PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 4, 1965

SOME CONSIDERATIONS ON CONVERGENCE IN ABELIAN LATTICE-GROUPS

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We define α-convergence in an abelian l-group as follows: The net $(x_i)_{i\in I}$ α-converges to x if x is the only element such that $x = \bigvee_{i\geqq i_0}(x_i \wedge x) = \bigwedge_{i\geqq i_0}(x_i \vee x)$ for every $i_0 \in I$. In an Archimedean l-group (x_i) α-converges to x if and only if for every a and b the net $(a\vee x_i)\wedge b$ order-converges (in the ordinary sense) to $(a \vee x) \wedge b$. In general α-convergence is weaker than this latter condition and is considerably more natural in the non-Archimedean case. The algebraic operations of an arbitrary abelian l-group G are continuous relative to α-convergence. If G is completely distributive its α-convergence derives from a Hausdorff group-topology. Three sufficient conditions are given for the preservation of the α-convergence of an l-group G when it is embedded in another l-group E. In an appendix, we formulate a necessary and sufficient condition in order that an abstract sequential convergence derive from a topology.

The present paper is supplementary to [9] and concludes the investigation begun there. We note here that α-convergence is weaker than the concepts of convergence studied in that paper. In the next few paragraphs we review briefly some of the basic definitions and recall some of the results of [9] which will be needed below. The elementary theory of lattice groups is assumed known; we refer the reader to [2, Chap. XIV] or [4]. We shall employ the additive notation and use the standard abbreviation "l-group" for "lattice-group."

If A is a subset of an l-group G and if A has a least upper bound in G, we shall denote this l.u.b. by $\bigvee^{(G)}_{a\in A} a$ or $\sup^{(G)} A$; dually the g.l.b. is denoted by $\bigwedge^{(G)}_{a\in A} a$ or $\inf^{(G)} A$. In the case of a family $(x_\alpha)_{\alpha\in I}$ the notation is $\bigvee^{(G)}_{\alpha\in I} x_\alpha$ or $\sup^{(G)}\{x_\alpha : \alpha \in I\}$ and dually for greatest lower bounds. We shall omit subscripts and superscripts whenever confusion is unlikely. The term "positive" will be used for "$\geqq 0$." Throughout the present paper R will denote the real line, R^X (where X is an arbitrary set) the l-group of all real functions on X. M will be reserved for the l-group of all *bounded* real functions on [0, 1].

In [9] we investigated several types of order-convergence, the main ones being o-convergence, natural convergence and L-convergence. We repeat here the definitions of the latter two. Let G be an abelian

Received October 8, 1964. This research was supported by the National Science Foundation under Grant NSF GP-96.

The author is indebted to the referee for many valuable remarks and for suggesting the inclusion of references [1], [3], [4] and [10].

l-group and let $(x_i)_{i\in I}$ be a directed net in G (in the sense of [6]). An element $u \in G$ is said to be a *superelement* of (x_i) if $x_i \leqq u$ eventually; a *subelement* is defined dually. The net (x_i) is said to be *eventually bounded in* G if it has a superelement and a subelement. For ordinary sequences "bounded" and "eventually bounded" are equivalent.

We say that an eventually bounded net (x_i) *converges naturally to* $x \in G$ *relative to* G (denoted: $\nu\text{-}\lim^{(G)}_{i\in I} x_i = x$) if $\inf^{(G)} U = \sup^{(G)} V = x$, where U is the set of superelements and V the set of subelements of (x_i) in G. The operations $+$, $\vee$, $\wedge$, etc., of the l-group G are continuous with respect to this convergence.

If M is the l-group of all bounded real functions on $[0, 1]$ and if we define $\sigma_n(x) = n^2x(1-x^2)^n$, $x \in [0, 1]$, $n = 1, 2, \cdots$ then the sequence (σ_n) is pointwise convergent to 0 but is not eventually bounded in M. It is therefore natural to extend our definition of convergence so as to obtain non-eventually-bounded convergent nets, and one way to do this (discussed in § 7 and § 8 of [9]) is the following:

1.1. DEFINITION. The net $(x_i)_{i\in I}$ *L-converges to x relative to G* (denoted: $L\text{-}\lim^{(G)}_{i\in I} x_i = x$) if and only if for each pair $a, b \in G$ $\nu\text{-}\lim^{(G)}_{i\in I} (a \vee x_i) \wedge b = (a \vee x) \wedge b$.

1.2. PROPOSITION. ([9, Prop. 7.2]). *$L\text{-}\lim^{(G)} x_i = x$ if and only if for every $b \geqq 0$ in G $\nu\text{-}\lim^{(G)} | x_i - x | \wedge b = 0$.*

Our L-convergence is not related to Rennie's L-topology ([10]).

The following lemma, which will be needed later, is contained in Lemma 6.3 of [9].

1.3. LEMMA. *Let G be an Archimedean l-group, a an element of G and $(x_i)_{i\in I}$ a net in G which is eventually bounded. Then the following statements are equivalent:*

(i) $\bigvee_{i \geqq i_0} (x_i \wedge a) = a$ *for every* $i_0 \in I$.

(ii) $a \leqq u$ *for every superelement* u *of* (x_i).

2. **The α-convergence in an abelian lattice-group.** The sequence (σ_n) defined above L-converges to 0 in M and in this respect Definition 1.1 is effective. Suppose however that we embed M in the non-Archimedean l-group $J \circ M$ (where J denotes the ordered group of integers and $\circ$ denotes lexicographic product) by means of the "canonical" mapping $f \to (0, f)$. The sequence $(0, \sigma_n)$ is now bounded in $J \circ M$ and our trick fails: $(0, \sigma_n)$ is not L-convergent in $J \circ M$.

There is however another, more intrinsic, way of describing the pointwise convergence of (σ_n) in M which remedies the defect in this particular case. This is achieved by means of Def. 2.1 below; this definition may seem a little sophisticated at first but is in fact very natural, as a closer examination will show.

2.1. DEFINITION. The net $(x_i)_{i\in I}$ *α-converges to $x \in G$ relative to G* (denoted: $\alpha\text{-}\lim^{(G)}_{i\in I} x_i = x$) if x is the only element of G satisfying:

$$(1) \qquad x = \bigvee_{i\geqq i_0}^{(G)} (x_i \wedge x) = \bigwedge_{i\geqq i_0}^{(G)} (x_i \vee x) \quad \text{for every } i_0 \in I .$$

Compare Def. 2.1 with Löwig's Thm. 42 in [7]; see also Lemma 1.3 above. It will be convenient to call an element x satisfying (1) a *central element of* (x_i) *relative to* G. The definition then reads: $\alpha\text{-}\lim^{(G)} x_i = x$ if and only if x is the only central element *of* (x_i) *relative to* G.

Before studying α-convergence and its connection to L-convergence, we introduce another concept of convergence for purposes of comparison only and as an auxiliary tool. In fact, it proves to be very defective in the case of abelian l-groups, despite the fact that it arises from a close imitation of the method so successfully employed by H. Löwig in the case of Boolean rings. If a net (x_i) is eventually bounded in G, then an element $x \in G$ is said to be an *interelement* of (x_i) relative to G if $v \leqq x \leqq u$ for every subelement v and every superelement u of (x_i) in G. If (x_i) is not eventually bounded, then x is said to be an interelement of (x_i) relative to G if and only if for every a, b $(a \vee x) \wedge b$ is an interelement of $(a \vee x_i) \wedge b$ in the preceding sense.

