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FREE COMPLETE EXTENSIONS OF BOOLEAN ALGEBRAS

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From considering questions about the existence of free α-complete Boolean algebras and free complete Boolean algebras, one is led naturally to the following problem: Given a Boolean algebra B, is it possible to embed B as a subalgebra in a complete Boolean algebra B^* in such a way that homomorphisms of B into complete Boolean algebras can be extended to complete homomorphisms on B^*? In general, the answer is "no"; this paper establishes that B can be so embedded if and only if every homomorphic image of B is atomic. Several other equivalent conditions on B are also developed.

To express these ideas more precisely, we say that the complete Boolean algebra B^* is a *free complete extension* of the Boolean B provided that there exist an isomorphism i of B into B^* such that

(i) if h is a homomorphism of B into a complete Boolean algebra C, then there is a complete homomorphism h^* of B^* into C such that $h^* \circ i = h$;

(ii) B^* has no regular complete proper subalgebra which contains $i[B]$ — that is, $i[B]$ completely generates B^*. A Boolean algebra B is said to be *superatomic* if every homomorphic image of B is atomic (or, equivalently, if every subalgebra of B is atomic). Our principal result, then, is that a Boolean algebra B has a free complete extension if and only if B is superatomic.

The problem of determining which Boolean algebras have free complete extensions arose from a conjecture made by L. Rieger in [7]. This conjecture is, in effect, that no infinite free Boolean algebra has a free complete extension; this has been verified by H. Gaifman in [3] and, independently and by different methods, by A. Hales, in [4].

Rieger proved, on the other hand, that for any cardinal numbers α and β there exists a unique free α-complete Boolean algebra on β generators. This result can be expressed by the statement that the free Boolean algebra on β generators has a unique free α-extension (a *free α-extension* of a Boolean algebra is defined in the same manner as a free complete extension, with the concepts of completeness and complete homomorphism replaced by those of α-completeness and

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α-homomorphism). F. M. Yaqub generalized this results in showing ([10], § 2) that every Boolean algebra has a unique free α-extension.

In addition, Yaqub found that the free α-extension of a Boolean algebra is α-representable if and only if it is isomorphic to an α-field of sets. He also proved that if $\alpha \geqq 2^{\aleph_0}$ and the free α-extension of a Boolean algebra B is α-representable, then B is superatomic. In § 4, we prove that the converse of the second of these results follows from the first, when the first is supplemented with our own results.

In § 2, we derive several characterizations of the superatomic Boolean algebras. In our third section it is shown that every superatomic Boolean algebra has a free complete extension. Then, using the result of Gaifman and Hales mentioned above, we show that every Boolean algebra having a free complete extension is superatomic.

Boolean concepts which are used without definition in this paper are defined in [2] and [8]. We denote the join and meet of two elements a and b of a Boolean algebra by $a + b$ and ab, respectively; the complement of b will be denoted by $\bar{b}$; the join and meet of a set S of elements of a Boolean algebra will be denoted by ΣS and ΠS, when they exist. To simplify certain arguments, we have assumed that if $S = \phi$, then ΣS and ΠS are the zero and unit, respectively, of the Boolean algebra in which the operations are performed.

2. Superatomic Boolean algebras.

As defined by Yaqub in [10], a Boolean algebra B is superatomic if every subalgebra and every homomorphic image of B is atomic. This concept had been formulated previously in a somewhat more general context by Mostowski and Tarski, who, in [6], defined a hereditarily atomic Boolean ring as a Boolean ring each of whose homomorphic images is atomic; they observed that a Boolean ring R has this property if and only if every subring of R is atomic. In addition, Mayer and Pierce, in § 3 of [5], discussed Boolean algebras with scattered ordered bases (a totally ordered set, or chain, is *scattered* if it contains no subset which is order-isomorphic to the chain of rational numbers); it follows from their work that every such Boolean algebra is superatomic. However, there do exist superatomic Boolean algebras which are not of this form — a simple example is the Boolean algebra of finite and co-finite subsets of an uncountably infinite set, ordered by set inclusion.

