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TENSOR PRODUCTS OVER H*-ALGEBRAS

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Throughout, A, B, and C denote (semi-simple) H^*-algebras whose respective decompositions into minimal closed ideals are $A = \Sigma \oplus A_\alpha$, $B = \Sigma \oplus B_\beta$, and $C = \Sigma \oplus C_\gamma$. It is assumed that A is a right C-module and B is a left C-module. We define a tensor product $A \otimes_C B$ that is again an H^*-algebra, and show that it is isometric and isomorphic with an ideal in $A \otimes B \otimes C$. As a corollary, $A \otimes_C B$ is strongly semi-simple if A, B, and C are each strongly semi-simple. The converse to the corollary is shown to be false. When A, B, and C are closed ideals in some H^*-algebra, with ordinary multiplication as the module action, then $A \otimes_C B$ is shown to be isomorphic with the direct sum of all the one-dimensional ideals in $A \cap B \cap C$. When $A = L^2(G)$, $B = L^2(H)$, and $C = L^2(K)$, for suitable related compact groups G, H, and K, then the module actions are defined, and $A \otimes_C B$ can be constructed. When $G = H = K$, it is shown that $A \otimes_C B \cong L^2(G/N)$, where N is the closure of the commutator subgroup of G. A conjecture is stated that would generalize this result to the case where K is a closed subgroup of $G \cap H$.

Since $A \otimes_C B$ will be represented in terms of ordinary tensor products $A \otimes B$ of H^*-algebras, the requisite facts concerning $A \otimes B$ are stated here (details may be found in [2]).

$A \otimes B$ is the Hilbert space completion of the space $A \otimes' B$ of all conjugate bilinear functionals T on $A \times B$ of the form $T = \sum_{i=1}^n a_i \otimes b_i$, where $T(a, b) = \Sigma (a_i, a)(b_i, b)$ (see [3]). We define $(a \otimes b)(c \otimes d) = ac \otimes bd$, and extend by linearity and continuity to multiplication on $A \otimes B$. Then

I. $A \otimes B$ is an H^*-algebra and each $A_\alpha \otimes B_\beta$ may be identified with a closed ideal in $A \otimes B$.

II. $A \otimes B = \Sigma \otimes (A_\alpha \otimes B_\beta)$ is the decomposition of $A \otimes B$ into minimal closed ideals.

III. $A \otimes B$ is strongly semi-simple (see [5], p. 59) if and only if both A and B are strongly semi-simple.

1. Tensor products.

DEFINITION. $F_C(A, B)$ will denote the collection of all finite formal

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sums of the form

$\sum_{i=1}^{n} c_i(a_i, b_i)$, with $a_i \in A, b_i \in B$, and $c_i \in C$; i.e. $F_C(A, B)$ is the free C-module generated by $A \times B$.

$F_C(A, B)$ becomes an algebra and a pseudo-inner product space if the operations are defined by the formulas:

$$(c(a, b))\cdot(c'(a', b')) = cc'(aa', bb') \ ,$$
$$\lambda \Sigma c_i(a_i, b_i) = \Sigma(\lambda c_i)(a_i, b_i),\ \lambda \text{ complex, and}$$
$$(c(a, b), c'(a', b')) = (c, c')(a, a')(b, b')$$

(the first and third must be extended by linearity). The positive semi-definiteness of the pseudo-inner product follows from the fact that $(c(a, b), c'(a', b')) = (a \otimes b \otimes c, a' \otimes b' \otimes c')$; the other properties required of an inner product obviously hold.

Let I_1' be the ideal in $F_C(A, B)$ spanned by the set of all elements of the following forms:

$$c(a_1 + a_2, b) - c(a_1, b) - c(a_2, b) \ , \tag{1}$$
$$c(a, b_1 + b_2) - c(a, b_1) - c(a, b_2) \ , \tag{2}$$
$$(c_1 + c_2)(a, b) - c_1(a, b) - c_2(a, b) \ , \tag{3}$$
$$\lambda c(a, b) - c(\lambda a, b) \ , \text{ and} \tag{4}$$
$$\lambda c(a, b) - c(a, \lambda b) \tag{5}$$

for arbitrary $a, a_i \in A$; $b, b_i \in B$; $c, c_i \in C$; and complex numbers λ. Let I_2' be the ideal in $F_C(A, B)$ generated by the set of all elements of the forms:

$$c_1c_2(a, b) - c_1(ac_2, b) \ , \text{ and} \tag{6}$$
$$c_1c_2(a, b) - c_1(a, c_2b) \tag{7}$$

for arbitrary $a \in A$, $b \in B$, and $c_i \in C$. Then let $I' = I_1' \vee I_2' = I_1' + I_2'$, the ideal generated by the set of all elements of the forms (1)–(7).