DEFINITION. The net (x_i) *L^*-converges to x relative to G* (denoted: $L^*\text{-}\lim^{(G)} x_i = x$) if x is the only interelement of (x_i) relative to G.

If (x_i) is eventually bounded then $\nu\text{-}\lim x_i = x$, $L\text{-}\lim x_i = x$ and $L^*\text{-}\lim x_i = x$ are of course equivalent.

2.2. LEMMA. *If x is an interelement of (x_i) and if $y \in G$ and $i_0 \in I$ are such that $x_i \wedge x \leqq y \leqq x$ for all $i \geqq i_0$, then y too is an interelement of (x_i) and dually.*

Proof. If (x_i) is eventually bounded, if u is a superelement and v a subelement of (x_i), then there is some $i \geqq i_0$ such that $v \leqq x_i \wedge x \leqq y \leqq x \leqq u$, hence $v \leqq y \leqq u$. Suppose now that (x_i) is not eventually bounded and fix a, b. If $v \leqq (a \vee x_i) \wedge b \leqq u$ for all $i \geqq k$ say, then $v \leqq (a \vee x) \wedge b \leqq u$ by the definition of interelement. We therefore have on one hand $(a \vee y) \wedge b \leqq (a \vee x) \wedge b \leqq u$ and on the other hand $v \leqq [(a \vee x_i) \wedge b] \wedge [(a \vee x) \wedge b] = [a \vee (x_i \wedge x)] \wedge b$ for all $i \geqq k$; if i is chosen to be $\geqq i_0, k$, then $v \leqq [a \vee (x_i \wedge x)] \wedge b \leqq (a \vee y) \wedge b$. Thus $v \leqq (a \vee y) \wedge b \leqq u$; we infer that $(a \vee y) \wedge b$ is an interelement of $(a \vee x_i) \wedge b$, $i \in I$.

2.3. LEMMA. *If x is a central element of (x_i) then x is an interelement of (x_i). If moreover (x_i) is not eventually bounded, then the converse is also true.*

Proof. Let x be a central element of (x_i). If (x_i) is eventually bounded, it is immediately seen that x is also an interelement of (x_i). If (x_i) is not eventually bounded we show, using the infinite distributive laws, that for every $a, b \in G$ $(a \vee x) \wedge b$ is a central element (and hence also an interelement) of $(a \vee x_i) \wedge b$, $i \in I$.

Assume now that (x_i) is not eventually bounded and let x be an interelement of (x_i). Fix $i_0 \in I$. Obviously $x \leqq x_i \vee x$ for all $i \geqq i_0$. If $y \leqq x_i \vee x$ for all $i \geqq i_0$, then $(x \vee x_i) \wedge y = y$ eventually. Since $(x \vee x) \wedge y = x \wedge y$ is, by the definition, an interelement of $(x \vee x_i) \wedge y$, $i \in I$, and since the latter net is eventually equal to y, we have $y = x \wedge y$, hence $y \leqq x$. Thus $x = \bigwedge_{i \geqq i_0} (x_i \vee x)$. Similarly we prove the dual equality.

2.4. COROLLARY. *If G is Archimedean, then x is an interelement of (x_i) if and only if it is a central element of (x_i).*

Proof. If (x_i) is eventually bounded this is a direct consequence of Lemma 1.3. If (x_i) is not eventually bounded the result is included in the above Lemma.

2.5. THEOREM. *L-$\lim x_i = x$ implies L^*-$\lim x_i = x$, and L^*-$\lim x_i = x$ implies α-$\lim x_i = x$. If (x_i) is eventually bounded then L^*-$\lim x_i = x$ is equivalent with L-$\lim x_i = x$ (and with ν-$\lim x_i = x$). If (x_i) is not eventually bounded L^*-$\lim x_i = x$ is equivalent with α-$\lim x_i = x$.*

Proof. If L-$\lim x_i = x$, where (x_i) is not eventually bounded, then ν-$\lim (a \vee x_i) \wedge b = (a \vee x) \wedge b$ for every a, b. Hence x is an interelement of (x_i). If y were another interelement, then (with $a = x \wedge y$, $b = x \vee y$ in the definition) $[(x \wedge y) \vee y] \wedge (x \vee y) = y$ would be an interelement of $[(x \wedge y) \vee x_i] \wedge (x \vee y)$, $i \in I$, which however converges naturally to $[(x \wedge y) \vee x] \wedge (x \vee y) = x$. Thus $y = x$. We have shown that L-$\lim x_i = x$ implies L^*-$\lim x_i = x$.

If $L^*\text{-lim}\, x_i = x$ then $x = \bigvee_{i \geqq i_0} (x_i \wedge x)$ for every i_0; in fact if $y \geqq x_i \wedge x$ for all $i \geqq i_0$, then $x \wedge y$ is an interelement of (x_i) by Lemma 2.2, therefore $x \wedge y = x$, i.e. $y \geqq x$. Dually we show that $x = \bigwedge_{i \geqq i_0} (x_i \vee x)$. That x is the only central element of (x_i) follows from Lemma 2.3. The final part of the theorem is also a consequence of Lemma 2.3.

None of the converse implications is valid. To show that $L^*\text{-lim}\, x_i = x$ does not imply $L\text{-lim}\, x_i = x$ consider the direct product $M \times (J \circ M)$ of the l-groups M and $J \circ M$ and set $x_n = (\sigma_n; 0, \sigma_n)$ (the sequence (σ_n) was defined above). It can easily be shown that $\alpha\text{-lim}\, x_n = 0$ and since (x_n) is not eventually bounded in $M \times (J \circ M)$ we infer from the preceding theorem that $L^*\text{-lim}\, x_n = 0$. However $L\text{-lim}\, x_n = 0$ is false; in fact if $x = (f; 1, g)$ then $(0 \vee x_n) \wedge x$ fails to converge naturally to $(0 \vee 0) \wedge x = 0$, since every superelement $u = (h; m, h')$ of $(0 \vee x_n) \wedge x = (\sigma_n \wedge f; 0, \sigma_n)$ must necessarily have $m \geqq 1$. Finally to show that $\alpha\text{-lim}\, x_i = x$ does not imply $L^*\text{-lim}\, x_i = x$ consider the sequence $(0, \sigma_n)$ in $J \circ M$. We thus see that α-convergence is in general weaker than L-convergence, both for bounded as well as for unbounded sequences. However, in an Archimedean l-group they are equivalent:

2.6. THEOREM. *In an Archimedean l-group L-convergence and α-convergence are equivalent.*

Proof. Assume $\alpha - \lim x_i = x$. Then, by Thm. 3.8 of the following section, for every $a, b \in G$ $\alpha - \lim (a \vee x_i) \wedge b = (a \vee x) \wedge b$. By Corollary 2.4 this means $(a \vee x) \wedge b$ is the only interelement of $(a \vee x_i) \wedge b$, $i \in I$ and since the latter net is bounded, $\nu - \lim (a \vee x_i) \wedge b = (a \vee x) \wedge b$. Hence $L - \lim x_i = x$.

