *the adjoint relative to* τ)

$$H_x R_x = L_{x\#}\,, \quad H_x L_x = R_{x\#} \tag{1.9}$$

$$H_x, R_x, L_{x\#} \textit{ commute}\,; \quad H_x, L_x, R_{x\#} \textit{ commute} \tag{1.10}$$

$$x\# \cdot x = x \cdot x\# = c \tag{1.11}$$

$$x\# \cdot x^2 = x^2 \cdot x\# = x \tag{1.12}$$

$$U_x H_x = I \textit{ where } U_x = R_x(R_x + L_x) - R_{x^2} = L_x(L_x + R_x) - L_{x^2} \tag{1.13}$$

$$\mathfrak{A} \textit{ has no nonzero ideals } \mathfrak{B} \textit{ with } \mathfrak{B}^2 = 0\,. \tag{1.14}$$

Proof. By our assumptions it suffices to pass to the algebraic closure Ω of Φ and prove the result there.

(1.8) is just the associativity (1.6) of τ.

$$\begin{aligned}\tau(H_x L_u x, v) &= \tau_x(L_u x, v)\\ &= \partial_{L(u)v} \log Q\,|_x = \tau(x\#, L_u v) = \tau(L_u^* x\#, v)\end{aligned}$$

from (1.4) so that

$$H_x R_x u = H_x(u \cdot x) = H_x L_u x = L_u^* x\# = R_u x\# = L_{x\#} u\,;$$

dually $H_x L_x = R_{x\#}$, so (1.9) is proven. Hence we have

$$R_x H_x = L_x^* H_x^* = (H_x L_x)^* = R_{x\#}^* = L_{x\#} = H_x R_x\,,$$

so R_x and H_x commute, hence both commute with their product $L_{x\#}$. The dual result holds, so (1.10) is proven.

(1.11) is a bit longer. First,

$$\begin{aligned}\partial_u\{x\# \cdot x\}\,|_x &= \{\partial_u x\#\,|_x\} \cdot x + x\# \cdot \{\partial_u x\,|_x\}\\ &= -H_x u \cdot x + x\# \cdot u = \{-R_x H_x + L_{x\#}\}u = 0\end{aligned}$$

by (1.9), (1.10) for all u, x. Hence

$$0 = \partial_x \tau(x\# \cdot x, v) = \partial_u \tau(x\#, x \cdot v) = \partial_u\{\partial_{x \cdot v} \log Q\,|_x\}$$

for all u, v, x. Thus relative to a basis for $\mathfrak{X}$ $\partial_{x\cdot v} \log Q|_x$ is in $\Omega(\xi_1^p, \cdots, \xi_n^p)$ where p is the characteristic of Ω; since Ω is perfect, it must be a pth power. If $Q = \prod Q_i^{q_i}$ for Q_i distinct irreducible factors then

$$\begin{aligned}\partial_{x\cdot v} \log Q &= \Sigma q_i \partial_{x\cdot v} \log Q_i \\ &= \Sigma q_i Q_i^{-1} \partial_{x\cdot v} Q_i = R^{-1} \Sigma q_i R_i \partial_{x\cdot v} Q_i = R^{-1} S\end{aligned}$$

for $R = \prod Q_i$, $R_i = Q_i^{-1}R$, S a polynomial. But in reduced form $R^{-1}S$ must be a pth power, and the denominator of the reduced form is composed of some of the distinct irreducible factors Q_i, so it can be a pth power only by being constant. Hence $\tau(x\#\cdot x, v) = \partial_{x\cdot v} \log Q|_x$ itself is a polynomial. Yet it is homogeneous of degree 0 since by the chain rule

$$\begin{aligned}\partial_{\lambda x\cdot v} \log Q|_{\lambda x} &= \partial_{x\cdot v} \log \{Q \circ \lambda\}|_x \\ &= \partial_{x\cdot v} \{\log \lambda^q + \log Q\}|_x = \partial_{x\cdot v} \log Q|_x ,\end{aligned}$$

so it must be constant. Thus $\tau(x\#\cdot x, v) = \tau(c\#\cdot c, v) = \tau(c, v)$ since $c\#\cdot c = L_{c\#}c = H_c R_c c = c$ by (1.9). This holds for all v, so by nondegeneracy $x\#\cdot x = c$ wherever $x\#$ is defined. The dual result is established similarly, so (1.11) is finished.

(1.12) follows from (1.10), (1.11) by

$$x\#\cdot x^2 = L_{x\#}R_x x = R_x L_{x\#} x = R_x(x\#\cdot x) = R_x c = x$$

and dually. (1.13) holds since for all u

$$\begin{aligned}Iu = u = \partial_u x|_x &= \partial_u\{x\#\cdot x^2\}|_x \\ &= (-H_x u)\cdot x^2 + x\#\cdot(x\cdot u + u\cdot x) \\ &= \{-R_{x^2}H_x + L_{x\#}(L_x + R_x)\}u \\ &= \{-R_{x^2}H_x + H_x R_x(L_x + R_x)\}u \\ &= \{R_x(R_x + L_x) - R_{x^2}\}H_x u = U_x H_x u\end{aligned}$$

by (1.9), (1.10) and dually.

For (1.14), suppose $u, v \in \mathfrak{B}$ where $\mathfrak{B}$ is an ideal with $\mathfrak{B}^2 = 0$. Then $\tau(u, v) = \tau(c, u\cdot v) = 0$, so $\tau(\mathfrak{B}, \mathfrak{B}) = 0$. If $Q(x) \neq 0$ then H_x is defined, and by (1.13) $U_x H_x = I$, so U_x is invertible. Since $\mathfrak{B}$ is an ideal and U_x is composed of multiplications, $U_x\mathfrak{B} \subset \mathfrak{B}$; by nonsingularity $U_x\mathfrak{B} = \mathfrak{B} = H_x\mathfrak{B}$. Thus $\tau_x(u, v) = \tau(H_x u, v) \in \tau(\mathfrak{B}, \mathfrak{B}) = 0$. If $\{x_1, \cdots, x_n\}$ is a basis for $\mathfrak{X}$ with $\{x_{m+1}, \cdots, x_n\}$ a basis for $\mathfrak{B}$, and if $\tilde{\Omega}$ denotes the algebraic closure of $\Omega(\xi_1, \cdots, \xi_m)$ then $\partial_u \partial_v \log Q|_x = -\tau_x(u, v) = 0$ for $u, v \in \mathfrak{B}$ implies $\partial_v \log Q \in \tilde{\Omega}(\xi_{m+1}^p, \cdots, \xi_n^p)$. Repeating the argument of (1.11) with all factorization taking place over $\tilde{\Omega}$ we see $\partial_v \log Q$ must be a polynomial in $\xi_{m+1}, \cdots, \xi_n$ with coefficients in $\tilde{\Omega}$. But for $v \in \mathfrak{B}$, $\deg Q > \deg \partial_v Q = \deg\{Q\cdot\partial_v \log Q\} = \deg Q + \deg \partial_v \log Q \geqq \deg Q$ unless

$\partial_v \log Q$ is the zero polynomial (where deg means the degree as a polynomial in $\xi_{m+1}, \cdots, \xi_n$). Hence we must have $\partial_v \log Q = 0$. Then

$$\tau(w, v) = -\partial_w \partial_v \log Q \,|_c = 0$$

for all $w \in \mathfrak{A}$, and by nondegeneracy $v = 0, \mathfrak{B} = 0$.

Notice that the proof would be greatly simplified if we assumed the characteristic of Φ to be 0 or greater than q. At least in this case, formulas (1.8) to (1.13) arise quite natually when one starts differentiating the relations (1.1). In fact, they appear (disguised) in the work of O. S. Rothaus [10, p. 210] on the differential geometry of ω-domains.

5. Norms and noncommutative Jordan algebras. Recall that a *noncommutative Jordan algebra* [11] is a nonassociative algebra in which $L_x, R_x, L_{x^2}, R_{x^2}$ commute for each x. An algebra is *separable* if it is a direct sum of simple ideals with separable centers, or equivalently if it is semisimple and remains so under any extension of the base field.

THEOREM 1.1. *If $\mathfrak{A}$ is a normed algebra then it is a separable noncommutative Jordan algebra, and the symmetrized algebra $\mathfrak{A}^+$ is a separable commutative Jordan algebra.*

Proof. We can apply Lemmas 1.2 and 1.3. By (1.6), τ is a nondegenerate associative symmetric bilinear form, and by (1.14) $\mathfrak{A}$ has no ideals $\mathfrak{B} \neq 0$ with $\mathfrak{B}^2 = 0$, so by Dieudonne's theorem $\mathfrak{A}$ is semisimple. Any extension of $\mathfrak{A}$ remains normed, hence semisimple, so $\mathfrak{A}$ is separable. If $Q(x) \neq 0$, from (1.13), (1.8), and the selfadjointness of H_x we get

$$\begin{aligned} R_x(R_x + L_x) - R_{x^2} &= H_x^{-1} = (H_x^{-1})^* \\ &= \{L_x(L_x + R_x) - L_{x^2}\}^* \\ &= (R_x + L_x)R_x - R_{x^2}\,, \end{aligned}$$

hence R_x commutes with L_x. From (1.10) we see H_x, L_x, R_x generate a commutative algebra of linear transformations which contains U_x by (1.13), hence also R_{x^2} and L_{x^2}. Thus R_x, L_x, R_{x^2}, L_{x^2} commute for those x where $Q(x) \neq 0$; since Φ is assumed to be infinite this set is dense, so they commute for all x, and $\mathfrak{A}$ is a noncommutative Jordan algebra.

In $\mathfrak{A}^+$ we have $L_x^+ = \frac{1}{2}(L_x + R_x) = R_x^+$, so $\mathfrak{A}^+$ is a commutative Jordan algebra. τ is still a nondegenerate associative form for $\mathfrak{A}^+$, and averaging (1.13) shows

$$U_x = \tfrac{1}{2}(R_x + L_x)^2 - \tfrac{1}{2}(R_{x^2} + L_{x^2}) = 2L_x^{+2} - L_{x^2}^+ = U_x^+$$

is composed of multiplications in $\mathfrak{A}^+$, so the proof of (1.14) carries over verbatim. Thus $\mathfrak{A}^+$ is semisimple by Dieudonne's theorem; the same holds for all extensions of $\mathfrak{A}^+$, and $\mathfrak{A}^+$ is separable.