PROPOSITION 1. $I_1' = \{X \in F_C(A, B): (X, X) = 0\}$.

Proof. Straightforward computations show that $(X, Y) = 0$ if X is of one of the forms (1)–(5) and $Y = c'(a', b')$. Extending by linearity we have immediately that $(X, Y) = 0$ for all $X \in I_1'$, $Y \in F_C(A, B)$. Suppose then that $X = \sum_{i=1}^{n} c_i(a_i, b_i)$ and that $(X, X) = 0$. It must be shown that $X \in I_1'$.

If $\{c_i\}_{i=1}^{n}$ is not linearly independent, then we may assume that $c_n = \sum_{i=1}^{n-1} \lambda_i c_i$, and so

$$\begin{aligned} X &= \sum_{i=1}^{n-1} c_i(a_i, b_i) + \Big(\sum_{i=1}^{n-1} \lambda_i c_i\Big)(a_n, b_n) \\ &= \sum_{i=1}^{n-1} c_i(a_i, b_i) + \sum_{i=1}^{n-1} c_i(\lambda_i a_n, b_n) \\ &\quad + \Big[\Big(\sum_{i=1}^{n-1} \lambda_i c_i\Big)(a_n, b_n) - \sum_{i=1}^{n-1} c_i(\lambda_i a_n, b_n)\Big] . \end{aligned}$$

The expression in brackets is clearly an element of I_1', call it γ_1. Thus we have

$$X = \sum_{j=1}^{2} \sum_{i=1}^{n-1} c_i(a_{ij}, b_{ij}) + \gamma_1 ,$$

where $a_{i1} = a_i$, $a_{i2} = \lambda_i a_n$, $b_{i1} = b_i$, $b_{i2} = b_n$. Repeating the process as many times as is necessary we obtain

$$X = \sum_{j=1}^{2^p} \Big(\sum_{i=1}^{n-p} c_i(a_{ij}, b_{ij})\Big) + \gamma_p ,$$

where $\gamma_p \in I_1'$ and $\{c_i\}_{i=1}^{n-p}$ is linearly independent. Then, for each fixed index i, by using an argument similar to the one above, we can write

$$\sum_{j=1}^{2^p} c_i(a_{ij}, b_{ij}) = \sum_{k=1}^{2^{q(i)}} \Big(\sum_{j=1}^{2^p - q(i)} c_i(a_{ij}, b_{ijk})\Big) + \gamma_{iq(i)} ,$$

where $\gamma_{iq(i)} \in I_1'$ and $\{a_{ij}: j = 1, \cdots, 2^p - q(i)\}$ is linearly independent. As a result, we have

$$X = \sum_{i=1}^{n-p} \sum_{j=1}^{2^p - q(i)} \sum_{k=1}^{2^{q(i)}} c_i(a_{ij}, b_{ijk}) + \gamma ,$$

where $\{c_i\}$ is linearly independent, $\{a_{ij}\}$ is linearly independent for each fixed i, and $\gamma \in I_1'$.