The entire machinery of [9, § 8] is now at our disposal for the "completion" of an Archimedean l-group relative to its α-convergence.

3. Continuity of the algebraic operations. The operations $+, -, \vee, \wedge$ etc. are continuous relative to L-convergence; this follows from Prop. 1.2 (see [9, Prop. 7.4]). It is much less trivial to show that they are continuous relative to α-convergence too. This will be our next goal: The proof of Thm. 3.8 below goes via a number of auxiliary propositions, most of them covering special cases. Let us however remark at this point that L^*-convergence violates this natural requirement of continuity. Setting $x_n = (\sigma_n; 0, \sigma_n)$ in the l-group $M \times (J \circ M)$ as before, and $c = (0; 1, 0)$ we see that $L^*\text{-lim}\, x_n = 0$ does not imply $L^*\text{-lim}\, x_i \wedge c = 0 \wedge c$. Hence the mapping $x \rightarrow x \wedge c$, with c fixed, may fail to be continuous. The mapping $G \times G \ni (x, y) \rightarrow x + y \in G$ may also fail to be jointly continuous as is seen from the consideration of the sequences $x_n = (\sigma_n; 0, \sigma_n)$ and $y_n = (-\sigma_n; 0, 0)$ in $M \times (J \circ M)$. These are overwhelming disadvantages and we have to reject L^*-convergence. It is of course true that $L^*\text{-lim}\, x_i = x$ implies $L^*\text{-lim}\,(-x_i) = -x$ and $L^*\text{-lim}\,(x_i + c) = x + c$ but this offers little consolation.

Notice the following useful facts:

(2) If $x = \bigwedge_{i \geqq i_0} (x_i \vee x)$ and if $x^* \geqq x$, then $x^* = \bigwedge_{i \geqq i_0} (x_i \vee x^*)$; and dually.

(3) If $\alpha\text{-lim}\, x_i = x$ and if $x_i \leqq y$ eventually, then $x \leqq y$; and dually.

(4) If x, y are central elements of (x_i), then so are $x \vee y$ and $x \wedge y$.

(5) If (y_j) is a subnet of (x_i) (in the sense of [6]) and if x is a central element of (y_j), then x is a central element of (x_i) also.

(6) The following three statements are equivalent:
 (i) x is a central element of (x_i);
 (ii) $-x$ is a central element of $(-x_i)$;
 (iii) $x + c$ is a central element of $(x_i + c)$.

The easy proofs are left to the reader. From (6) in particular it follows that $\alpha\text{-lim}\, x_i = x$, $\alpha\text{-lim}\,(-x_i) = -x$ and $\alpha\text{-lim}\,(x_i + c) = x + c$ are equivalent.

3.1. PROPOSITION. *If* $\alpha\text{-lim}_{i \in I}\, x_i = x$ *and if* $(y_j)_{j \in J}$ *is a subnet of* $(x_i)_{i \in I}$, *then* $\alpha\text{-lim}_{j \in J}\, y_j = x$.

Proof. We first show that x is a central element of (y_j), i.e.

$$x = \bigvee_{j \geqq j_0} (y_j \wedge x) = \bigwedge_{j \geqq j_0} (y_j \vee x) \quad \text{for each } j_0 \in J\,. \tag{7}$$

Assume $z \geqq y_j \wedge x$ for every $j \geqq j_0$ and define $y = x \wedge z$. We shall show that y is a central element of (x_i). In fact fix i_0. We have $x \geqq y$, hence by (2) $y = \bigvee_{i \geqq i_0} (x_i \wedge y)$. To show the dual equality $y = \bigwedge_{i \geqq i_0} (x_i \vee y)$ assume

$$u \leqq x_i \vee y \quad \text{for all } i \geqq i_0\,. \tag{8}$$

Then $u \leqq x_i \vee y \vee x = x_i \vee x$ for all $i \geqq i_0$, hence

$$u \leqq \bigwedge_{i \geqq i_0} (x_i \vee x) = x\,. \tag{9}$$

Suppose now $y_j = x_{n(j)}$, $j \in J$, and choose $j' \geqq j_0$ such that $n(j') \geqq i_0$. Then by (8) $u \leqq x_{n(j')} \vee y$ and combining with (9)

$$u \leqq x \wedge (x_{n(j')} \vee y) = y \vee (x_{n(j')} \wedge x) = y\ ,$$

since $y = z \wedge x \geqq y_j \wedge x$ for all $j \geqq j_0$. Thus $y = \bigwedge_{i \geqq i_0}(x_i \vee y)$.

We infer that y is a central element of (x_i), hence $y = x$, i.e. $x \wedge z = x$, $z \geqq x$ and this proves the first half of (7). The dual is proved analogously. It follows now from (5) that x is the only central element of (y_j).

3.2. PROPOSITION. If $x_i \geqq 0$ for every i, then the following are equivalent:

(i) α-$\lim x_i = 0$

(ii) If $x \geqq 0$ and $x = \bigvee_{i \geqq i_0} (x_i \wedge x)$ for every i_0, then $x = 0$

(iii) For each $x > 0$ there exist $i_0 \in I$ and $u_0 \in G$ such that $x > u_0 \geqq x_i \wedge x$ for all $i \geqq i_0$.

Proof. That (i) implies (ii) is obvious. (iii) is only a restatement of (ii). We now show that (ii) implies (i). Assume (ii) is true. Then 0 is a central element of (x_i). In fact fix i_1; obviously $0 = \bigvee_{i \geqq i_1}(x_i \wedge 0)$. To verify the dual equality $0 = \bigwedge_{i \geqq i_1}(x_i \vee 0) = \bigwedge_{i \geqq i_1} x_i$ suppose $y \leqq x_i$ for all $i \geqq i_1$. Defining $x = y \vee 0$ we have $0 \leqq x \leqq x_i$ for all $i \geqq i_1$, i.e., $x = x_i \wedge x$ for all $i \geqq i_1$. But then $x = \bigvee_{i \geqq i_0}(x_i \wedge x)$ for *any* i_0, since there is always an $i \geqq i_0, i_1$. By hypothesis (ii) $x = 0$, i.e., $y \leqq 0$ and this means $0 = \bigwedge_{i \geqq i_1} x_i$.

If x were another central element of (x_i), then $x = \bigwedge_{i \geqq i_0}(x_i \vee x) \geqq 0$ and on the other hand $x = \bigvee_{i \geqq i_0}(x_i \wedge x)$ for every i_0. By (ii) $x = 0$.

3.3. PROPOSITION. If $x_i \geqq 0\ (i \in I)$, $y_j \geqq 0\ (j \in J)$, α-$\lim x_i = 0$ and α-$\lim y_j = 0$, then α-$\lim_{(i,j) \in I \times J}(x_i + y_j) = 0$. (Here $I \times J$ is directed by the cartesian ordering).