THEOREM. *If B is a Boolean algebra, then the following are equivalent:*

(i) *every homomorphic image of B is atomic;*

(ii) *every subalgebra of B is atomic;*

(iii) *the Stone space of B is clairsemé (has no nonempty dense-*

in-itself set);

(vi) *every chain of elements of B is scattered;*

(v) *there is a sublattice S of B, whose elements generate B, such that every chain of elements of S is scattered;*

(vi) *no subalgebra of B is an infinite free Boolean algebra.*

Proof. The equivalence of (i) and (ii) is proven in [6], Theorem 3.12. We could, in fact, include in our collection of equivalent statements,

(i′) no homomorphic image of B is atomless; since it is easily shown that this is equivalent to (i).

It is well-known that in the natural duality between Boolean algebras and their Stone spaces, the homomorphic images and atoms of a Boolean algebra correspond to the closed subspaces and isolated points, respectively, of its Stone space. Thus, (iii) is equivalent to (i) (this was observed in [5]).

A Boolean algebra generated by a chain of elements which is order-isomorphic to the rationals is necessarily atomless, and, in fact, is isomorphic to the free Boolean algebra on $\aleph_0$ generators (as is shown on page 54 of [2]). Thus, (ii) implies (iv), and (vi) implies (iv). Moreover, if B is a nonatomic Boolean algebra, then B has an atomless element b; if a is any element of B such that $a < b$, then for some element c of B, $a < c < b$. Every nonatomic Boolean algebra can thus be shown to have a chain of elements which is order isomorphic to the rationals; hence, (iv) implies both (ii) and (vi).

It is clear that (iv) implies (v). We shall conclude our proof by showing that (v) implies (i′). Let us assume that B is a Boolean algebra generated by a sublattice S. Suppose that h is a homomorphism of B onto the Boolean algebra B'. If a and b are elements of S such that $a < b$ and $h(\bar{a}b)$ is neither an atom nor the zero element of B', then there are elements s and t of S such that $s < t$, $h(\bar{s}t)$ is not the zero of B', and $h(\bar{s}t) < h(\bar{a}b)$. This implies that either

$$h(a) < h(a + bs) < h(b)$$

or $h(a) < h(a + bt) < h(b)$. In either case, we have found an element c of S such that $a < c < b$ and neither $h(\bar{a}c)$ nor $h(\bar{c}b)$ is the zero of B'. Thus we may conclude that either B' has an atom or S has a chain of elements which is order-isomorphic to the rationals. Hence, if B is generated by a sublattice S such that every chain of elements of S is scattered, then no homomorphic image of B is atomless.

REMARK. The equivalence of conditions (i) and (iii) of the theorem indicates that every clairsemé compact Hausdorff space whose topology has a base of open-and-closed sets is the Stone space of a superatomic Boolean algebra. We can, in fact, make a stronger statement. If a compact Hausdorff space is clairsemé, then it is clearly totally disconnected; that is, it has only one-point maximal connected components. It is shown in Chapter I of [1] that every locally compact totally disconnected space has a base of open-and-closed sets. Thus, every clairsemé compact Hausdorff space is the Stone space of some superatomic Boolean algebra. (The author is indebted to Prof. M. Henriksen of Purde University for this argument.)

3. Free complete extensions of Boolean algebras. In this section, we show that a Boolean algebra has a free complete extension if and only if it is superatomic. Let us observe first that the definition of the free complete extension of a Boolean algebra implies that if a Boolean algebra has a free complete extension, then it is unique in the following sense: if B_1^* and B_2^* are free complete extensions of the Boolean algebra B, and i_1 and i_2 are the isomorphisms of the definition which carry B into B_1^* and B_2^* respectively, then there is an isomorphism i^* of B_1^* onto B_2^* such that $i^* \circ i_1 = i_2$.