It is known [1, p. 585 and argument on pp. 590–593] that this implies the simple summands of $\mathfrak{A}$ are either commutative Jordan algebras, quasiassociative algebras, or are of degree 2. We will see in Theorems 2.8 and 2.4 that the generic norm of a separable commutative Jordan algebra is nondegenerate and admits Jordan composition $N(U_x y) = N(x)^2 N(y)$ and therefore such algebras are normed. If $\mathfrak{A}$ is a separable quasiassociative algebra and Ω the extension of Φ such that $\mathfrak{A}_\Omega = \tilde{\mathfrak{A}}(\lambda)$ for $\tilde{\mathfrak{A}}$ associative, $\lambda \in \Omega$, then $\tilde{\mathfrak{A}}$ must be separable too and Theorem 2.8 shows that the generic norm is nondegenerate. By the remarks at the beginning of § 2 we see that $\mathfrak{A}$ is a normed algebra. However, it is not known if all separable flexible algebras of degree 2 are normed, so the converse of Theorem 1.1 is incomplete.

THEOREM 1.2. *If Q is a form on a vector space $\mathfrak{X}$, $c \in \mathfrak{X}$ a point where the trace form τ_c is nondegenerate and relative to which $Q(H_x y) = h(x)Q(y)$ whenever both sides are defined (where $\tau_c(H_x u, v) = \tau_x(u, v)$ and h is some rational function) then the Koecher algebra $\mathfrak{A}(Q, c)$ is a separable commutative Jordan algebra. If normalized, Q admits Jordan composition $Q(U_x y) = Q(x)^2 Q(y)$ for $U_x = 2L_x^2 - L_{x^2}$.*

Proof. We have observed before that $\mathfrak{A}$ is normed; since it is commutative, Theorem 1.1 shows that it is a separable commutative Jordan algebra. (1.13) shows that $U_x = H_x^{-1}$, so $Q(U_x y) = h(x)^{-1}Q(y)$. If Q is normalized, putting $y = c$ gives $Q(x^2) = Q(U_x c) = h(x)^{-1}Q(c) = h(x)^{-1}$, so $Q(U_x y) = Q(x^2)Q(y)$. Putting $y = x^2$ gives $Q((x^2)^2) = Q(x^2)^2$, so $Q(z^2) = Q(z)^2$ if z is of the form x^2. But the differential of $x \to x^2$ at c is $2I$ since $\partial_u x^2 |_c = 2u$; by Lemma 1.1, $Q(z^2) = Q(z)^2$ for all z. Thus $Q(U_x y) = Q(x)^2 Q(y)$.

REMARK. This is related to a result of Koecher's [8, Satz 5]. We note also that the above result holds over the field of real numbers if Q is a positive homogeneous real-analytic function ω on an open subset $\mathfrak{Y}$ of $\mathfrak{X}$ such that $c \in \mathfrak{Y}$ is a point where the Hessian of $\log \omega$ is nondegenerate and $\omega(H_x y) = \det H_x \cdot \omega(y)$ for all $x, y \in \mathfrak{Y}$. The Koecher algebra can be defined as before and the formulas of Lemmas 1.2 and 1.3 remain valid on $\mathfrak{Y}$. We can conclude that L_y commutes with L_{y^2} for $y \in \mathfrak{Y}$, and the commutativity extends to all $x \in \mathfrak{X}$ by analytic continuation. This gives an algebraic proof that the algebra of an ω-domain is a Jordan algebra [9, p. 44].

5. Some results of R. D. Schafer. In this section we will ont assume that Φ is infinite, so we cannot immediately apply the methods of the differential calculus as previously formulated. A polynomial $Q \in \Phi[\xi_1, \cdots, \xi_n]$ together with a choice of basis $\{x_1, \cdots, x_n\}$ for a vector space $\mathfrak{X}$ determine in a canonical way a function on $\mathfrak{X}$; if this induced function $Q(u) = Q(\mu_1, \cdots, \mu_n)$ vanishes for all $u = \mu_1 x_1 + \cdots + \mu_n x_n$ in $\mathfrak{X}$ and if $|\Phi| > \deg Q$ then by the usual specialization theorem we conclude Q must be the zero polynomial. For $u = \Sigma \mu_i x_i$ we set $\partial_u Q = \Sigma \mu_i \partial_i Q$ where ∂_i is formal partial derivation with respect to the indeterminate ξ_i. Then we can define a bilinear form

$$\begin{aligned}\tau_x(u, v) &= -\partial_u \partial_v \log Q(x) \\ &= Q(x)^{-2}\{Q(x)\partial_u\partial_v Q(x) - \partial_u Q(x)\partial_v Q(x)\}\end{aligned}$$

whenever $Q(x) \neq 0$. Note that if Φ is infinite these definition agree with the usual ones of the differential calculus.

THEOREM 1.3. *Let $\mathfrak{A}$ be an algebra with identity c. If $\{x_1, \cdots, x_n\}$ is a basis for $\mathfrak{A}$ and $Q \in \Phi[\xi_1, \cdots, \xi_n]$ a homogeneous polynomial of degree q such that the trace form τ_c is defined and nondegenerate and $Q(ab) = Q(a)Q(b)$ for all $a, b \in \mathfrak{A}$, and if $|\Phi| > q$ then $\mathfrak{A}$ is a separable alternative algebra. Conversely, if $\mathfrak{A}$ is a separable alternative algebra and $\{x_1, \cdots, x_n\}$ a basis then the generic norm $N(\xi_1, \cdots, \xi_n)$ is nondegenerate and $N(ab) = N(a)N(b)$.*

Proof. The assumption that $|\Phi| > q$ implies that $Q(xy) = Q(x)Q(y)$ holds under extension of Φ to an infinite field Ω. As we have noted before, taking $E_x = L_x$, $F_x = R_x$ makes $\mathfrak{A}_\Omega$ a normed algebra. Hence by Theorem 1.1 it is separable. By (1.3), $\tau(u, v) = \tau_{E(x)c}(E_x u, E_x v) = \tau_x(L_x u, L_x v)$ so $I = L_x^* H_x L_x = R_x H_x L_x = R_x L_x H_x$ by (1.8), (1.10). But form (1.13) $I = U_x H_x$; thus $R_x L_x = U_x = R_x(R_x + L_x) - R_{x^2}$, whence $R_x^2 = R_{x^2}$. Dually $L_x^2 = L_{x^2}$, so $\mathfrak{A}_\Omega$ is alternative (again, strictly speaking this has been proved only for those x where $Q(x) \neq 0$, but since Ω is infinite this set is dense and hence the identities hold everywhere). Since $\mathfrak{A}_\Omega$ is a separable alternative algebra, so is $\mathfrak{A}$.

Conversely, if $\mathfrak{A}$ is a separable alternative algebra then by Theorem 2.8 and the Corollary to Theorem 2.5 the generic norm has nondegenerate trace form and $N(xy) = N(x)N(y)$.

THEOREM 1.4. *Let $\mathfrak{A}$ be an algebra with identity c. If $\{x_1, \cdots, x_n\}$ is a basis for $\mathfrak{A}$ and $Q \in \Phi[\xi_1, \cdots, \xi_n]$ a homogeneous polynomial of degree q such that the trace form τ_c is defined and nondegenerate and $Q(\{aba\}_1) = Q(a)^2 Q(b) = Q(\{aba\}_2)$ for all $a, b \in \mathfrak{A}$ where $\{aba\}_1 = 2a(ba) - ba^2$, $\{aba\}_2 = 2(ab)a - a^2 b$, and if $|\Phi| > 2q$ then*

(a) *$\mathfrak{A}$ is a noncommutative Jordan algebra*

(b) $\mathfrak{A}$ *and* $\mathfrak{A}^+$ *are separable*

(c) $(L_a - R_a)^2 = 0$ *for all* $a \in \mathfrak{A}$.

Conversely, if $\mathfrak{A}$ *satisfies* (a), (b), (c) *and if* $\{x_1, \cdots, x_n\}$ *is a basis then the generic norm* $N(\xi_1, \cdots, \xi_n)$ *has nondegenerate trace form and admits the above compositions.*

Proof. For the first part we may pass to an infinite extension Ω of Φ by our assumptions on $|\Phi|$. Let $E_x = 2L_xR_x - R_{x^2}$, $F_x = 2R_xL_x - L_{x^2}$, $\alpha = \beta = 2$, $e = f = Q^2$; then $E_c = F_c = I$, $\partial_u E_x|_c = 2L_u$, $\partial_u F_x|_c = 2R_u$, and $Q(E_x y) = Q(x)^2 Q(y) = Q(F_x y)$ since $\{aba\}_1 = E_a b$, $\{aba\}_2 = F_a b$. By (1.1) we see that Q is a norm on $\mathfrak{A}_\Omega$. By Theorem 1.1, $\mathfrak{A}_\Omega$ and $\mathfrak{A}_\Omega^+$ are separable noncommutative and commutative Jordan algebras respectively. From (1.3) we have $\tau(u, v) = \tau_{E(x)c}(E_x u, E_x v)$ so $I = E_x^* H_{E(x)c} E_x = F_x H_{x^2} E_x$ by (1.8), and $E_x F_x = H_{x^2}^{-1}$. As in the proof of Theorem 1.1, $H_x^{-1} = U_x^+$; now $\mathfrak{A}_\Omega^+$ is a commutative Jordan algebra, so $U_{x^2}^+ = (U_x^+)^2$ (see (2.1) below). Thus $E_x F_x = (U_x^+)^2$. Applying $\partial_c|_x$ we get

$$\begin{aligned}\partial_c\{E_x F_x\}|_x &= \{2L_x + 2R_x - 2R_x\}F_x + E_x\{2R_x + 2L_x - 2L_x\}\\ &= 2\{L_x F_x + E_x R_x\}\end{aligned}$$

and $\partial_c(U_x^+)^2|_x = 2L_x^+ U_x^+ + 2U_x^+ L_x^+$; hence $L_x F_x + E_x R_x = L_x^+ U_x^+ + U_x^+ L_x^+$. Applying $\partial_c|_x$ again, the left side becomes

$$\begin{aligned}F_x + E_x + 4L_xR_x &= 4L_xR_x + 4R_xL_x - (L_{x^2} + R_{x^2})\\ &= 8L_xR_x - 2L_{x^2}^+\end{aligned}$$

and the right side becomes $2U_x^+ + 4L_x^{+2} = 8L_x^{+2} - 2L_{x^2}^+$, so equating gives $L_x^{+2} = L_xR_x$, $(L_x + R_x)^2 = 4L_xR_x$, and finally $(L_x - R_x)^2 = 0$. Since these results hold for $\mathfrak{A}_\Omega$ they hold for $\mathfrak{A}$.