Proof. We shall apply the preceding proposition. Let $z \geqq 0$ be such that

$$z = \bigvee_{i \geqq i_0,\ j \geqq j_0} [(x_i + y_j) \wedge z] \quad \text{for every } i_0, j_0\ . \tag{10}$$

We shall first show that

$$z = \bigvee_{i \geqq i_0} (x_i \wedge z) \quad \text{for every } i_0\ . \tag{11}$$

Let $y \geqq x_i \wedge z$ for all $i \geqq i_0$. Define $y_0 = z \wedge y$. Then $z \geqq y_0 \geqq x_i \wedge z$ for all $i \geqq i_0$. From this we shall deduce that

$$z - y_0 = \bigvee_{j \geqq j_0} [y_j \wedge (z - y_0)] \quad \text{for every } j_0 . \tag{12}$$

In fact if $u \geqq y_j \wedge (z - y_0)$ for all $j \geqq j_0$, then

$$u + y_0 \geqq (y_j + y_0) \wedge z \geqq (y_j + x_i \wedge z) \wedge z$$

for all $i \geqq i_0$ and all $j \geqq j_0$, hence $u + y_0 \geqq (y_j + x_i) \wedge (y_j + z) \wedge z = (y_j + x_i) \wedge z$ for all $i \geqq i_0$ and all $j \geqq j_0$. By (10) $u + y_0 \geqq z$, i.e., $u \geqq z - y_0$ and thus (12) is established. This implies, by the preceding proposition, $z - y_0 = 0$, $z = y_0 = z \wedge y$, $y \geqq z$ which proves (11). Finally, by the preceding proposition again, (11) implies $z = 0$.

3.4. PROPOSITION. If $\alpha\text{-lim}\, x_i = 0$ then $\alpha\text{-lim}\, x_i^+ = 0$ and $\alpha\text{-lim}\, x_i^- = 0$ (where $x_i^+ = x_i \vee 0$, $x_i^- = (-x) \vee 0$).

Proof. $x_i^+ \geqq 0$, therefore we can apply Prop. 3.2. Let $x \geqq 0$ be such that

$$x = \bigvee_{i \geqq i_0} (x_i^+ \wedge x) \quad \text{for every } i_0 . \tag{13}$$

We shall show that

$$x = \bigvee_{i \geqq i_0} (x_i \wedge x) \quad \text{for every } i_0 . \tag{14}$$

If $y \geqq x_i \wedge x$ for all $i \geqq i_0$, then $y \geqq x_i \wedge 0 \wedge x$ for all $i \geqq i_0$, $y \geqq [\bigvee_{i \geqq i_0} (x_i \wedge 0)] \wedge x = 0 \wedge x (= 0)$. Thus $y \geqq (x_i \wedge x) \vee (0 \wedge x) = x_i^+ \wedge x$ for all $i \geqq i_0$, hence $y \geqq x$ by (13); (14) is established. That $x = \bigwedge_{i \geqq i_0} (x_i \vee x)$ follows from (2) and the relation $x \geqq 0$. Now x being a central element of (x_i) must be equal to 0.

Finally $\alpha\text{-lim}\, x_i = 0$ implies $\alpha\text{-lim}\,(-x_i) = 0$ and by what was proved $\alpha\text{-lim}\, x_i^- = 0$.

3.5. COROLLARY. *$\alpha\text{-lim}\, x_i = 0$ implies $\alpha\text{-lim}\, |x_i| = 0$.*

Proof. $\alpha\text{-lim}\, x_i = 0$ implies $\alpha\text{-lim}\, x_i^+ = 0$ and $\alpha\text{-lim}\, x_j^- = 0$. By Prop. 3.3 $\alpha\text{-lim}_{i,j}\,(x_i^+ + x_j^-) = 0$. Now $|x_i| = x_i^+ + x_i^-$, $i \in I$, being a subnet of $(x_i^+ + x_j^-)$, $(i, j) \in I \times J$, α-converges to 0.

3.6. PROPOSITION. $\alpha\text{-lim}\, |x_i| = 0$ implies $\alpha\text{-lim}\, x_i = 0$.

Proof. $\alpha\text{-lim}\, |x_i| = 0$ implies $\alpha\text{-lim}\,(-|x_i|) = 0$. Since $0 \leqq x_i \vee 0 \leqq |x_i| \vee 0$ we have $0 = \bigwedge_{i \geqq i_0} (x_i \vee 0)$ for every i_0. Dually $0 \geqq x_i \wedge 0 \geqq (-|x_i|) \wedge 0$ implies $0 = \bigvee_{i \geqq i_0} (x_i \wedge 0)$ and thus 0 is a central element of (x_i). Let x be another central element. Then $-x$ is a central element of $(-x_i)$, therefore

(15) $$x = \bigvee_{i \geqq i_0} (x_i \wedge x) \quad \text{for every } i_0$$

(16) $$-x = \bigvee_{i \geqq i_0} (-x_i \wedge -x) \quad \text{for every } i_0 .$$

We infer from (15) and (16) that

(17) $$|x| = \bigvee_{i \geqq i_0} (|x_i| \wedge |x|) \quad \text{for every } i_0 .$$

(In fact if $y \geqq |x_i| \wedge |x|$ then $y \geqq x_i \wedge x$ and $y \geqq (-x_i) \wedge (-x)$, hence $y \geqq x$ and $y \geqq -x$, i.e. $y \geqq x \vee -x = |x|$).
By Prop. 3.2 $|x| = 0, x = 0$.

3.7. THEOREM. *α-$\lim x_i = x$ if and only if α-$\lim |x_i - x| = 0$.*

In fact both are equivalent to α-$\lim (x_i - x) = 0$. Notice that, by Prop. 3.2 α-$\lim_{i \in I} x_i = 0$ and $0 \leqq y_i \leqq x_i$ for all $i \in I$ imply α-$\lim_{i \in I} y_i = 0$.

3.8. THEOREM. *If α-$\lim_{i \in I} x_i = x$ and α-$\lim_{j \in J} y_j = 0$, then α-$\lim_{i \in I} (-x_i) = -x$, α-$\lim_{i \in I} |x_i| = |x|$, α-$\lim_{(i,j) \in I \times J} (x_i + y_j) = x + y$, α-$\lim_{(i,j) \in I \times J} (x_i \vee y_j) = x \vee y$ and dually.*

Proof. $0 \leqq |(x_i + y_j) - (x + y)| \leqq |x_i - x| + |y_j - y|$. By Prop. 3.3, Thm. 3.7 and the remark preceding the present theorem:

$$\alpha\text{-}\lim_{i,j} (x_i + y_j) = x + y .$$

Similarly

$$\begin{aligned} |x_i \vee y_j - x \vee y| &= |x_i \vee y_j - x \vee y_j + x \vee y_j - x \vee y| \\ &\leqq |x_i \vee y_j - x \vee y_j| + |x \vee y_j - x \vee y| \leqq |x_i - x| + |y_j - y| \end{aligned}$$

etc.

4. Subspaces and product spaces. If an abelian l-group G is embedded in another abelian l-group E with preservation of all existing joins and meets, then the α-convergence of E can be relativized to G. It is natural to ask under what conditions this relative convergence coincides with the α-convergence of G itself. Theorem 4.1 below gives three sufficient conditions.