Fix any pair $< i, j >$ of indices. By the Hahn-Banach Theorem and the Riesz Theorem there exist $a' \in A$ and $c' \in C$ such that

$$\| c' \| = \| a' \| = 1,\ (c_i, c') = d_i > 0, (a_{ij}, a') = d_{ij} > 0 ,$$

$(c_{i'}, c') = 0$ if $i' \neq i$, and $(a_{ij'}, a') = 0$ if $j' \neq j$. Since $F_C(A, B)$ is a pseudo-inner product space, the Schwarz inequality holds. Thus if we let $b' = \Sigma\{b_{ijk}: k = 1, \cdots, 2^{q(i)}\}$, we have

$$| (X, c'(a', b') | \leqq (X, X)(c'(a', b'), c'(a', b')) = 0 .$$

On the other hand,

$$\begin{aligned} (X, c'(a', b')) &= \textstyle\sum_{m,n,k}(c_m, c')(a_{mn}, a')(b_{mnk}, b') \\ &= d_i d_{ij} \| b' \|^2 = 0 , \end{aligned}$$

so that $b' = 0$. If we now write

$$\begin{aligned}\textstyle\sum_k c_i(a_{ij}, b_{ijk}) &= c_i(a_{ij}, \textstyle\sum_k b_{ijk}) + [\textstyle\sum_k c_i(a_{ij}, b_{ijk}) - c_i(a_{ij}, \textstyle\sum_k b_{ijk})] \\ &= c_i(a_{ij}, 0) + \gamma'_{ij} ,\end{aligned}$$

where γ'_{ij} is the expression in brackets, which is clearly an element of I'_1, then we have

$$X = \textstyle\sum_{i,j} c_i(a_{ij}, 0) + \gamma' ,$$

where $\gamma' = \sum_{i,j}\gamma'_{ij}$, and so $X \in I'_1$.

$F_C(A, B)$ is a pseudo-normed space, with $\| X \|^2 = (X, X)$. Let us denote by $\mathscr{F}_C(A, B)$ its pseudo-normed completion, i.e. the collection of all Cauchy sequences from $F_C(A, B)$. Define a mapping

$$\varphi\colon F_C(A, B) \to A \otimes B \otimes C$$

as follows:

$$\varphi(\Sigma\, c_i(a_i, b_i)) = \Sigma\, a_i \otimes b_i \otimes c_i .$$

It is immediate that φ is a linear, homogeneous, multiplicative isometry, and that its range is dense. Thus φ can be extended to an isometric homomorphism on $\mathscr{F}_C(A, B)$ onto $A \otimes B \otimes C$. Note that $\| XY \| \leqq \| X \| \| Y \|$ for all $X, Y \in F_C(A, B)$, since $A \otimes B \otimes C$ is a Banach algebra. Thus the operations defined on $F_C(A, B)$ can be extended to $\mathscr{F}_C(A, B)$, as usual.

Let I_1, I_2, and I denote the closures, in $\mathscr{F}_C(A, B)$, of I'_1, I'_2, and I', respectively. It is obvious from Proposition 1 that $I_1 = \{X \in \mathscr{F}_C(A, B)\colon \| X \| = 0\}$, i.e. i.e. I_1 is the closure of (0). Thus I_1 is a subset of every closed subspace of $\mathscr{F}_C(A, B)$, which means, in particular, that $I = I_2$. In other words, I can be described quite simply as the closed ideal of $\mathscr{F}_C(A, B)$ generated by the collection of all elements of the forms (6) and (7).

DEFINITION. $A \otimes_C B$, the tensor product of A and B, over C, is the quotient algebra $\mathscr{F}_C(A, B)/I$.

$A \otimes_C B$ is a normed space (as is always the case when a pseudo-normed space is factored by a closed subspace). We proceed to identify it with an ideal in $A \otimes B \otimes C$. Let $D = \varphi(I)$ and define a map $\gamma\colon A \otimes_C B \to (A \otimes B \otimes C)/D$ by the formula $\gamma(X + I) = \varphi(X) + D$. It is clear that γ is linear, and since $\gamma(I) = \varphi(0) + D = D, \gamma$ is well defined; it is multiplicative since φ is multiplicative. Finally, γ is an isometry. For if $T = X + I \in A \otimes_C B$, then

$$\begin{aligned}\| \gamma T \| = \| \varphi X + D \| &= \inf \{\| \varphi X + Z \|\colon Z \in D\} \\ &= \inf \{\| \varphi X + \varphi Y \|\colon Y \in I\} \\ &= \inf \{\| X + Y \|\colon Y \in I\} = \| T \| ,\end{aligned}$$

since φ is an isometric homomorphism.