Conversely, suppose $\mathfrak{A}$, $\mathfrak{A}^+$ are separable; since the generic norm of $\mathfrak{A} = \bigoplus \mathfrak{A}_i$ is $N = \prod N_i$ it suffices to consider simple algebras $\mathfrak{A}$, and as we remarked after Theorem 1.1 such an $\mathfrak{A}$ is either a commutative Jordan algebra, a quasiassociative algebra, or is of degree 2. In the first case $E_x = F_x = U_x$, and the result follows from Theorems 2.4 and 2.8 below. Suppose $\mathfrak{A}$ is a separable quasiassociative algebra with $(L_a - R_a)^2 = 0$. Since this identity is of degree 2 it is valid on the extension $\mathfrak{A}_\Omega$, Ω the algebraic closure of Φ. Now $\mathfrak{A}_\Omega = \widetilde{\mathfrak{A}}(\lambda)$ for $\lambda \in \Omega$, so if $\widetilde{R}_a$, $\widetilde{L}_a$ denote the multiplications in the associative algebra $\widetilde{\mathfrak{A}}$ we have

$$\begin{aligned}L_a = \lambda\widetilde{L}_a + (1-\lambda)\widetilde{R}_a\,, \quad R_a &= \lambda\widetilde{R}_a + (1-\lambda)\widetilde{L}_a\,,\\ (L_a - R_a)^2 &= (2\lambda - 1)^2(\widetilde{L}_a - \widetilde{R}_a)^2\,.\end{aligned}$$

If $\lambda = \frac{1}{2}$, $\mathfrak{A}$ is a Jordan algebra, and we just saw the result holds in that case. Otherwise we must have $(\widetilde{L}_a - \widetilde{R}_a)^2 = 0$. This is impossible in matrix algebras of degree > 1 (take a, b such that $ab = b \neq 0$, $ba = 0$), and since $\tilde{\mathfrak{A}}$ is separable it is a direct sum of matrix algebras; hence these must be of degree 1, and since Ω is algebraically closed $\tilde{\mathfrak{A}}$ is a direct sum of fields, so again $\mathfrak{A}$ is a commutative Jordan algebra.

It remains to consider the case where $\mathfrak{A}$ is a separable flexible algebra of degree 2 satisfying (b) and (c); these hypotheses remain valid on an infinite extension Ω of Φ, and it will suffice to show the extension of N is nondegenerate and admits the given composition on $\mathfrak{A}_\Omega$. Now (b) and Theorem 2.8 imply N is nondegenerate (since $\mathfrak{A}_\Omega$ and $\mathfrak{A}_\Omega^+$ have the same generic norm), so we only have to show N admits E_x, F_x as composition. Since $\mathfrak{A}_\Omega$ is flexible,

$$R_x(R_x + L_x) - R_{x^2} = L_x(L_x + R_x) - L_{x^2} = 2L_x^{+2} - L_{x^2}^+ = U_x^+$$

by averaging (see (2.6) below). Hence $E_x = U_x^+ + R_x(L_x - R_x)$, $F_x = U_x^+ + L_x(L_x - R_x)$. Write $\mathfrak{A}_\Omega = \Omega c \oplus \mathfrak{M}$ where $\mathfrak{M} = c^\perp$ is the orthogonal complement of Ωc under τ; since τ is the generic trace and $\mathfrak{A}_\Omega^+$ is of degree 2, $xy + yx \in \Omega c$ if $x, y \in \mathfrak{M}$. Thus if $xy = \alpha c + z$ ($z \in \mathfrak{M}$) then $yx = \beta c - z$. By flexibility,

$$\alpha x + zx = (xy)x = x(yx) = \beta x - xz \,,$$
$$(\beta - \alpha)x = zx + xz \in \Omega c \,.$$

But $x \in \mathfrak{M}$, so $\beta = \alpha$. Then $\tau(c, [x, y]) = \tau(c, 2z) = 0$. Since

$$[\lambda c + x, \mu c + y] = [x, y]$$

we have $\tau(c, [u, v]) = 0$ for any $u, v \in \mathfrak{A}_\Omega$. Since

$$E_x w = U_x^+ w + (L_x - R_x)R_x w = U_x^+ w + [x, wx]$$

we have

$$\tau(c, E_x w) = \tau(c, U_x^+ w) \,.$$

Next,

$$\begin{aligned}\tau(\{L_x - R_x\}y, y) &= \tau(\{R_y - L_y\}x, R_y^+ c) \\ &= \tau(R_y^+\{R_y - L_y\}x, c) = \tau(\{R_y - L_y\}R_y^+ x, c) \\ &= \tau([R_y^+ x, y], c) = 0\end{aligned}$$

by the above since Theorem 2.4 and Lemma 1.2 show τ is associative for $\mathfrak{A}_\Omega^+$. Polarizing gives

$$(L_x - R_x) + (L_x - R_x)^* = 0 \,,$$

hence $(L_x - R_x)^*(L_x - R_x) = 0$ by (c). Now (c) also implies

$$L_x(L_x - R_x) = R_x(L_x - R_x) = L_x^+(L_x - R_x) \,;$$

all operators commute here, so

$$\begin{aligned} E_x^* E_x &= \{U_x^+ + L_x^+(L_x - R_x)^*\}\{U_x^+ + L_x^+(L_x - R_x)\} \\ &= U_x^{+2} + U_x^+ L_x^+\{(L_x - R_x) + (L_x - R_x)^*\} \\ &\quad + L_x^{+2}(L_x - R_x)^*(L_x - R_x) \\ &= U_x^{+2} = U_x^{+*} U_x^+ \end{aligned}$$

and finally

$$\tau(E_x u, E_x v) = \tau(U_x^+ u, U_x^+ v) .$$

But from $N(c) = 1$ and $\deg N = 2$ we get

$$\begin{aligned} \tau(u, v) &= N(c)^{-2}\{\partial_u N|_c \, \partial_v N|_c - N(c)\partial_u \partial_v N|_c\} \\ &= \tau(c\sharp, u)\tau(c\sharp, v) - \partial_u N|_v . \end{aligned}$$

By (1.11), $c\sharp = c\sharp \cdot c = c$; therefore

$$\begin{aligned} \partial_u\{N \circ E_x\}|_v &= \partial_{E(x)u} N|_{E(x)v} \\ &= \tau(c, E_x u)\tau(c, E_x v) - \tau(E_x u, E_x v) \\ &= \tau(c, U_x^+ u)\tau(c, U_x^+ v) - \tau(U_x^+ u, U_x^+ v) \\ &= \partial_{U^+(x)u} N|_{U^+(x)v} = \partial_u\{N \circ U_x^+\}|_v \end{aligned}$$

for all u, v; from $\partial_u\{N \circ E_x - N \circ U_x^+\}|_v = 0$ we see $N \circ E_x - N \circ U_x^+$ is in $\Omega[\xi_1^p, \cdots, \xi_n^p]$. But it is homogeneous of degree 2, so $N \circ E_x = N \circ U_x^+$, and by Theorem 2.4 $N(E_x y) = N(U_x^+ y) = N(x)^2 N(y)$. Similarly for F_x, so N admits composition.

These theorems are due to Schafer [11, 12]. He assumed that Q was given as a function $Q: x \to [x, \cdots, x]$ on $\mathfrak{A}$ where $[, \cdots,]$ was a q-linear form. It is easy to see that a basis $\{x_1, \cdots, x_n\}$ for $\mathfrak{A}$ determines a unique homogeneous polynomial $Q \in \Phi[\xi_1, \cdots, \xi_n]$ of degree q such that the functional relation $Q(x) = [x, \cdots, x]$ remains valid under all extensions of Φ, so our assumptions are essentially equivalent to his. His formulation in terms of functions has the advantage of being intrinsic, but it requires that Φ have characteristic 0 or $p > q$, while the above proofs hold if only $|\Phi| > q$ or $|\Phi| > 2q$.

Schafer proved Theorem 1.4 only for the special cases $q = 2, 3$.[2] Different choices for the noncommutative ternary compositions lead to slightly different classes of algebras; for example,

$$E_x = 2L_x^2 - L_{x^2} , \quad F_x = 2R_x^2 - R_{x^2}$$

leads to the algebras satisfying $(L_x - R_x)^2(L_x^2 - L_{x^2}) = 0$, which includes all alternative algebras.

[2] Professor L. Paige informs me that Mrs. E. Papousek has independently extended Schafer's results to arbitrary q.

When $p = 0$ or $p > q$ analogous results hold even if the dimension of $\mathfrak{A}$ is infinite. The differential calculus, Lemma 1.2, and the discussion of nondegeneracy carry over straightforwardly; in the analogue of Lemma 1.3 we cannot define H_x or $x\sharp$, so the notation is less convenient, but the proof is not much longer. Although we cannot draw any conclusions about separability, we can conclude that an infinite dimensional normed algebra is a noncommutative Jordan algebra (Theorem 1.1) and from this we can get Schafer's result that infinite-dimensional algebras admitting associative or Jordan composition are alternative or Jordan algebras respectively (Theorems 1.3 and 1.4). However, it is conjectured that all normed algebras are necessarily finite-dimensional, so for convenience we have restricted ourselves to the finite-dimensional case.