Let E be an l-group and G an l-subgroup. G is said to be *regular in E* (equivalently E is said to be *regular over G* or a *regular extension of G*) if $A \subset G$ and $\inf^{(G)} A = 0$ imply $\inf^{(E)} A = 0$. It is then true that $x = \sup^{(G)} X (X \subset G)$ implies $x = \sup^{(E)} X$, and dually.

4.1. THEOREM. *If the abelian l-group E is a regular extension of G, (x_i) and x are in G, then $\alpha\text{-}\lim^{(E)} x_i = x$ implies $\alpha\text{-}\lim^{(G)} x_i = x$. If moreover either*

(i) *G is Archimedean*

or (ii) *E is completely distributive (see Definition 5.1 below)*

or (iii) *for every $e \in E$, $e > 0$ there is $g \in G$ such that $0 < g \leqq e$ then $\alpha\text{-}\lim^{(G)} x_i = x$ and $\alpha\text{-}\lim^{(E)} x_i = x$ are equivalent.*

Proof. That $\alpha\text{-}\lim^{(E)} x_i = x$ implies $\alpha\text{-}\lim^{(G)} x_i = x$ follows from the definition of α-convergence and the regularity of E over G. We here prove the sufficiency of conditions (i) and (iii) for the converse implication; the sufficiency of condition (ii) will be proved below (§ 5).

Let $\alpha\text{-}\lim^{(G)} x_i = x$. Without loss of generality we can assume $x = 0, x_i \geqq 0$. Then 0 is a central element of (x_i) relative to E too. Let $e_0 \in E$ be another central element of (x_i) in E; then $e_0 = \bigwedge^{(E)}_{i \geqq i_0} (x_i \vee e_0) \geqq 0$ and

$$e_0 = \bigvee_{i \geqq i_0}^{(E)} (x_i \wedge e_0) \quad \text{for every } i_0 . \tag{18}$$

Case (i) (G Archimedean). We first show that $e_0 \wedge a = 0$ for every positive $a \in G$. In fact $\alpha\text{-}\lim^{(G)} x_i = 0$ implies $L\text{-}\lim^{(G)} x_i = 0$ by Thm. 2.6, hence $\nu\text{-}\lim^{(G)} x_i \wedge a = 0$. Then $\nu\text{-}\lim^{(E)} x_i \wedge a = 0$ (see [9, Prop. 6.2]), therefore $\nu\text{-}\lim^{(E)} x_i \wedge e_0 \wedge a = 0$. If $\tilde{u}$ is any superelement of $x_i \wedge e_0 \wedge a, i \in I$, in E then by (18)

$$e_0 \wedge a = \bigvee_{i \geqq i_0}^{(E)} (x_i \wedge e_0 \wedge a) \leqq \tilde{u}$$

for suitable i_0. Thus $e_0 \wedge a$, being a lower bound to the set of superelements of $x_i \wedge e_0 \wedge a$, $i \in I$, in E, must be 0.

In particular $e_0 \wedge x_i = 0$ for every $i \in I$, hence by (18) $e_0 = 0$.

Case (iii). (For every $e > 0$, $e \in E$, there is $g \in G$ such that $0 < g \leqq e$). Assume $e_0 > 0$ in (18) and let $g \in G$ be such that $0 < g \leqq e_0$. Then

$$g = e_0 \wedge g = \left[\bigvee_{i \geqq i_0}^{(E)} (x_i \wedge e_0)\right] \wedge g = \bigvee_{i \geqq i_0}^{(E)} (x_i \wedge e_0 \wedge g) = \bigvee_{i \geqq i_0}^{(E)} (x_i \wedge g)$$

hence $g = \bigvee^{(G)}_{i \geqq i_0} (x_i \wedge g)$. By Prop. 3.2 we then have $g = 0$, a contradiction. Thus e_0 must be 0.

Condition (iii) in the above theorem covers the case of the Everett extension G^* of G by means of "Cauchy" cuts, as well as the extension $\tilde{G}$ (see [9]). It seems improbable that the implication $\alpha\text{-}\lim^{(G)} x_i = x \Rightarrow \alpha\text{-}\lim^{(E)} x_i = x$ remains valid if we merely assume that E is regular over G.

We close this section with a theorem on cartesian products, whose proof is easy.

4.2. THEOREM. *If G is a direct union $G = \times_{\tau \in T} G^\tau$ of abelian l-groups G^τ, $\tau \in T$ and if $x \in G$, $x_i \in G$, then $\alpha\text{-}\lim_i^{(G)} x_i = x$ if and only if $\alpha\text{-}\lim_i^{(G^\tau)} x_i^\tau = x^\tau$ for every $\tau \in T$.* (Here x^τ denotes the τ-th "coordinate" of x).

5. The case of completely distributive abelian l-groups. There is a very neat characterization of α-convergence in a completely distributive abelian l-group.

5.1. DEFINITION. An abelian l-group G is said to be *completely distributive* if it satisfies the following condition:

(P) *If, for each index α in a set A, $(x_{\alpha j})_{j \in I_\alpha}$ is a family in G and if all joins and meets exhibited in equality* (19) *below exist, then this equality is valid:*

$$\bigwedge_{\alpha \in A} \bigvee_{j \in J_\alpha} x_{\alpha, j} = \bigvee_{\varphi \in \phi} \bigwedge_{\alpha \in A} x_{\alpha, \varphi(\alpha)} \; ; \tag{19}$$

here $\phi \equiv \times_{\alpha \in A} J_\alpha$, i.e., ϕ is the set of all choice-functions $\varphi(.)$ on A with $\varphi(\alpha) \in J_\alpha$ for each $\alpha \in A$.

For equivalent formulations of complete distributivity see [12, Thm. 2.6].

5.2. THEOREM. *If G is completely distributive and $(x_i)_{i \in I}$ is a directed net in G, then the following are equivalent:*

(i) $\alpha\text{-}\lim\limits_{i \in I} x_i = 0$

(ii) *For each cofinal subset J of I* $\bigwedge\limits_{j \in J} |x_j| = 0$.

Proof. If $\alpha\text{-}\lim_{i \in I} x_i = 0$ then $\alpha\text{-}\lim_{i \in I} |x_i| = 0$ and since $(x_j)_{j \in J}$ is a subnet of $(x_i)_{i \in I}$ we must have $\alpha\text{-}\lim_{j \in J} |x_j| = 0$, hence $\bigwedge_{j \in J} |x_j| = 0$. Conversely assume that (ii) holds. To show that $\alpha\text{-}\lim x_i = 0$ (equivalently $\alpha\text{-}\lim |x_i| = 0$) it is sufficient, by Prop. 3.2, to show that if $x \geqq 0$ and $x = \bigvee_{j \geqq i} (|x_j| \wedge x)$ for each $i \in I$ then $x = 0$. But if $x = \bigvee_{j \geqq i} (|x_j| \wedge x)$ for each $i \in I$ then

$$x = \bigwedge_{i \in I} \bigvee_{j \geqq i} (|x_j| \wedge x) \,.$$

On the other hand by hypothesis (ii) $\bigwedge_{i \in I} (|x_{\varphi(i)}| \wedge x) = 0$ for each choice function $\varphi(.) \in \times_{i \in I} J_i$, where $J_i = \{j \in I : j \geqq i\}$. Thus

$$\bigvee_{\varphi} \bigwedge_{i \in I} (|x_{\varphi(i)}| \wedge x) = 0$$

and by the complete distributivity of G, $x = 0$.

