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ON A CONJECTURE OF R. J. KOCH

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Dedicated to Professor Alexander Doniphan Wallace on the occasion of his sixtieth birthday

R. J. Koch proved that if X is a compact, continuously partially ordered space and if W is an open subset of X which has no local minima, then each point of W is the supremum of an order arc which meets $X - W$. More recently he extended this result to quasi ordered spaces in which the sets $E(x) = \{y: x \leqq y \leqq x\}$ are assumed to be totally disconnected and W is a chain. He conjectured that the latter hypothesis is superfluous, and we show here that Koch's conjecture is correct.

As a corollary it follows that if X is a compact, continuously quasi ordered space with zero (i.e., a unique minimal element), if each set $E(x)$ is totally disconnected, and if each set $L(x) = \{y: y \leqq x\}$ is connected, then X is arcwise connected.

We begin by recalling a few definitions (see [1], [2], [3] and [4]). We say that $X = (X, \Gamma)$ is a continuously quasi ordered space provided X is a Hausdorff space, Γ is a quasi order (= reflexive, transitive relation) on X and the graph of Γ is a closed subset of $X \times X$. We identify Γ with its graph and regard the symbols $x \leqq y$, and $x \Gamma y$ and $(x, y) \in \Gamma$ as synonyms.

A *chain* of a quasi ordered space X is a subset C of X such that $a \leqq b$ or $b \leqq a$ holds for each a and b in C. We also define

$$L(a, \Gamma) = \{x \in X: (x, a) \in \Gamma\} ,$$
$$M(a, \Gamma) = \{x \in X: (a, x) \in \Gamma\} ,$$
$$E(a, \Gamma) = L(a, \Gamma) \cap M(a, \Gamma) ,$$

for each $a \in X$. It is also convenient to define

$$I(a, b, \Gamma) = M(a, \Gamma) \cap L(b, \Gamma) ,$$

the closed "interval" from a to b. Where there is no ambiguity we shall write $(L(a)$ (resp., $M(a)$, $E(a)$, $I(a, b))$ for $L(a, \Gamma)$, (resp., $M(a, \Gamma)$, $E(a, \Gamma)$, $I(a, b, \Gamma)$). It is well known [3] that if X is a continuously quasi ordered space then the sets $L(a)$, $M(a)$, $E(a)$ and $I(a, b)$ are closed and, if X is compact, then X contains a minimal element, that is, an element m such that $L(m) - E(m)$ is empty.

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A subset Y of the quasi ordered space (X, Γ) is said to *have no local Γ-minima* if, for each $x \in Y$ and each neighborhood U of x, the set

$$Y \cap U \cap L(x, \Gamma) - E(x, \Gamma)$$

is nonempty. This definition is due to Koch [2].

In case the relation Γ is a partial order, it is known that a connected chain joining two distinct points is an arc. (Here we use the term *arc* to describe a continuum with precisely two non-cutpoints.) An arc which is also a chain is termed an *order arc*.

The following two lemmas will be of later use.

LEMMA 1. *Let X be a compact, continuously quasi ordered space, let a and b be members of X, and let K be a closed subset of X such that $I(a, b) \cap K = 0$. Then there exist open sets U and V such that $a \in U$, $b \in V$ and for each $a' \in U$ and $b' \in V$ it follows that $I(a', b') \cap K = 0$.*

Proof. Suppose, on the contrary, that for all neighborhoods U and V of a and b, respectively, there exists $a' \in U$ and $b' \in V$ such that $I(a' b') \cap K \neq 0$. Then

$$\Gamma \cap (\bar{U} \times K) \cap (K \times \bar{V}) \neq 0.$$

These sets form a family of nonempty closed sets with the finite intersection property and hence their intersection is nonempty:

$$\Gamma \cap (\{a\} \times K) \cap (K \times \{b\}) \neq 0 ,$$

that is to say, $I(a, b) \cap K \neq 0$, contrary to the hypothesis.

LEMMA 2. *If R is an open subset of the compact, continuously quasi ordered space X, then the set*

$$F = \{(a, b) \in X \times X\colon I(a, b) - R \neq 0\}$$

is closed.

Proof. If $(a, b) \notin F$ then $I(a, b) \cap (X - R) = 0$. By Lemma 1, there are open sets U and V with $a \in U$ and $b \in V$ such that for each $a' \in U$ and $b' \in V$ it follows that $I(a', b') \subset R$, and hence $(U \times V) \cap F = 0$. Therefore, F is closed.

2. Koch's theorem for quasi ordered spaces. The crux of our proof is embodied in the following theorem.

THEOREM. *Let $X = (X, \Gamma)$ be a compact, continuously quasi* *ordered space and let W be an open subset of X. If*

(i) *$E(x, \Gamma)$ is totally disconnected for each $x \in X$,*

(ii) *W has no local Γ-minima, then X admits a minimal quasi order which has a closed graph and satisfies* (i) *and* (ii). *Moreover, this minimal quasi order is a partial order.*

Proof. Let $\{\Gamma_\alpha\}$ be a maximal nest of quasi orders on X such that each Γ_α has a closed graph and satisfies (i) and (ii), and let $\Gamma = \cap \{\Gamma_\alpha\}$. Clearly (X, Γ) is a continuously quasi ordered space and $E(x, \Gamma)$ is totally disconnected. We will show that W has no local Γ-minima.