CHAPTER II

1. **Inverses.** In this chapter we do not assume that Φ is infinite, and in this first section we do not assume that the algebras are finite-dimensional. Let $\mathfrak{A}$ be a commutative Jordan algebra; then we have the following identities [5, pp. 1155–1156]:

$$U_{U(x)y} = U_x U_y U_x \qquad (U_x = 2L_x^2 - L_{x^2}) \tag{2.1}$$

$$[L_x, L_{yz}] + [L_y, L_{zx}] + [L_z, L_{xy}] = 0 \tag{2.2}$$

$$L_z L_y L_x + L_x L_y L_z + L_{(xz)y} = L_{yz} L_x + L_{xz} L_y + L_{yx} L_z \tag{2.3}$$

$$L_{x^n} = 2L_{x^{n-1}} L_x - L_{x^{n-2}} U_x \qquad (n \geqq 2)\,. \tag{2.4}$$

We say that an element $a \in \mathfrak{A}$ is *regular* with *inverse* b if $ab = c$ and $a^2 b = a$ (where c is the identity of $\mathfrak{A}$).

THEOREM 2.1. *Let $\mathfrak{A}$ be a commutative Jordan algebra with identity c. Then for $a \in \mathfrak{A}$ the following are equivalent:*
(a) *a is regular*
(b) *c is in the range of U_a*
(c) *U_a is an invertible transformation.*
The inverse is unique; if a has inverse b then b has inverse a and $U_a U_b = U_b U_a = I$. In this case $L_b = U_a^{-1} L_a$, all L_{a^k}, L_{b^j} commute, and $\Phi(a, b)$ is a commutative associative algebra.

Proof. If U_a is an invertible transformation then clearly c is in its range; conversely, if c is in the range of U_a, say $U_a d = c$, then by (2.1) $I = U_c = U_{U(a)d} = U_a U_d U_a$, so U_a is invertible.

If U_a is invertible and $b = U_a^{-1} a$ then $U_a c = a^2 = L_a a = L_a U_a b =$

$U_aL_ab = U_a(ab)$ (U_a commutes with L_a since L_{a^2} commutes with L_a in a Jordan algebra). Since U_a is one-to-one, $ab = c$. We get $U_aa = U_a(a^2b)$ in the same way, so $a^2b = a$ (which also follows from

$$a = U_ab = 2a(ab) - a^2b = 2a - a^2b) .$$

Hence a is regular with inverse b.

If a is regular with inverse b, then $L_{ab} = I$, so (2.2) shows $[L_a, L_{b^2}] = [L_b, L_{a^2}] = 0$. Thus

$$\begin{aligned} U_ab^2 &= 2L_a^2(L_{b^2}c) - L_{a^2}(b^2) = 2L_{b^2}(L_a^2c) - a^2b^2 \\ &= a^2b^2 = L_{a^2}L_bb = L_bL_{a^2}b = L_ba = c \end{aligned}$$

is in the range of U_a.

$U_ab = 2a(ab) - a^2b = a$ shows the inverse b is uniquely determined as $U_a^{-1}a$; it also shows $U_a = U_{U(a)b} = U_aU_bU_a$ by (2.1), so $U_aU_b = U_bU_a = I$. Then b has inverse $U_b^{-1}b = U_ab = a$.

In this case $[L_b, L_{a^2}] = [L_b, U_a] = 0$ (since $U_a = U_b^{-1}$), so L_b commutes with $2L_a^2 = U_a + L_{a^2}$. Thus putting $x = y = a, z = b$ in (2.3) yields $L_a = (2L_a^2 - L_{a^2})L_b = U_aL_b, L_b = U_a^{-1}L_a$. Hence U_b, L_b are in the commutative algebra of linear transformations generated by L_a, L_{a^2}, U_a^{-1}; by (2.4) this includes all L_{a^k}, L_{b^j}, so they all commute. Then it is easy to see that $\Phi(a, b)$ is a commutative associative algebra.

This theorem is due to N. Jacobson [4, 5]. The above proof is simpler than his; the simplification comes from repeated use of the "fundamental formula" (2.1), which was unknown when he first proved the theorem.

Now let $\mathfrak{A}$ be a (possibly infinite-dimensional) noncommutative Jordan algebra. The following identities hold [1, pp. 573–575]:

$$[L_x, R_z] = [R_x, L_z] \tag{2.5}$$

$$R_{yx} + L_xL_y = L_{xy} + R_xR_y \tag{2.6}$$

$$R_xL_y(R_x + L_x) + L_{yx^2} = R_{x^2}L_y + L_{yx}(R_x + L_x) . \tag{2.7}$$

We define $a \in \mathfrak{A}$ to be *regular* with *inverse* b if $ab = ba = c, a^2b = ba^2 = a$ (c the indentity of $\mathfrak{A}$).

THEOREM 2.2. *If $\mathfrak{A}$ is a noncommutative Jordan algebra then $a \in \mathfrak{A}$ has an inverse b if and only if b is the inverse of a in the commutative Jordan algebra $\mathfrak{A}^+$. Then $L_b = U_a^{-1}R_a, R_b = U_a^{-1}L_a$ where $U_a = R_a(R_a + L_a) - R_{a^2} = L_a(L_a + R_a) - L_{a^2} = U_a^+$. All $L_{a^k}, R_{a^k}, L_{b^j}, R_{b^j}$ commute, and $\Phi(a, b)$ is a commutative associative algebra.*

Proof. Clearly if $ab = ba = c, a^2b = ba^2 = a$ then a has inverse b in $\mathfrak{A}^+$. Conversely, suppose a has b as inverse in $\mathfrak{A}^+$; then $ab + ba =$

$2c$, $a^2b + ba^2 = 2a$. From (2.6)

$$\begin{aligned}(L_{a^2} - R_{a^2})b &= (L_a^2 - R_a^2)b \\ &= (L_a - R_a)(L_a + R_a)b = 2(L_a - R_a)c = 0\end{aligned}$$

so $a^2b = ba^2 = a$. Then $ba = b(a^2b) = (ba^2)b = ab$, so $ab = ba = c$, and b is the inverse of a in $\mathfrak{A}$.

By (2.5), $[L_a, R_b] = [R_a, L_b]$; by (2.6), $ab = ba = c$ implies $R_aR_b = L_aL_b$, $R_bR_a = L_bL_a$, so $[R_a, R_b] = [L_a, L_b]$. Then

$$[R_a + L_a, L_b] = \tfrac{1}{2}[R_a + L_a, R_b + L_b] = 2[L_a^+, L_b^+] = 0$$

from Theorem 2.1. Hence putting $x = a, y = b$ in (2.7) yields

$$R_a = \{R_a(R_a + L_a) - R_{a^2}\}L_b = U_aL_b\ .$$

Now (2.6) implies $R_a^2 - R_{a^2} = L_a^2 - L_{a^2}$, so averaging gives

$$\begin{aligned}U_a &= R_a(R_a + L_a) - R_{a^2} \\ &= L_a(L_a + R_a) - L_{a^2} = 2L_a^{+2} - L_{a^2}^+ = U_a^+\ .\end{aligned}$$

By Theorem 2.1, U_a^+ is invertible, so $L_b = U_a^{-1}R_a$. $R_b = U_a^{-1}L_a$ follows by duality. It is standard that $R_x, L_x, R_{x^2}\, L_{x^2}$ generate all R_{x^k}, L_{x^k} (by induction using (2.7), its dual, and power-associativity) so the rest of the proof proceeds as in Theorem 2.1.

COROLLARY. *An algebraic element a in a noncommutative Jordan algebra has an inverse if and only if it has an inverse in the associative algebra $\Phi[a]$.*

Notice that formulas (1.9) to (1.13) of Lemma 1.3 show again that in a normed algebra $x\# = x^{-1}$, $H_x = U_x^{-1}$, $L_{x^{-1}} = U_x^{-1}R_x$, $R_{x^{-1}} = U_x^{-1}L_x$, and $U_x, L_x, R_x, L_{x^{-1}}, R_{x^{-1}}$ all commute.

2. Standard properties of the generic norm. In this section again all algebras are finite-dimensional, but Φ is not assumed to be infinite. We will give alternate proofs of the results of N. Jacobson [3] concerning the generic norm. After first proving a general lemma about polynomials admitting some kind of composition, an application of a technique involving the theory of inverses and the Hilbert Nullstellensatz (due to N. Jacobson [3, p. 37]) yields the desired results in a direct fashion.

LEMMA 2.1. *If $Q \in \Phi[\eta_1, \cdots, \eta_m]$ is a polynomial and $M_1, \cdots, M_m$ are rational expressions in $\Phi(\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_m)$ which are linear in the indeterminates η_j and such that*

(a) $Q(M_1(\xi, \eta), \cdots, M_m(\xi, \eta)) = m(\xi)Q(\eta)$ *for some rational expres-*

sion $m(\xi_1, \cdots, \xi_n) \in \Phi(\xi_1, \cdots, \xi_n)$

(b) $M_i(c, \eta) = \eta_i$ *for some* $c = (c_1, \cdots, c_n)$ *where* $c_i \in \Phi$ *then each irreducible factor* Q_i *of* Q *admits the composition*

$$Q_i(M_1(\xi, \eta), \cdots, M_m(\xi, \eta)) = m_i(\xi)Q_i(\eta)$$

for some rational expression m_i.

Proof. For convenience let $M(\xi, \eta)$ denote $(M_1(\xi, \eta), \cdots, M_m(\xi, \eta))$. Let $Q = \prod Q_i^{q_i}$ be the factorization of Q into distinct irreducible factors. From (a) we have

$$\prod Q_i(M(\xi, \eta))^{q_i} = Q(M(\xi, \eta)) = m(\xi)Q(\eta) = m(\xi) \prod Q_i(\eta)^{q_i} .$$

Since each Q_i divides the right side and is irreducible as a polynomial over $\Phi(\xi_1, \cdots, \xi_n)$ it must divide some $Q_j(M(\xi, \eta))$, say $Q_j(M(\xi, \eta)) = m_i(\xi, \eta)Q_i(\xi)$ where m_i is rational in ξ and a polynomial in η. From (b), specializing $\xi \rightarrow c$ gives $Q_j(\eta) = m_i(c, \eta)Q_i(\eta)$. Since the Q_k are distinct and irreducible, $i = j$; then $Q_i(M(\xi, \eta)) = m_i(\xi, \eta)Q_i(\eta)$ and the assumed linearity of the $M_k(\xi, \eta)$ in η imply m_i is of degree 0 in η, $m_i(\xi, \eta) = m_i(\xi)$. Hence $Q_i(M(\xi, \eta)) = m_i(\xi)Q_i(\eta)$ as desired.