Notice that (i) implies (ii) in any abelian l-group. We can now proceed to the

Completion of the proof of Theorem 4.1. Let E be a completely distributive regular extension of G. If $\alpha\text{-}\lim_{i\in I}^{(G)} x_i = 0$ then $\bigwedge_{j\in J}^{(G)} |x_j| = 0$ for every cofinal subset $J \subset I$. By the regularity of E over G $\bigwedge_{j\in J}^{(E)} |x_j| = 0$ for every cofinal $J \subset I$, hence by Thm. 5.2 $\alpha\text{-}\lim^{(E)} x_i = 0$.

Our next result is that in a completely distributive abelian l-group the α-convergence derives from a group topology.

5.3. THEOREM. *If G is completely distributive then its α-convergence derives from a topology $\mathfrak{T}$ on G which makes G into a Hausdorff topological group.*

This means that $\alpha\text{-}\lim x_i = x$ if and only if (x_i) is eventually in each $\mathfrak{T}$-neighborhood of x.

Proof. To show that the α-convergence is a topological convergence, it is sufficient, by [6, p. 74] to show that it has the following properties:

(i) If $x_i = x$ for every $i \in I$, then $\alpha\text{-}\lim x_i = x$.

(ii) If $\alpha\text{-}\lim x_i = x$ and if (y_j) is a subnet of (x_i), then $\alpha\text{-}\lim y_j = x$.

(iii) If $\alpha\text{-}\lim x_i = x$ is false, then there is a subnet (y_j) of (x_i) no subnet of which α-converges to x.

(iv) If, for each i in a directed set I, $(x_{i,j})_{j\in R_i}$ is a net in G such that $\alpha\text{-}\lim_{j\in R_i} x_{i,j} = x_i$ and if $\alpha\text{-}\lim x_i = x$ then

$$\alpha\text{-}\lim y_{(i,f)} = x$$

where $y_{(i,f)} \equiv y_{(i,f(\cdot))}$, $(i, f(.)) \in I \times \bigtimes_{i\in I} R_i \equiv \sum$ is the net defined by $y_{(i,f(\cdot))} = x_{i,f(i)}$ $\left(\sum \equiv I \times \bigtimes_{i\in I} R_i \text{ is directed coordinatewise}\right)$.

For a variation on these conditions see [1].

(i) is obvious and (ii) was proved earlier (Prop. 3.1). To show (iii) we assume (without loss of generality) that $x = 0$. If $\alpha - \lim x_i = 0$ is false then by Thm. 5.2 there is a cofinal subset J of I and some $z \in G$ such that $0 < z \leqq |x_j|$ for every $j \in J$. Then $(x_j)_{j\in J}$ is a subnet of $(x_i)_{i\in I}$ no subnet of which can α-converge to 0.

Finally to establish (iv) we need a lemma.

LEMMA. *Let D be a cofinal subset of $\sum \equiv I \times \bigtimes_{i\in I} R_i$. For each*

$i_0 \in I$ let $A_{i_0} = \{j \in R_{i_0}$: there exists a choice-function $f(.) \in \times_{i \in I} R_i$ with $f(i_0) = j$ and $(i_0, f(.)) \in D\}$, i.e. $A_{i_0} = \{f(i_0): f$ is such that $(i_0, f) \in D\}$. Then the set

$$\Omega = \{i_0 \in I : A_{i_0} \text{ is cofinal in } R_{i_0}\}$$

is cofinal in I.

In fact suppose there is $k_0 \in I$ such that, for every $i \geqq k_0$ A_i is not cofinal in R_i. Then for each $i \geqq k_0$ there is $j(i) \in R_i$ such that no element j of A_i is $\geqq j(i)$. Define $f_0(.)$ in $\times_{i \in I} R_i$ by:

$$f_0(i) = \begin{cases} \text{an arbitrary element of } R_i \text{ if } i \ngeqq k_0 \\ j(i) \text{ if } i \geqq k_0 \,. \end{cases}$$

Then there is no element $(i, f(.)) \in D$ with $(i, f(.)) \geqq (k_0, f_0(.))$, which contradicts the fact that D is cofinal in $I \times \times_{i \in I} R_i$.

Having established the lemma we now turn to the proof of (iv). Suppose that $\alpha\text{-}\lim_{j \in R_i} x_{i,j} = x_i$ for each $i \in I$ and that $\alpha\text{-}\lim_{i \in I} x_i = 0$. (There is no loss of generality in assuming $x = 0$). We shall show that $\alpha\text{-}\lim_{(i,f(\cdot)) \in \Sigma} y_{(i,f(\cdot))} = 0$ by applying Thm. 5.2. If D is cofinal in $\sum$ and $z \leqq | y_{i,f(\cdot)} |$ for every $(i, f(.)) \in D$, i.e. if $z \leqq | x_{i,f(i)} |$ for every i and $f(.)$ such that $(i, f(.)) \in D$, then in particular, for a fixed i in Ω (Ω is defined as in the lemma) $z \leqq | x_{i,f(i)} |$ for every $f(.) \in \times_{i \in I} R_i$ such that $(i, f(.)) \in D$, hence

$$z - | x_i | \leqq | x_{i,f(i)} | - | x_i | \leqq | x_{i,f(i)} - x_i |$$

$$z - | x_i | \leqq \inf \left\{ | x_{i,f(i)} - x_i | : f(.) \in \bigtimes_{i \in I} R_i \quad \text{and} \quad (i, f(.)) \in D \right\} = 0 \,.$$

(That the *inf* is zero is a consequence of the equality $\alpha\text{-}\lim_{j \in R_i} | x_{i,j} - x_i | = 0$ and the fact that $A_i = \{f(i) : f(.) \in \times_{i \in I} R_i$ and $(i, f(.)) \in D\}$ is cofinal in R_i).

Thus $z \leqq | x_i |$ for each $i \in \Omega$ and since Ω is cofinal in I, $z \leqq \bigwedge_{i \in \Omega} | x_i | = 0$ by the same reasoning. In other words

$$\bigwedge_{(i,f(\cdot)) \in D} | y_{(i,f(\cdot))} | = 0$$

for each cofinal subset D of $\sum$; hence $\alpha\text{-}\lim_{(i,f(\cdot)) \in \Sigma} y_{(i,f(\cdot))} = 0$.

We conclude that there is a topology $\mathfrak{T}$ on G such that $\alpha\text{-}\lim x_i = x$ if and only if, for each $\mathfrak{T}$-neighborhood U of x, $x_i \in U$ eventually. This topology is Hausdorff since limits of arbitrary α-convergent nets are unique. Finally by Thm. 3.8 the operations of the l-group are continuous and the proof is complete.