Since D is a closed ideal in the H^*-algebra $A \otimes B \otimes C$, $(A \otimes B \otimes C)/D$ is isomorphic and isometric with the closed ideal $D^{\perp}$, which we shall denote by E. We summarize the foregoing information in the next theorem.

THEOREM. *There is an isometric isomorphism from $A \otimes_C B$ into $A \otimes B \otimes C$; its range is the closed ideal E which is the orthogonal complement of the closed ideal D generated by all elements of the forms*

(i) $a \otimes b \otimes c_1 c_2 - a c_2 \otimes b \otimes c_1$,

(ii) $a \otimes b \otimes c_1 c_2 - a \otimes c_2 b \otimes c_1$.

Consequently, $A \otimes_C B$ is an H^*-algebra; its minimal closed ideals can be identified with those minimal closed ideals $A_\alpha \otimes B_\beta \otimes C_\gamma$ of $A \otimes B \otimes C$ that are orthogonal to D.

COROLLARY. *If A, B, and C are strongly semi-simple, then $A \otimes_C B$ is strongly semi-simple.*

The following proposition provides means by which it is easy to construct examples for which the converse to the above corollary is false.

PROPOSITION 2. If $A_\alpha \otimes B_\beta \otimes C_\gamma$ is a minimal closed ideal in E, then C_γ is of dimension one.

Proof. Choose a canonical basis $\{a_{ij} \otimes b_{kl} \otimes c_{mn}\}$ for $A_\alpha \otimes B_\beta \otimes C_\gamma$ (see [2]). Since $a_{ij} \otimes b_{kl} \otimes c_{mn} \in E$, it must be orthogonal to

$$a_{ij} \otimes b_{kl} \otimes c_{mp} c_{pn} - a_{ij} c_{pn} \otimes b_{kl} \otimes c_{mp} .$$

If the dimension of C_γ were greater than one, then it would be possible to choose $n \neq p$, and we would have

$$\begin{aligned} 0 &= (a_{ij} \otimes b_{kl} \otimes c_{mn}, a_{ij} \otimes b_{kl} \otimes c_{mn} - a_{ij} c_{pn} \otimes b_{kl} \otimes c_{mp} \\ &= \| a_{ij} \|^2 \| b_{kl} \|^2 \| c_{mn} \|^2 , \end{aligned}$$

since $(c_{mn}, c_{mp}) = 0$. This, of course, is a contradiction.

COROLLARY. *If C has no one-dimensional minimal ideals, then $A \otimes_C B = (0)$.*

2. Examples. Perhaps the easiest method of obtaining examples of H^*-algebras A, B, and C related as above is to let A, B, and C be closed ideals in some H^*-algebra $\mathscr{A}$. The structure of $A \otimes_C B$, under such circumstances, is described in the next proposition.

PROPOSITION 3. Suppose that A, B and C are closed ideals in an H^*-algebra $\mathscr{A}$. If A and B are viewed as C-modules with ordinary multiplication in $\mathscr{A}$ as the module action, then $A \otimes_C B$ is isomorphic with the direct sum of all the one-dimensional minimal ideals in $A \cap B \cap C$. The isomorphism is an isometry if and only if the identity of each one-dimensional minimal ideal in $A \cap B \cap C$ has norm one.

Proof. Choose a canonical basis $\{u_{pq}^{\delta}\}$ for $\mathscr{A}$. Then $\{a_{ij}\} = A = \cap \{u_{pq}^{\delta}\}$, $\{b_{kl}^{\beta}\} = B \cap \{u_{pq}^{\delta}\}$, and $\{c_{mn}^{\gamma}\} = C \cap \{u_{pq}^{\delta}\}$ are canonical bases for A, B, and C, respectively and $\{a_{ij}^{\alpha} \otimes b_{kl}^{\beta} \otimes c_{mn}^{\gamma}\}$ is a canonical basis for $A \otimes B \otimes C$. If $a_{ij}^{\alpha} \otimes b_{kl}^{\beta} \otimes c_{mn}^{\gamma} \in E$, then, by Proposition 2, $c_{mn}^{\gamma} = c^{\gamma}$ is the identity of a one-dimensional minimal ideal. If $\alpha \neq \gamma$, then

$$a_{ij}^{\alpha} \otimes b_{kl}^{\beta} \otimes c^{\gamma} c^{\gamma} - a_{ij}^{\alpha} c^{\gamma} \otimes b_{kl}^{\beta} \otimes c^{\gamma} = a_{ij}^{\alpha} \otimes b_{kl}^{\beta} \otimes c^{\gamma} \in D .$$