As a corollary, if Q is homogeneous we may take $n = 1$, $M(\lambda, \eta) = \lambda\eta$; then from

$$Q(M(\lambda, \eta)) = Q(\lambda\eta) = \lambda^q Q(\eta)$$

we conclude by the lemma that each irreducible factor Q_i satisfies $Q_i(\lambda\eta) = m_i(\lambda)Q_i(\eta)$. But $\prod m_i(\lambda) = \lambda^q$, so $m_j(\lambda) = \lambda^{q_i}$ and we see that each irreducible factor of a form is again a form. Of course, this is obvious anyway.

If $\mathfrak{A}$ is a strictly power-associative algebra over Φ (ie. remains power-associative under any extension of the base field) the *generic minimum polynomial* $m_\xi(\lambda)$ of $\mathfrak{A}$ is the mimimum polynomial of the generic element $\xi = \xi_1 x_1 + \cdots + \xi_n x_n$ of $\mathfrak{A}_\Sigma$, $\Sigma = \Phi(\xi_1, \cdots, \xi_n)$, relative to some basis $\{x_1, \cdots, x_n\}$ for $\mathfrak{A}$ over Φ. Specializing $\xi \rightarrow x$ gives a polynomial $m_x(\lambda)$ which has the same irreducible factors as the minimum polynomial $\mu_x(\lambda)$ of x in $\mathfrak{A}$ (this and the following assertions are found in [3, pp. 27–28]). The *generic norm* is the constant term $N(\xi_1, \cdots, \xi_n)$ of $m_\xi(\lambda)$; it is a homogeneous polynomial in the ξ_i. As a polynomial function the generic norm is independent of the basis for $\mathfrak{A}$, and the generic norm of an extension algebra $\mathfrak{A}_\Omega$ is just the natural extension of N to $\mathfrak{A}_\Omega$. Note, however, that N is given initially by a polynomial and a choice of basis rather than by a polynomial function (compare with Theorems 1.3, 1.4).

THEOREM 2.3. *If $\mathfrak{A}$ is a commutative Jordan algebra, the generic norm $N(x)$ has the same irreducible factors as* $K(x) = \det U_x$.

Proof. It suffices to prove everything over the algebraic closure Ω of Φ: the K, N of $\mathfrak{A}_\Omega$ are just the natural extensions of those of $\mathfrak{A}$, so if the former have the same irreducible factors over Ω the latter must have the same irreducible factors over Φ. By Theorem 2.1 and the Corollary to Theorem 2.2 $K(x) = 0 \Leftrightarrow x$ has no inverse in $\Omega[x]$. Since $\Omega[x]$ is associative, x has no inverse $\Leftrightarrow$ the constant term of $\mu_x(\lambda)$ is zero. But $\mu_x(\lambda)$ and $m_x(\lambda)$ have the same irreducible factors, so their constant terms vanish on the same set. The constant term of $m_x(\lambda)$ is $N(x)$, so $K(x) = 0 \Leftrightarrow N(x) = 0$. Because Ω is algebraically closed we can apply the Hilbert Nullstellensatz to conclude that K and N have the same irreducible factors.

THEOREM 2.4. *If $\mathfrak{A}$ is a commutative Jordan algebra, all normalized irreducible factors M of the generic norm admit Jordan composition $M(U_xy) = M(x)^2M(y)$.*

Proof. By the fundamental formula (2.1), $U_{U(x)y} = U_xU_yU_x$, so taking determinants gives $K(U_xy) = K(x)^2K(y)$. Since this holds for an infinite extension of Φ, it becomes an identity between polynomials relative to a choice of basis for $\mathfrak{A}$, and since $U_c = I$ we can apply Lemma 2.1 and Theorem 2.3 to conclude that each irreducible factor M of K or N admits composition $M(U_xy) = m(x)M(y)$. From $M(c) = 1$ we can conclude $M(U_xy) = M(x)^2M(y)$ as in the proof of Theorem 1.2.

THEOREM 2.5. *Let $\mathfrak{A}$ be a strictly power-associative algebra, $\mathfrak{B}$ a subalgebra. Then the restriction $N|_{\mathfrak{B}}$ of the generic norm N of $\mathfrak{A}$ to $\mathfrak{B}$ is a polynomial on $\mathfrak{B}$ having the same irreducible factors as the generic norm $N_{\mathfrak{B}}$ of $\mathfrak{B}$. If M is any normalized factor of N then $M(xy) = M(x)M(y)$ if x, y are contained in an associative subalgebra.*

Proof. For the first assertion, if $x \in \mathfrak{B}$ then

$$N|_{\mathfrak{B}}(x) = 0 \Longleftrightarrow N(x) = 0 \Longleftrightarrow x \text{ has no inverse in } \Phi[x];$$

since $\Phi[x] \subset \mathfrak{B} \subset \mathfrak{A}$ the same reasoning shows $N_{\mathfrak{B}}(x) = 0 \Leftrightarrow x$ has no inverse in $\Phi[x]$. Thus $N|_{\mathfrak{B}}(x) = 0 \Leftrightarrow N_{\mathfrak{B}}(x) = 0$; this remains valid on extension of Φ to its algebraic closure, so as in Theorem 2.3 we conclude $N|_{\mathfrak{B}}, N_{\mathfrak{B}}$ have the same irreducible factors.

Next, let M be a normalized irreducible factor of N and assume $\mathfrak{B}$ is an associative subalgebra of $\mathfrak{A}$. If $D(x)$ is the determinant of left multiplication by x on $\mathfrak{B}$ then the associativity $L_{xy} = L_x L_y$ of $\mathfrak{B}$ gives $D(xy) = D(x)D(y)$. Since this remains valid on an infinite extension of Φ, relative to a basis for $\mathfrak{B}$ it becomes an identity between polynomials. Applying Lemma 2.1 we see that every irreducible factor of D admits composition. But by the same argument as before, $N|_{\mathfrak{B}}$ has the same irreducible factors as D. Hence $M|_{\mathfrak{B}}$, as a factor of $N|_{\mathfrak{B}}$, admits composition $M|_{\mathfrak{B}}(xy) = m(x)M|_{\mathfrak{B}}(y)$; taking $y = c$, the assumption that M is normalized gives $m(x) = M|_{\mathfrak{B}}(x)$. Hence if x, y are in an associative subalgebra $\mathfrak{B}$ we have

$$M(xy) = M|_{\mathfrak{B}}(xy) = M|_{\mathfrak{B}}(x)M|_{\mathfrak{B}}(y) = M(x)M(y) .$$

COROLLARY. *If $\mathfrak{A}$ is an alternative algebra, all normalized irreducible factors M of the generic norm admit associative composition $M(xy) = M(x)M(y)$.*

THEOREM 2.6. *Let $\mathfrak{A}$ be an alternative or Jordan algebra. If $\{x_1, \cdots, x_n\}$ is a basis and $Q \in \Phi[\xi_1, \cdots, \xi_n]$ a homogeneous polynomial of degree q such that $Q(ab) = Q(a)Q(b)$ or $Q(U_a b) = Q(a)^2 Q(b)$ respectively for all $a, b \in \mathfrak{A}$, and if $|\Phi| > q$ or $|\Phi| > 2q$ then in either case Q is a product of irreducible factors of the generic norm N.*

Proof. The assumptions on $|\Phi|$ guarantee that we can pass to the algebraic closure of Φ and prove the result there. We saw $Q(c) \neq 0$ in § 2 of Chapter I (c the identity of $\mathfrak{A}$). If $N(x) \neq 0$ then x has an inverse y in $\Phi[x]$, $xy = U_x y = c$. Thus $Q(c)$ equals $Q(x)Q(y)$ or $Q(x)^2 Q(y)$, and in either case $Q(x)$ cannot be 0. Thus $Q(x) = 0 \Rightarrow N(x) = 0$, and the Hilbert Nullstellensatz yields the result.

THEOREM 2.7. *The generic norm of a simple alternative or Jordan algebra is irreducible.*

Proof. We refer to [3, pp. 33 and 39].

THEOREM 2.8. *The generic norm of an alternative or Jordan algebra $\mathfrak{A}$ with identity c has nondegenerate trace form τ_c if and only if $\mathfrak{A}$ is separable.*

Proof. By the Corollary to Theorem 2.5 or by Theorem 2.4 N admits R_x, L_x or U_x on $\mathfrak{A}_\Omega$, Ω an infinite extension of Φ. Thus if τ_c is nondegenerate $\mathfrak{A}_\Omega$ is normed, so is separable by Theorem 1.1, and hence $\mathfrak{A}$ is too. Conversely, assume $\mathfrak{A}$ is separable; it will suffice to prove N_Ω is nondegenerate for Ω the algebraic closure of Φ. Then $\mathfrak{A}_\Omega = \bigoplus \mathfrak{A}_i$ for $\mathfrak{A}_i$ simple alternative or Jordan algebras respectively, and $N_\Omega = \prod N_i$, $\tau = \bigoplus \tau_i$, so it will suffice to prove N_i is nondegenerate. We are thus reduced to considering the case of a simple algebra $\mathfrak{A}$ over an algebraically closed field Ω. By Lemma 1.2 τ is a well-defined associative biliner form, hence by simplicity it is either nondegenerate or the zero form. Assume $\tau = 0$; from (1.4) with $y = c$, $E_x = L_x$ or U_x we get $\tau_x(L_x u, L_x v) = \tau(u, v) = 0$ or $\tau_{x^2}(U_x u, U_x v) = \tau(u, v) = 0$ respectively. Then $\tau_x = 0$ or $\tau_{x^2} = 0$ on the sets where L_x or U_x are nonsingular. Since Ω is infinite, $\tau_z = 0$ for z in a dense set in either case, so $\tau_x = 0$ for all x; in other words $\partial_u \partial_v \log N = 0$ for all u, v. Thus $\partial_v \log N \in \Omega(\xi_1^p, \cdots, \xi_n^p)$; but it is homogeneous of degree -1, so it is zero, and $\partial_v N = N \partial_v \log N = 0$ for all v. Then $N \in \Omega[\xi_1^p, \cdots, \xi_n^p]$; since Ω is perfect, N is a pth power, which contradicts Theorem 2.7. Thus $\tau \neq 0$, so it must be nondegenerate.