5.4. COROLLARY. *If G is a regular l-subgroup of a direct union*

$\times_{\tau \in T} G^\tau$ *of simply ordered abelian l-groups, then its α-convergence derives from a Hausdorff group-topology. In particular in R^X (X any set) α-convergence is pointwise convergence.*

In fact such an l-group G is completely distributive. It is a little absurd to derive this corollary from Thm. 5.3, since we can prove it directly and in fact determine the topology. In each G^τ α-convergence is equivalent to the topological convergence which is defined by means of open intervals. By Theorem 4.2 the α-convergence of $\times_{\tau \in T} G^\tau$ derives from the product topology. Finally by Thm. 4.1, case (ii), ($\times_{\tau \in T} G^\tau$ is completely distributive) the α-convergence of G derives from the relative topology of the subspace $G \subset \times_{\tau \in T} G^\tau$. Notice that this argument serves to establish Thm. 5.3 in the particular case that G is Archimedean, for if G is Archimedean and completely distributive then it is representable as a regular l-subgroup of a direct union $\times_{\tau \in T} G^\tau$ of simply ordered (abelian) l-groups $G^\tau, \tau \in T$ (in fact as a "regular subdirect union"). See [13, Thm. 2.2].

Similar results are of course valid for Boolean algebras, where the analogue of α-convergence is simply the natural convergence as defined in §1. For instance the "pathological" examples given in [7, p. 1192–93] and [3] are not completely distributive. K. Matthes [8] has given a condition on a lattice L which is necessary and sufficient in order that the natural convergence of L derive from a topology. If R^R is the l-group of all real functions on the real line, then the natural convergence of R^R does not derive from any topology (R^R is not "$\aleph_0$-regulär" [8]), whereas its α-convergence (L-convergence is a topological convergence. Notice however that for *sequences* natural convergence and α-convergence are equivalent in R^R.

An abelian l-group G is said to be $(\aleph_0, \aleph_0)$-distributive if it satisfies condition (P) of Def. 5.1 whenever the set A as well as each J_α are countable.

5.5. Proposition. If G is $(\aleph_0, \aleph_0)$-distributive then its α-convergence of sequences derives from a T_1-topology $\mathfrak{T}(G)$.

This follows from the discussion of § 6 and the fact that in an $(\aleph_0, \aleph_0)$-distributive l-group the characterization of Thm. 5.2 is valid for ordinary sequences. As far as continuity of the group operations is concerned we can affirm that the mapping $G \ni x \to -x \in G$ is continuous relative to $\mathfrak{T}(G)$ and that for each $y \in G$ the mapping $G \ni x \to x + y \in G$ is also continuous. The l-group $G \times G$ is $(\aleph_0, \aleph_0)$-distributive, hence its sequential α-convergence derives from a T_1-topology $\mathfrak{T}(G \times G)$. If $G \times G$ is topologized with $\mathfrak{T}(G \times G)$ and G is topologized with $\mathfrak{T}(G)$ then the mapping $\varphi: G \times G \ni (x, y) \to x + y \in G$ is continuous. In fact using Thm. 4.2

we can easily show that the inverse image $\varphi^{-1}(K) \subset G \times G$ of a $\mathfrak{T}(G)$-closed set $K \subset G$ is $\mathfrak{T}(G \times G)$-closed.

6. Appendix on abstract sequential convergence. In this section we give an elementary theorem on a necessary and sufficient condition in order that a given abstract sequential convergence be equivalent to a sequential convergence defined by means of a topology. The argument establishing this result is essentially due to Löwig [7]. Though the present appendix is only loosely connected with the rest of the paper, it is attached here because the main result is used in establishing Proposition 5.5.

Let X be an arbitrary set and $\mathfrak{C}$ an assignment of "limits" to certain sequences of elements of X. If the element $x \in X$ is assigned to the sequence (x_n), we write $\mathfrak{C}\text{-}\lim_n x_n = x$ and say that (x_n) $\mathfrak{C}$-converges to x. We say that $\mathfrak{C}$ is an *abstract sequential convergence in X with unique limits* if it satisfies the following conditions:

(20) To each sequence at most one "limit" is assigned.

(21) If $x_n = x$ for every n, then $\mathfrak{C}\text{-}\lim x_n = x$.

(22) If $\mathfrak{C}\text{-}\lim\limits_n x_n = x$ and if $(x_{k(n)})$ is a subsequence of (x_n), then

$$\mathfrak{C}\text{-}\lim_n x_{k(n)} = x.$$

The *star-convergence* corresponding to $\mathfrak{C}$ is defined as follows: $\mathscr{C}^*\text{-}\lim x_n = x$ if and only if every subsequence $(x_{k(n)})$ of (x_n) contains a sub-subsequence $(x_{k(\lambda(n))}) = x$ such that $\mathscr{C}\text{-}\lim x_{k(\lambda(n))} = x$. The notion of star-convergence was introduced by Urysohn in [11]. A T_1-topology $\mathfrak{T}(\mathscr{C})$ can also de defined in X by means of $\mathfrak{C}$; it is called the *derivative topology* of $\mathfrak{C}$:

(τ) A set $K \subset X$ is closed relative to $\mathfrak{T}(\mathfrak{C})$ if $\mathfrak{C}\text{-}\lim x_n = x$ and $x_n \in K$ for all n imply $x \in K$.

$\mathfrak{T}(\mathfrak{C})$ determines a new sequential convergence which we shall call the *derivative topological convergence*: $\mathfrak{T}(\mathfrak{C})\text{-}\lim x_n = x$ if and only if, for each $\mathfrak{T}(\mathfrak{C})$-neighborhood U of x, (x_n) is eventually in U. In ordinary cases star-convergence is known to be equivalent with the derivative topological convergence. According to theorem 6.1 below this is actually true in the most general case, provided we stick to our reasonable assumption of uniqueness of limits. The first to observe the connection between star-convergence and derivative topological convergence was P. Urysohn [11] who proved a restricted form of Thm. 6.1, under the severe assumption that $\mathfrak{C}$ satisfies the following condition:

(I) The operator $A \to \tilde{A}$ defined by $\tilde{A} = \{x \in X$: there exists a sequence (x_n) in A with $\mathfrak{C}$-$\lim x_n = x\}$ for every $A \subset X$, is idempotent, i.e. $\tilde{\tilde{A}} = \tilde{A}$.

The same condition was involved in the proof of Satz 29 of [5], which dealt with a particular kind of order-convergence in lattice groups introduced by Kantorovitch in the same paper. Löwig [7, pp. 1191–1192] removed condition (I) proving the same equivalence for another particular concept of order-convergence in Boolean algebras. However, it is easily seen that Löwig's argument, with slight modifications, can serve to establish the following general theorem.

6.1. THEOREM. (Urysohn [11], Löwig [7]). *Let $\mathfrak{C}$ be an abstract sequential convergence with unique limits in a set X. Then the corresponding star-convergence is equivalent to the derivative topological convergence, i.e.,* $\mathfrak{C}^*$-$\lim x_n = x$ *if and only if* $\mathfrak{T}(\mathfrak{C})$-$\lim x_n = x$.