Similarly, if $\beta \neq \gamma$, then $a_{ij}^{\alpha} \otimes b_{kl}^{\beta} \otimes c^{\gamma} \in D$. Thus if an element of a canonical basis is to be in E it must be of the form $c^{\gamma} \otimes c^{\gamma} \otimes c^{\gamma}$. Relatively straightforward computations show that each such basis element is orthogonal to D, and the proof is completed.

Suppose now that G, H, and K are compact groups, and that $\theta\colon K \to G$ and $\varphi\colon K \to H$ are continuous homomorphisms. Then $\theta(K)$ and $\varphi(K)$ are closed subgroups of G and H, respectively, $L^2(G)$ and $L^2(H)$ become modules over $L^2(K)$, with the module action defined by:

$$g * k(x) = \int_K g(x(\theta z)^{-1}) k(z) dz \ ,$$

$$k * h(y) = \int_K k(z) h((\varphi z)^{-1} y) dz \ ,$$

for all $g \in L^2(G)$, $h \in L^2(H)$, $k \in L^2(K)$, $x \in G$, and $y \in H$ (all integrations are with respect to normalized Haar measures). If we let $A = L^2(G)$, $B = L^2(H)$, $C = L^2(K)$, then $A \otimes_C B$ is a well-defined H^*-algebra. As was remarked in [2], $A \otimes B \otimes C$ can be identified with $L^2(G \times H \times K)$, and so, by the Theorem of §1, $A \otimes_C B$ can be identified with a closed ideal J in $L^2(G \times H \times K)$. At one extreme, suppose θ and φ map K onto the identities of G and H, respectively. It is not difficult to see that in this case $A \otimes_C B$ can be identified with $L^2(G \times H)$.

At what might be considered another extreme, suppose that G and H are closed subgroups of some compact group, that K is a closed subgroup of $G \cap H$, and that θ and φ are the inclusion maps. Define an equivalence relation on $G \times H \times K$ as follows: $(x, y, z) \sim (u, v, w)$

if and only if $F(x, y, z) = F(u, v, w)$ for all $F \in J$. Then $M = \{(x, y, z): (x, y, z) \sim (e, e, e)\}$ is a closed normal subgroup of $G \times H \times K$, and its cosets are the equivalence classes of $\sim$. All functions $F \in J$ are thus constant on the cosets of M, providing a mapping ψ from J to $L^2((G \times H \times K)/M)$. The map ψ is an isometric isomorphism and its image is an ideal. On the basis of the Tannaka Duality Theorem (see [4], p. 439) it seems reasonable to conjecture that ψ is surjective, so that $A \otimes_C B$ can be identified with $L^2((G \times H \times K)/M)$. The conjecture has not been settled in general, but let us consider the very special case where $G = H = K$. Then, by Proposition 3, $A \otimes_C B$ can be identified with the direct sum of all one-dimensional minimal ideals in $L^2(G)$, which in turn is isomorphic and isometric with $L^2(G/N)$, where N is the closure of the commutator subgroup of G. Since G/N and $(G \times G \times G)/M$ are isomorphic via the mapping $xN \to (x, e, e)M$, the conjecture is verified in this special case.

REFERENCES

1. B. Gelbaum, *Tensor products over Banach algebras*, to appear in Trans. Amer. Math. Soc.

2. L. Grove, *A generalized group algebra for compact groups*, to appear in Studia Mathematica, vol. **26**, Fascicule 1.

3. F. Murray, and J. von Neumann. *On rings of operators I*, Ann. of Math. **37** (1936), 116–229.

4. M. Naimark, *Normed Rings*, P. Noordhoff, Groningen, the Netherlands, 1959.

5. C. Rickart, *General Theory of Banach Algebras*, Van Nostrand, New York, 1960.

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