4. Applications to normed algebras.

In this section we use the foregoing results to characterize all possible norms on a normed algebra. Throughout the section we will assume that Φ is infinite.

LEMMA 2.2. *If $Q \in \Phi[\eta_1, \cdots, \eta_m]$ is a polynomial and $M_1, \cdots, M_m$ are rational expressions in $\Phi(\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_m)$ which are linear in the indeterminates η_k and such that*

(a) *$\partial_k \log Q(M_1(\xi, \eta), \cdots, M_m(\xi, \eta)) = \partial_k \log Q(\eta)$ for all k where ∂_k is formal partial derivation with respect to η_k*

(b) *$M_i(c, \eta) = \eta_i$ for some $c = (c_1, \cdots, c_n)$ where $c_i \in \Phi$*

then $Q = Q'Q''$ where each irreducible factor Q_i of Q' admits the composition $Q_i(M_1(\xi, \eta), \cdots, M_m(\xi, \eta)) = m_i((\xi Q_i(\eta)$ for some rational expression m_i and where $Q'' \in \Phi[\eta_1^p, \cdots, \eta_m^p]$.

Proof. For convenience we set $M(\xi, \eta) = (M_1(\xi, \eta), \cdots, M_m(\xi, \eta))$. Let $Q = \prod Q_i^{q_i}$ be the factorization of Q as in Lemma 2.1 Suppose $Q_1, \cdots, Q_j$ are the Q_i which admit composition. Write $Q = Q'Q''$ for $Q' = \prod_{i \leq j} Q_i^{q_i}$. Then $Q'(M(\xi, \eta)) = m'(\xi)Q'(\eta)$, so

$$\begin{aligned} \partial_k \log Q'(M(\xi, \eta)) &= \partial_k \log m'(\xi) + \partial_k \log Q'(\eta) \\ &= \partial_k \log Q'(\eta) \, . \end{aligned}$$

Since $\partial_k \log Q = \partial_k \log Q' + \partial_k \log Q''$, subtraction from (a) gives

$$\partial_k \log Q''(M(\xi, \eta)) = \partial_k \log Q''(\eta) \, .$$

Now

$$\begin{aligned} \partial_k \log Q'' &= \sum_{i>j} q_i \partial_k \log Q_i \\ &= \sum q_i Q_i^{-1} \partial_k Q_i = R^{-1} \sum q_i R_i \partial_k Q_i = R^{-1} S \end{aligned}$$

where $R = \prod_{i>j} Q_i$, $R_i = Q_i^{-1} R$. In reduced form the denominator of

$\partial_k \log Q''$ thus consists of some of the distinct irreducible factors Q_i of R with $i > j$, while the denominator of

$$\begin{aligned}\partial_k \log Q''(M(\xi, \eta)) \\ &= R(M(\xi, \eta))^{-1} \sum q_i R_i(M(\xi, \eta)) \partial_k Q_i(M(\xi, \eta)) \\ &= R(M(\xi, \eta))^{-1} T(\xi, \eta)\end{aligned}$$

considered as a rational expression in η with coefficients in $\Phi(\xi_1, \cdots, \xi_n)$ is just $R(M(\xi, \eta)) = \prod_{i>j} Q_i(M(\xi, \eta))$. If Q_i is a factor of the former denominator it must be a factor of the latter; then as in Lemma 2.1 we see that Q_i admits composition. This contradicts our assumption that Q_i does not admit composition for $i > j$, so no such Q_i exists, and the denominator of $R^{-1}S$ in reduced form must be a constant. Then $\partial_k \log Q''$ is a polynomial; we noticed after Lemma 2.1 that each Q_i is homogeneous, so $\partial_k \log Q''$ is homogeneous of degree -1, and thus must be identically zero. Then $\partial_k Q'' = Q'' \partial_k \log Q'' = 0$ for all k, and $Q'' \in \Phi[\eta_1^p, \cdots, \eta_m^p]$.

THEOREM 2.9. *If Q is a norm on a normed algebra $\mathfrak{A}$ with identity c then Q is a product of irreducible factors of the generic norm N of $\mathfrak{A}$. The symmetrized algebra $\mathfrak{A}^+$ is the Koecher algebra $\mathfrak{A}(Q, c)$. In particular, any commutative normed algebra is the Koecher algebra of its norm.*

Proof. Q remains a norm on $\mathfrak{A}_\Omega$, Ω the algebraic closure of Φ, and N remains the generic norm, so it suffices to prove the result over Ω.

We have

$$\begin{aligned}\partial_u \partial_v \log \{Q \circ H_x\}|_y &= \partial_{H(x)u} \partial_{H(x)v} \log Q \,|_{H(x)y} \\ &= -\tau_{H(x)y}(H_x u, H_x v) \\ &= -\tau(H_x^* H_{H(x)y} H_x u, v) .\end{aligned}$$

Now $H_x^* = H_x$ and by averaging (1.13) $H_x^{-1} = 2L_x^{+2} - L_{x^2}^+ = U_x^+$, so from the fundamental formula (2.1) for the commutative Jordan algebra $\mathfrak{A}^+$

$$\begin{aligned}H_x H_{H(x)y} H_x &= (U_x U_{H(x)y} U_x)^{-1} \\ &= (U_{U(x)H(x)y})^{-1} = U_y^{-1} = H_y .\end{aligned}$$

Therefore

$$\begin{aligned}\partial_u \partial_v \log \{Q \circ H_y\}|_y &= -\tau(H_y u, v) \\ &= -\tau_y(u, v) = \partial_u \partial_v \log Q \,|_y ,\end{aligned}$$

and

$$\partial_u\{\partial_v \log \{Q \circ H_x\} - \partial_v \log Q\}|_y = 0 .$$

The term in braces is homogeneous of degree -1, so taking $u = y$ gives $\partial_v \log \{Q \circ H_x\}|_y = \partial_v \log Q|_y$ by Euler's equations. Since Ω is infinite, the relation $\partial_v \log Q(H_x y) = \partial_v \log Q(y)$ becomes a relation between polynomials relative to a basis for $\mathfrak{A}$, and we can apply Lemma 2.2 to write $Q = Q'Q''$ where all the irreducible factors Q_i of Q' admit H_x as composition and where Q'' is a pth power (since Ω is perfect). If Q_i admits H_x it admits $U_x = H_x^{-1}$; if we normalize it as in Theorem 1.2 we can apply Theorem 2.6 to conclude Q_i is an irreducible factor of N. Let $Q = \prod Q_i^{q_i}$ be the factorization of Q. If p does not divide q_i then $Q_i^{q_i}$ is not a pth power, hence not all of it can appear in Q'', so Q_i must appear in Q' and hence be an irreducible factor of N. To complete the proof we need only show that for each j we can alter the exponents so that $\widetilde{Q} = \prod Q_i^{\widetilde{q}_i}$ is still a norm but p does not divide $\widetilde{q}_j$. By Lemma 2.1, all Q_i admit E_x, F_x of (1.1), so any $\widetilde{Q}$ does too; hence $\widetilde{Q}$ will be a norm as soon as its trace form $\widetilde{\tau}$ is nondegenerate.

If $\mathfrak{A}_\Omega = \bigoplus \mathfrak{A}_i$ is the decomposition of $\mathfrak{A}_\Omega$ into simple ideals guaranteed by Theorem 1.1 then $N = \prod N_i$, N_i the generic norm of $\mathfrak{A}_i$. By Theorem 2.7 the N_i are the irreducible factors of N over Ω, so if p does not divide q_i then Q_i is an N_j. If p divides q_i then $\partial Q_i^{q_i} = 0$ and hence Q_i does not contribute to the trace form τ of Q. Thus τ is a linear combination of the traces of the N_j. The latter are concentrated on the $\mathfrak{A}_j$'s, so by the nondegeneracy of τ all the N_j must appear among the Q_i with q_i not divisible by p. We may renumber so that $Q_i = N_i$ for $1 \leqq i \leqq m$. Thus the Q_j for $j > m$ are the ones p divides q_j. By Theorem 2.8 the trace form τ_i of $Q_i = N_i$ is nondegenerate on $\mathfrak{A}_i$. By Lemma 1.2, τ_j for $j > m$ is a (perhaps degenerate) associative bilinear form on $\mathfrak{A}_\Omega$, so $\tau_j = \bigoplus_{i \leqq m} \lambda_i \tau_i$ for some $\lambda_i \in \Omega$ (because any associative form on the simple algebra $\mathfrak{A}_i$ over the algebraically closed field Ω is a scalar multiple of the given nondegenerate form τ_i). If we pick integers $\bar{q}_i$ such that $\bar{q}_i 1 + \lambda_i \neq 0$ in Ω and set $\bar{Q} = \prod_{i \leqq m} Q_i^{\bar{q}_i}$ then $\widetilde{Q} = Q_j \bar{Q}$ has trace form $\widetilde{\tau} = \bigoplus_{i \leqq m} (\bar{q}_i + \lambda_i)\tau_i$ which is nondegenerate. The exponent of Q_j in $\widetilde{Q}$ is 1, so $\widetilde{Q}$ is the desired norm. (As a matter of fact, since this implies Q_j is an irreducible factor of N it is one of the N_i and hence there actually is no Q_j for $j > m$).

For the last statement of the theorem, observe that the Koecher algebra is defined because Q admits H_x. Multiplication in $\mathfrak{A}^+$ is $L_u^+ = \frac{1}{2}(L_u + R_u)$, so

$$\begin{aligned}\tau(L_u^+ v, w) &= \tfrac{1}{2}\tau(\{L_u + R_u\}v, w)\\ &= \tfrac{1}{2}\{\tau(L_u v, w) + \tau(v, L_u w)\}\\ &= \tfrac{1}{2}\partial_u\partial_v\partial_w \log Q|_c\end{aligned}$$

by (1.8) and (1.5). But this defines the multiplication in the Koecher algebra, so $\mathfrak{A}^+ = \mathfrak{A}(Q, c)$.

Note that the norms on $\mathfrak{A}$ are precisely all $Q = \prod N_i^{n_i}$ where no n_i is divisible by p (in particular no n_i is 0).