Löwig's argument can be found in [7, pp. 1191–92]. What follows here is an outline of this argument, with some modifications necessitated by the fact that the property expressed by condition (22) above is assumed here for subsequences only and not for rearrangements (see Löwig's Thm. 29). First, one shows that if $\mathfrak{C}$-$\lim x_n = x$, then $\mathfrak{T}(\mathfrak{C})$-$\lim x_n = x$ and x is the only limit of (x_n) under the topological convergence $\mathfrak{T}(\mathfrak{C})$. This latter assertion (uniqueness) is established as follows: If $y \neq x$ then the set $K = \{x_n : x_n \neq y\} \cup \{x\}$ is $\mathfrak{T}(\mathfrak{C})$-closed; in fact if (a_n) is a sequence in K with $\mathfrak{C}$-$\lim a_n = a$ and if the range of (a_n) is infinite then by the definition of K there is a subsequence of (a_n) which is of the form $x_{\lambda(1)}, x_{\lambda(2)}, x_{\lambda(3)}, \cdots$ with $\lambda(1) < \lambda(2) < \lambda(3) < \cdots$; in other words there exists a sequence which is both a subsequence of (a_n) and a subsequence of (x_n). This implies $a = x$, hence $a \in K$. If the range of (a_n) is finite the same conclusion (i.e. $a \in K$) is trivial. Thus K is indeed $\mathfrak{T}(\mathfrak{C})$-closed. The complement of K is a $\mathfrak{T}(\mathfrak{C})$-neighborhood of y which fails to contain eventually the terms of the sequence (x_n).

Next we show that $\mathfrak{T}(\mathfrak{C})$-$\lim x_n = x$ implies $\mathfrak{C}^*$-$\lim x_n = x$. Assume, by way of contradiction, that there exists a subsequence $(x_{k(n)})$ of (x_n) no sub-subsequence of which $\mathfrak{C}$-converges to x. The above result then can be seen to imply that no subsequence (or rearrangement of a subsequence) of $(x_{k(n)})$ $\mathfrak{C}$-converges at all. Then the set $K = \{x_{k(n)} : x_{k(n)} \neq x\}$ is $\mathfrak{T}(\mathfrak{C})$-closed and its complement is a $\mathfrak{T}(\mathfrak{C})$-neighborhood of x, but $(x_{k(n)})$ is eventually outside this neighborhood; a contradiction.

Finally the implication $\mathfrak{C}^*$-$\lim x_n = x \Rightarrow \mathfrak{T}(\mathfrak{C})$-$\lim x_n = x$ is easy to establish. This completes the argument.

It follows from Thm. 6.1 that limits of sequences are unique under the derivative topological convergence. Notice however that the topology $\mathfrak{T}(\mathfrak{C})$ is not necessarily Hausdorff, i.e., limits of nets may fail to be unique. Consider for instance the extended real line $\bar{R}$ and set $\mathfrak{C}$-$\lim x_n = x$ whenever $x_n = x$ for every n (x may be $+\infty$ or $-\infty$), $\mathfrak{C}$-$\lim x_n = +\infty$ whenever (x_n) is strictly increasing and $\mathscr{C}$-$\lim x_n = -\infty$ whenever (x_n) is strictly decreasing. Let A, B be $\mathfrak{T}(\mathfrak{C})$-open sets containing $+\infty$ and $-\infty$ respectively; then $A \cap B \neq \varnothing$ since the complements A^c and B^c are countable. In fact an uncountable set contains both a strictly increasing and a strictly decreasing sequence, hence an uncountable $\mathfrak{T}(\mathfrak{C})$-closed set contains both $-\infty$ and $-\infty$. An immediate consequence of 6.1 is the following theorem:

6.2. THEOREM. *An abstract sequential convergence $\mathscr{C}$ with unique limits derives from a topology if and only if it satisfies the following condition*

(23) *If (x_n) does not converge to x under $\mathfrak{C}$, then there is a subsequence $(x_{k(n)})$ no sub-subsequence of which converges under $\mathfrak{C}$ to x.*

Comparing this theorem with analogous results of Arnold [1] and Kelley [6] on convergence of arbitrary directed nets we see that, surprisingly, in the case of ordinary sequences the extra assumption of uniqueness of limits renders the condition on iterated limits (condition (iv) at the beginning of the proof of Thm. 5.3) superfluous.

Theorems 6.1 and 6.2 have been recorded here because the author has been unable to find an explicit statement of these general results in the literature. It seems that only the obvious implication $\mathfrak{C}^*$-$\lim x_n = x \Rightarrow \mathfrak{T}(\mathfrak{C})$-$\lim x_n = x$ is widely known. For instance it is stated in [2, p. 62] that if (x_n) star-converges to a then "it certainly converges to a in the star *topology*; moreover . . . this special case is sufficient for the applications of star-convergence which we have in mind."

In connection with Thm. 6.2 we observe that in general there are more than one topologies determining the sequential convergence $\mathfrak{C}$. For instance if $\mathfrak{C}$ is pointwise convergence of sequences of real functions on $[0, 1]$, then the class of Baire functions is $\mathfrak{T}(\mathfrak{C})$-closed but not closed relative to pointwise convergence of nets (i.e., relative to the product topology of $R^{[0,1]}$). The topology $\mathfrak{T}(\mathfrak{C})$ is the strongest topology determining $\mathfrak{C}$ and is T_1. If there is at least one Hausdorff topology determining $\mathfrak{C}$ then a fortiori $\mathfrak{T}(\mathfrak{C})$ is Hausdorff.

If $\mathfrak{C}$ is an abstract sequential convergence with unique limits on X which satisfies condition (23), then the sequential convergence $\mathfrak{C} \times \mathfrak{C}$

on $X \times X$ defined by:

$$\text{(a)} \quad \mathfrak{C} \times \mathfrak{C} - \lim (x_n, y_n) = (x, y) \text{ if and only if}$$
$$\mathfrak{C} - \lim x_n = x \text{ and } \mathfrak{C} - \lim y_n = y,$$

also satisfies condition (23) and hence it derives from a T_1-topology $\mathfrak{T}(\mathfrak{C} \times \mathfrak{C})$. This topology is stronger than the product topology $\mathfrak{T}(\mathfrak{C}) \times \mathfrak{T}(\mathfrak{C})$ on $X \times X$. (Observe that if A, B are $\mathfrak{T}(\mathfrak{C})$-open in X then $A \times B$ is $\mathfrak{T}(\mathfrak{C} \times \mathfrak{C})$-open in $X \times X$). It may be strictly stronger. For instance if $\mathfrak{T}(\mathfrak{C})$ is not Hausdorff, then the diagonal of $X \times X$ is not closed under $\mathfrak{T}(\mathfrak{C}) \times \mathfrak{T}(\mathfrak{C})$ though it is obviously $\mathfrak{T}(\mathfrak{C} \times \mathfrak{C})$-closed.

The assumption of uniqueness of limits plays an important role in the considerations of the present section. (Let X be an infinite set and set $\mathfrak{C}$-lim $x_n = x$ whenever the "range" of (x_n) is finite and x is any element of X. Under $\mathfrak{T}(\mathfrak{C})$ every sequence converges to every element).

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