Let $x \in W$ and let U be a neighborhood of x; since W is open and $E(x, \Gamma)$ is totally disconnected, we may assume that $U \subset W$ and that $E(x, \Gamma) \cap U$ is closed. Since X is normal there exist open sets V and R such that

$$E(x, \Gamma) \cap U \subset V \subset \bar{V} \subset U\,,$$
$$X - U \subset R \subset \bar{R} \subset X - \bar{V}\,.$$

For each α, the compact set $L(x, \Gamma_\alpha) \cap \bar{V}$ has a Γ_α-minimal element which we denote x_α. And since W has no local Γ_α-minima there exists

$$y_\alpha \in (X - \bar{R}) \cap L(x_\alpha, \Gamma_\alpha) - E(x_\alpha, \Gamma_\alpha)\,.$$

It follows that

$$y_\alpha \in L(x, \Gamma_\alpha) - \bar{R} \cup \bar{V}$$

so that the sets $L(x, \Gamma_\alpha) - R \cup V$ are compact, nonempty and nested. Consequently there exists

$$y \in L(x, \Gamma) - R \cup V$$

and it is clear that $y \notin E(x, \Gamma)$. That is, W has no local Γ-minima.

Now suppose that Γ is not a partial order; then there exists a nondegenerate set $E(x, \Gamma)$. Since $E(x, \Gamma)$ is compact and totally disconnected, there exist nonempty, closed and disjoint sets A and B whose union is $E(x, \Gamma)$. Since X is normal there exist disjoint open sets P and Q such that $A \subset P$ and $B \subset Q$. Let

$$F = \{(a, b) : I(a, b) - P \cup Q \neq 0\}\,.$$

By Lemma 2, F is a closed subset of $X \times X$ and hence

$$\Delta = \Gamma - ((P \times Q) - F)$$

is also closed. Since P and Q are disjoint, Δ is a reflexive relation on X.

We claim that Δ is a quasi order. For suppose $p \Delta q$ and $q \Delta r$ but $(p, r) \in (X \times X) - \Delta$. Now $(p, r) \in \Gamma$ so that $(p, r) \in (P \times Q) - F$ and hence $q \in P$ or $q \in Q$. If $q \in P$ then, since $r \in Q$ and $(q, r) \in \Delta$ we infer that $(q, r) \in F$ and thus $I(q, r) - P \cup Q \neq 0$. But $I(q, r) \subset I(p, r)$ and hence $I(p, r) - P \cup Q \neq 0$, contrary to the fact that $(p, r) \in (P \times Q) - F$. A similar contradiction ensues if $q \in Q$, and thus Δ is a quasi order.

Since $\Delta \subset \Gamma$ it is obvious that each set $E(x, \Delta)$ is totally disconnected. Now suppose $z \in W$ and that O is a neighborhood of z, $O \subset W$. If $z \in W - Q$ then

$$L(z, \Delta) = L(z, \Gamma)$$

and hence there exists

$$y \in O \cap L(z, \Delta) - E(z, \Delta) .$$

And if $z \in Q$, the fact that W has no local Γ-minima insures the existence of

$$y \in O \cap Q \cap L(z, \Gamma) - E(z, \Gamma) .$$

But $y \notin P$ implies $y \in L(z, \Delta)$, so that in any event W has no local Δ-minima.

Finally we note that Δ contradicts the minimality of Γ, for if $a \in A$ and $b \in B$ then $(a, b) \in \Gamma - \Delta$. Therefore Γ is a partial order.

COROLLARY 1. *Let X be a compact, continuously quasi ordered space and let W be an open subset of X. If conditions* (i) *and* (ii) *of the theorem are satisfied, then each point of W is the supremum of an order arc which meets $X - W$.*

Proof. By the preceeding theorem we may assume that the quasi order is a partial order. Thus Koch's theorem for partially ordered spaces applies.

An element 0 of the quasi ordered space X is a *zero* of X provided

$$0 = E(0) = \cap \{L(x) \colon x \in X\} .$$

COROLLARY 2. *If X is a compact, continuously quasi ordered space with zero, if each set $E(x)$ is totally disconnected and if each set $L(x)$ is connected, then X is arcwise connected.*

Proof. Let $W = X - \{0\}$; the connectedness of the sets $L(x)$ guarantees that W has no local minima and therefore each point of W lies in arc containing 0.

Following Koch we say that a subset C of the quasi ordered space X is *biconnected* if C is connected and if each of the sets $E(x) \cap C$ is connected.

COROLLARY 3. *Let X be a compact, continuously quasi ordered space and suppose there exists $a \in X$ such that*

$$E(a) = \cap \{L(x): x \in X\} .$$

If $X - E(a)$ has no local minima then each element of X can be joined to $E(a)$ by a biconnected chain.

Proof. Let Z denote the compact, continuously partially ordered space which is obtained when $E(x)$ is identified with a point, for each $x \in X$. Let $\phi(X) = Z$ be the canonical quotient map and let

$$X \xrightarrow{m} Y \xrightarrow{l} Z$$

be the monotone-light factorization of ϕ. It is easy to see that Y inherits a quasi order from Z which has a closed graph and is such that $E(y)$ is totally disconnected, for each $y \in Y$. Moreover, Y-$m(E(\mathrm{a}))$ has no local minima and hence, by the theorem, there are order arcs joining points of Y to $m(E(a))$. Since m is monotone, the corollary follows at once.

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