CHAPTER III

1. Isotopes. Throughout this chapter $\mathfrak{A}$ will denote a finite-dimensional commutative Jordan algebra (not necessarily semisimple and Φ not necessarily infinite). We first recall the standard results about isotopes [5], which are all consequences of MacDonald's Theorem. Let u be any element of $\mathfrak{A} = (\mathfrak{X}, \cdot)$. We define a new multiplication on $\mathfrak{X}$ by $x \cdot_u y = (x \cdot u) \cdot y + (u \cdot y) \cdot x - (x \cdot y) \cdot u$. The algebra $\mathfrak{A}^{(u)} = (\mathfrak{X}, \cdot_u)$ has an identity if and only if u is regular, in which case the identity is u^{-1}. In this case we say $\mathfrak{A}^{(u)}$ is the *u-isotope* of $\mathfrak{A}$; $\mathfrak{A}^{(u)}$ is again a Jordan algebra, and the operator $U_x^{(u)}$ is given by

$$U_x^{(u)} = U_x U_u \, . \tag{3.1}$$

The relation of isotopy is symmetric and transitive.

THEOREM 3.1. *Let Q be a form on a finite-dimensional vector space $\mathfrak{X}$ over an infinite field Φ and $c \in \mathfrak{X}$ a point where the Koecher algebra $\mathfrak{A} = \mathfrak{A}(Q, c)$ is defined: thus the trace form τ_c is nondegenerate and $Q(H_x y) = h(x)Q(y)$. If $\tilde{c} \in \mathfrak{X}$ is any point where $\tau_{\tilde{c}}$ is nondegenerate then the Koecher algebra $\tilde{\mathfrak{A}} = \mathfrak{A}(Q, \tilde{c})$ is defined and is the u-isotope of $\mathfrak{A}$ for $u = \tilde{c}^{-1}$. Thus the Koecher algebras corresponding to different basepoints are isotopic.*

Proof. Let $\tau = \tau_c$, $\tilde{\tau} = \tau_{\tilde{c}}$ where $\tau_x(u, v) = -\partial_u \partial_v \log Q\,|_x$ as usual; then $\tau(H_y u, v) = \tau_y(u, v) = \tilde{\tau}(\tilde{H}_y u, v) = \tau_{\tilde{c}}(\tilde{H}_y u, v) = \tau(H_{\tilde{c}} \tilde{H}_y u, v)$ implies $\tilde{H}_y = H_{\tilde{c}}^{-1} H_y$ by the assumed nondegeneracy of $\tau, \tilde{\tau}$. Then $Q(\tilde{H}_x y) = Q(H_{\tilde{c}}^{-1} H_x y) = h(\tilde{c})^{-1} h(x) Q(y) = \tilde{h}(x) Q(y)$ and $\tilde{\mathfrak{A}}$ is defined. From (1.13) we see $\tilde{U}_y = \tilde{H}_y^{-1} = H_y^{-1} H_{\tilde{c}} = U_y U_{\tilde{c}}^{-1}$. Since $\tilde{\tau}$ is defined we must have $Q(\tilde{c}) \neq 0$, so we know $u = \tilde{c}^{-1}$ exists and equals $\tilde{c}\#$ by (1.11), (1.12). Thus $\mathfrak{A}^{(u)}$ is a Jordan algebra with identity $u^{-1} = \tilde{c}$ and $U_y^{(u)} = U_y U_u = U_y U_{\tilde{c}^{-1}} = U_y U_{\tilde{c}}^{-1}$ using (3.1) and Theorem 2.1. But $\tilde{\mathfrak{A}}$ has the same identity and $\tilde{U}_y = U_y U_{\tilde{c}}^{-1} = U_y^{(u)}$, so $\tilde{\mathfrak{A}} = \mathfrak{A}^{(u)}$.

COROLLARY. *If $\mathfrak{A}, \tilde{\mathfrak{A}}$ are commutative normed algebras on $\mathfrak{X}, \tilde{\mathfrak{X}}$ over an infinite field Φ with norms $Q, \tilde{Q}$ respectively, and if $W \colon \mathfrak{X} \to \tilde{\mathfrak{X}}$ is a one-to-one linear transformation such that $\tilde{Q}(Wx) = Q(x)$, then W is an isomorphism of $\mathfrak{A}$ with an isotope of $\tilde{\mathfrak{A}}$.*

Proof. W carries $\mathfrak{A}$ in the natural way isomorphically onto an algebra $\bar{\mathfrak{A}}$ on $\tilde{\mathfrak{X}}$ with identity $\bar{c} = Wc$ and norm $\bar{Q} = Q \circ W^{-1}$. By Theorem 2.9, $\bar{\mathfrak{A}}$ is the Koecher algebra $\mathfrak{A}(\bar{Q}, \bar{c})$ of $\bar{Q}$ with basepoint $\bar{c}$, and $\tilde{\mathfrak{A}}$ is the Koecher algebra $\mathfrak{A}(\tilde{Q}, \tilde{c})$ where $\tilde{c}$ is the identity of $\tilde{\mathfrak{A}}$. But $\bar{Q}(Wx) = Q(x) = \tilde{Q}(Wx)$ and Φ is infinite, so $\bar{Q} = \tilde{Q}$; by Theorem 3.1 $\bar{\mathfrak{A}}$ is the $\bar{c}^{-1}$ isotope of $\tilde{\mathfrak{A}}$. Hence W is an isomorphism of $\mathfrak{A}$ with an isotope of $\tilde{\mathfrak{A}}$.

COROLLARY. *If $\mathfrak{A}$ is a separable Jordan algebra with generic norm $N, |\Phi| > \deg N$, then a one-to-one linear transformation W of $\mathfrak{A}$ into itself is an automorphism of $\mathfrak{A}$ if and only if $Wc = c$ and $N(Wa) = N(a)$ for all $a \in \mathfrak{A}$.*

Proof. Clearly any automorphism satisfies the conditions. Conversely, any such W satisfies $N(Wx) = N(x)$ on an infinite extension by our assumption on $|\Phi|$; if we take $\mathfrak{A} = \tilde{\mathfrak{A}}, Q = \tilde{Q} = N$ the proof of the preceding corollary and the fact that $(Wc)^{-1} = c^{-1} = c$ show that W is an isomorphism of $\mathfrak{A}$ with $\mathfrak{A}^{(c)} = \mathfrak{A}$.

Note that $Q(Wa) = Q(a)$ need not hold for an automorphism W if Q is an arbitrary norm on the separable algebra $\mathfrak{A}$.

2. The group $\mathscr{G}(\mathfrak{A})$. We here generalize some results of Koecher [9, pp. 70–73]; remember that $\mathfrak{A}$ need not be semisimple and Φ need not be infinite. Let $\mathscr{G}(\mathfrak{A})$ be the set of all nonsingular linear transformations W on $\mathfrak{A}$ such that

$$U_{Wx} = WU_xW^*$$

for some linear transformation W^* and all x, and let $\mathscr{U}(\mathfrak{A})$ be the group generated by the U_x for x regular (see Theorem 2.1). One easily verifies that $\mathscr{G}(\mathfrak{A})$ is a group, $W^* \in \mathscr{G}(\mathfrak{A})$ for all $W \in \mathscr{G}(\mathfrak{A})$, and (by (2.1)) $\mathscr{U}(\mathfrak{A})$ is a subgroup; since

$$\begin{aligned} WU_xW^{-1} &= U_{Wx}W^{*-1}W^{-1} \\ &= U_{Wx}(WU_cW^*)^{-1} = U_{Wx}U_{Wc}^{-1} \in \mathscr{U}(\mathfrak{A}) \end{aligned}$$

it is a normal subgroup. If $W \in \mathscr{G}(\mathfrak{A})$ then

$$U_{U(Wx)y} = U_{WU(x)W^*y} = WU_xW^*U_yW^{**}U_xW^* ;$$

but from (2.1) $U_{U(Wx)y} = U_{Wx}U_yU_{Wx} = WU_xW^*U_yWU_xW^*$, so equating gives $W^{**} = W$. $(W_1W_2)^* = W_2^*W_1^*$ is clear, so * is an involution on $\mathscr{G}(\mathfrak{A})$ which leaves the elements U_x fixed. Geometrically, $W^* = \sharp \circ W^{-1} \circ \sharp$ where $\sharp$ is the mapping $x \rightarrow x^{-1} = U_x^{-1}x$:

$$\begin{aligned} W^*x &= W^* U_{x^{-1}}^{-1} x^{-1} \\ &= W^*(W U_{W^{-1}x^{-1}} W^*)^{-1} x^{-1} = U_{W^{-1}x^{-1}}^{-1} W^{-1} x^{-1} \\ &= (W^{-1}x^{-1})^{-1} = \{\# \circ W^{-1} \circ \#\} x \,. \end{aligned}$$

Thus $W^*v = u \Leftrightarrow Wu^{-1} = v^{-1}$.

THEOREM 3.2. *Two isotope $\mathfrak{A}^{(u)}, \mathfrak{A}^{(v)}$ are isomorphic under a linear transformation W if and only if $W \in \mathscr{G}(\mathfrak{A})$ and $W^*v = u$. The group of automorphisms of $\mathfrak{A}$ is the subgroup of $\mathscr{G}(\mathfrak{A})$ fixing the identity c.*

Proof. Suppose $W: \mathfrak{A}^{(u)} \to \mathfrak{A}^{(v)}$ is an isomorphism. Then W takes the identity u^{-1} of $\mathfrak{A}^{(u)}$ onto the identity v^{-1} of $\mathfrak{A}^{(v)}$, so $Wu^{-1} = v^{-1} \Rightarrow W^*v = u$, and by the multiplicative property

$$WL_x^{(u)} = L_{Wx}^{(v)} W \Rightarrow WU_x^{(u)} = U_{Wx}^{(v)} W \Rightarrow WU_x U_u = U_{Wx} U_v W$$

(by (3.1)) $\Rightarrow U_{Wx} = WU_x W^*$ for $W^* = U_u W^{-1} U_v^{-1}$ and all x, so $W \in \mathscr{G}(\mathfrak{A})$.

Conversely, if $W \in \mathscr{G}(A)$ and $W^*v = u$ then for all x $u = W^*v = (U_x^{-1} W^{-1} U_{Wx})v$, so $Wx \cdot_v Wx = U_{Wx}v = WU_x u = W(x \cdot_u x)$; by commutativity this linearizes to $W(x \cdot_u y) = Wx \cdot_v Wy$, and W is an isomorphism.

Applying this to the case $u = v = c$ and noting that $W^*c = c \Leftrightarrow Wc = c$ since $c^{-1} = c$, we see W is an automorphism of $\mathfrak{A} = \mathfrak{A}^{(c)}$ if and only if $W \in \mathscr{G}(\mathfrak{A})$ and $Wc = c$.

Isotopes are not always isomorphic; for example, the algebras $\mathfrak{H}(\mathfrak{D}_n, \gamma)$ for different γ are isotopic but not necessarily isomorphic. However, if Φ is algebraically closed then there is essentially only one $\mathfrak{H}(\mathfrak{D}_n, \gamma)$. This is a special case of the following general result.

THEOREM 3.3. *If Φ is algebraically closed, all isotopes $\mathfrak{A}^{(u)}$ are isomorphic.*

Proof. Since Φ is algebraically closed, every regular element has a regular square root [3, p. 43]. If $\mathfrak{A}^{(u)}$ is an isotope, u is regular by definition, so there is a regular $v \in \mathfrak{A}$ with $v^2 = u$. Then $U_v c = v^2 = u$, and $U_v = U_v^* \in \mathscr{U}(\mathfrak{A}) \subset \mathscr{G}(\mathfrak{A})$ since v is regular, so by Theorem 3.2 U_v is an isomorphism of $\mathfrak{A}^{(u)}$ onto $\mathfrak{A}^{(c)} = \mathfrak{A}$.

This can also be seen without recourse to Theorem 3.2. Let $v^2 = u$ as above; by bilinearity and commutativity it will suffice to demonstrate $U_v(x \cdot_u x) = (U_v x) \cdot (U_v x)$ in order to prove U_v is an isomorphism of $\mathfrak{A}^{(u)}$ onto $\mathfrak{A}$. But $x \cdot_u x = 2x \cdot (x \cdot u) - x^2 \cdot u = U_x u$, so $U_v(x \cdot_u x) = U_v U_x u = U_v U_x v^2 = U_v U_x U_v c = U_{U(v)x} c = (U_v x)^2$ by (2.1).

As a corollary of this result we get a quick proof of the following proposition [3, p. 43].

COROLLARY. *The generic norm of an isotope* $\mathfrak{A}^{(u)}$ *is* $N^{(u)}(x) = N(u)N(x)$.

Proof. It suffices to pass to the algebraic closure. There we can find a square root v of u, and by the above U_v is an isomorphism of $\mathfrak{A}^{(u)}$ onto $\mathfrak{A}$. Under an isomorphism corresponding elements clearly have the same generic norm, so

$$N^{(u)}(x) = N(U_v x) = N(v)^2 N(x) = N(v^2)N(x) = N(u)N(x) .$$

COROLLARY. *If* Φ *is algebraically closed, every* $W \in \mathscr{G}(\mathfrak{A})$ *can be written* $W = UV$ *where* $U \in \mathscr{U}(\mathfrak{A})$ *and* V *is an automorphism.*

Proof. Since $U_{Wc} = WU_c W^*$ is nonsingular, Wc is regular, and hence as above has a regular square root v. Then $U = U_v \in \mathscr{U}(\mathfrak{A})$, and $V = U^{-1}W$ has $V \in \mathscr{G}(\mathfrak{A})$ and $Vc = U^{-1}Wc = U_v^{-1}v^2 = c$, so V is an automorphism by Theorem 3.2.

Suppose now that $\mathfrak{A}$ is separable over an infinite field Φ with generic norm N. We claim that $\mathscr{G}(\mathfrak{A})$ is just the group of normpreserving transformations W such that $N(Wx) = \omega N(x)$ identically with $\omega \in \Phi$. If $N(Wx) = \omega N(x)$ we can apply the differential calculus as in (1.3) to get $\tau_{Wx}(Wu, Wv) = \tau_x(u, v)$, hence $W^* H_{Wx} W = H_x, H_{Wx} = W^{*-1} H_x W^{-1}$, $U_{Wx} = WU_x W^*$ and $W \in \mathscr{G}(\mathfrak{A})$; note that $*$ here is the adjoint relative to the nondegenerate form τ. Conversely, if $W \in \mathscr{G}(\mathfrak{A})$ then by Theorem 3.2 W is an isomorphism of $\mathfrak{A}^{(u)}$ onto $\mathfrak{A}$ for $u = W^*c$. Then $N(Wx) = N^{(u)}(x) = N(u)N(x)$ by the first Corollary, and W is normpreserving.

The group $\mathscr{G}(\mathfrak{A})$ has been computed by N. Jacobson [6, I, II, III] for central simple $\mathfrak{A}$; in each case it is easy to compute W^*.

3. The Lie algebra of $\mathscr{G}(\mathfrak{A})$. As a final example of the usefulness of the differential calculus we give short proofs of some results of N. Jacobson [3, pp. 42, 47, 48] on the Lie algebra of the algebraic group $\mathscr{G}(\mathfrak{A})$. We will assume throughout the section that Φ is infinite. We say a polynomial Q has a linear transformation W as *Lie invariant* if

$$\partial_{Wx} Q \mid_x = 0$$

identically. Such W form a Lie algebra $\mathscr{L}(\mathfrak{A}, Q)$ of linear transformations. Indeed, applying $\partial_y \mid_x$ to the defining relation gives

$$\partial_{Wy}Q\,|_x + \partial_y\partial_{Wx}Q\,|_x = 0\ ;$$

if $W, V \in \mathscr{L}(\mathfrak{A}, Q)$ then

$$\partial_{WVx}Q\,|_x = -\partial_{Vx}\partial_{Wx}Q\,|_x = -\partial_{Wx}\partial_{Vx}Q\,| = \partial_{VWx}Q\,|_x\ ,$$

and $\partial_{[W,V]x}Q\,|_x = 0$ implies $[W, V] \in \mathscr{L}(\mathfrak{A}, Q)$.

THEOREM 3.4. *If N is the generic norm of a Jordan algebra $\mathfrak{A}$ then $L_a \in \mathscr{L}(\mathfrak{A}, N)$ if and only if a has trace zero, ie. $\tau(c, a) = 0$. If $\mathfrak{A}$ is separable then*

$$\mathscr{L}(\mathfrak{A}, N) = \mathscr{L}(\mathfrak{A}') \oplus \mathscr{D}(\mathfrak{A})$$

where $\mathscr{L}(\mathfrak{A}')$ is the space of multiplications L_a by elements of trace zero and $\mathscr{D}(\mathfrak{A})$ is the Lie algebra of derivations.

Proof. For the first assertion it will suffice to prove $\partial_{L(a)y}N\,|_y = N(y)\tau(c, a)$. We apply $\partial_a\,|_c$ to $N(U_xy) = N(x)^2N(y)$ as a function of x. Using the chain rule the left side becomes

$$\partial_aN(U_xy)\,|_c = \partial_bN\,|_{U(c)y} = \partial_bN\,|_y$$

for $b = \partial_a\{U_xy\}\,|_c = 2L_ay$. Using $N(c) = 1$ and the Euler equations the right side becomes

$$\begin{aligned}2N(c)\partial_aN\,|_c\,N(y) &= 2N(y)N(c)^{-1}\partial_aN\,|_c = 2N(y)\partial_a\log N\,|_c\\ &= -2N(y)\partial_c\partial_a\log N\,|_c = 2N(y)\tau(c, a)\ .\end{aligned}$$

Equating gives $2\partial_{L(a)y}N\,|_y = 2N(y)\tau(c, a)$.

For the second assertion, let $W \in \mathscr{L}(\mathfrak{A}, N)$, $Wc = a$. Since $\mathfrak{A}$ is now assumed separable, τ is nondegenerate by Theorem 2.8. Hence we can define $\#$ as in § 4 of Chapter I, and

$$0 = N(x)^{-1}\partial_{Wx}N\,|_x = \partial_{Wx}\log N\,|_x = \tau(x\#, Wx)\ .$$

Putting $x = c$ we get $\tau(c, a) = 0$, so $L_a \in \mathscr{L}(\mathfrak{A}') \subset \mathscr{L}(\mathfrak{A}, N)$. Therefore $D = W - L_a \in \mathscr{L}(\mathfrak{A}, N)$, $Dc = 0$. We will prove D is a derivation. As above $0 = \tau(x\#, Dx)$, so $0 = \partial_y\{\tau(x\#, Dx)\}\,|_x = \tau(x\#, Dy) - \tau(H_xy, Dx)$. By nondegeneracy $D^*x\# = H_xDx$, and $U_xD^*x\# = Dx$ by (1.13). From $U_{x^2} = U_x^2$ (by (2.1)) and $(x^2)\# = x^{-2} = U_x^{-1}c = H_xc$ (by (1.11)-(1.13)) we get

$$\begin{aligned}D(x\cdot x) - 2x\cdot Dx &= U_{x^2}D^*(x^2)\# - 2L_xU_xD^*x\#\\ &= U_x\{U_xD^*H_xc - 2L_xD^*x\#\}\\ &= -U_x\{\partial_c\{U_xD^*x\#\}\,|_x\}\\ &= -U_x\{\partial_cDx\,|_x\} = -U_xDc = 0\ ;\end{aligned}$$

linearizing shows D is a derivation. Hence

$$W = L_a + D \in \mathscr{L}(\mathfrak{A}') + \mathscr{D}(\mathfrak{A}),$$

and we have shown $\mathscr{L}(\mathfrak{A}, N) \subset \mathscr{L}(\mathfrak{A}') + \mathscr{D}(\mathfrak{A})$; but $\mathscr{L}(\mathfrak{A}') \subset \mathscr{L}(\mathfrak{A}, N)$, and $\mathscr{D}(\mathfrak{A}) \subset \mathscr{L}(\mathfrak{A}, N)$ is known [14], so $\mathscr{L}(\mathfrak{A}, N) = L(\mathfrak{A}') + D(\mathfrak{A})$. Finally, $\mathscr{L}(\mathfrak{A}') \cap \mathscr{D}(\mathfrak{A}) = 0$ since $L_a c = a$, $Dc = 0$, so the sum is direct.

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