THEOREM. *If B is a superatomic Boolean algebra, then the field of all subsets of the Stone space of B is a free complete extension of B, with respect to the natural isomorphism i of B onto the field of open-and-closed subsets of its Stone space.*

Proof. Suppose that B is superatomic and that X is the Stone space of B. Let B^* denote the field of all subsets of X. It is clear that $i[B]$ completely generates B^*, since X is Hausdorff and its open-and-closed sets form a base for its topology. It remains to be proven that if h is a homomorphism of B into a complete Boolean algebra C, then there is a complete homomorphism h^* of B^* into C such that $h^* \circ i = h$.

Recall that for ordinal β, we define $X^{(\beta)}$, the β-th derivative of X, as follows: $X^{(0)} = X$; $X^{(\beta+1)}$ is the set of all limit points of $X^{(\beta)}$; and if β is a limit ordinal, then $X^{(\beta)} = \cap \{X^{(\alpha)}: \alpha < \beta\}$. Since X is clairsemé, every point of X is an isolated point of some derivative of X; moreover, X has empty derivatives.

If $p \in X$ and p is an isolated point of $X^{(\beta)}$, let b_p be an element of B such that $i(b_p) \cap X^{(\beta)} = \{p\}$.

Now suppose that h is a homomorphism of B into a complete Boolean algebra C. We define a mapping h^* of B^* into C inductively (for convenience, we will write $h^*(p)$ instead of $h^*(\{p\})$). If p is an isolated point of X, let $h^*(p) = h(b_p)$. If p is an isolated point of

$X^{(\beta)}$ and $h^*(q)$ is defined for each $q \in X \sim X^{(\beta)}$, let

$$h^*(p) = h(b_p)\Pi\{\overline{h^*(q)} : q \in i(b_p) \sim X^{(\beta)}\} .$$

If $S \subset X$, let $h^*(S) = \Sigma\{h^*(p) : p \in S\}$.

Next, we prove that for every ordinal β,

S_β : (i) if $b \in B$ and $i(b) \cap X^{(\beta)} = \phi$, then $h^*(i(b)) = h(b)$;

(ii) if $b \in B$ and $i(b) \cap X^{(\beta)} = \{p\}$, then

$$h(b)\Pi\{\overline{h^*(q)} : q \in i(b) \sim X^{(\beta)}\} = h^*(p) .$$

Since X is Hausdorff, the truth of (ii) implies that if $p, q \in X$ and $p \neq q$, then $h^*(p)$ is disjoint from $h^*(q)$. Thus, if S_1 and S_2 are subsets of X and $S_1 \cap S_2 = \phi$, then $h^*(S_1)$ and $h^*(S_2)$ are disjoint. Since h^* is completely additive by definition, (i) then implies that h^* is a complete homomorphism of B^* into C, and $h^* \circ i = h$.

S_0 is clearly true, since $X^{(0)} = X$. Now suppose that β is an ordinal such that S_α is true for every ordinal α such that $\alpha < \beta$, and that b is an element of B such that $i(b) \cap X^{(\beta)} = \phi$. If β is a limit ordinal, then since $i(b)$ is compact, there is an ordinal $\alpha, \alpha < B$, such that $i(b) \cap X^{(\alpha)} = \phi$; hence, $h^*(i(b)) = h(b)$. If β is not a limit ordinal, and $i(b) \cap X^{(\beta-1)} \neq \phi$, then $i(b) \cap X^{(\beta-1)}$is finite, since it is compact and discrete. We can, without loss of generality, assume that $i(b) \cap X^{(\beta-1)} = \{p\}$. Then, by $S_{\beta-1}(ii), h^*(p) = h(b)\Pi\{\overline{h^*(q)}: q \in i(b) \sim X^{(\beta-1)}\}$; it follows that

$$h(b) \leqq h^*(p) + \Sigma\{h^*(q) : q \in i(b) \sim X^{(\beta-1)}\} ,$$

and thus, $h^*(p) \leqq h(b) \leqq h^*(i(b))$. Moreover, if $q \in i(b) \sim X^{(\beta-1)}$, there is an ordinal $\alpha, \alpha < \beta - 1$, and a $c \in B$, $c \leqq b$, such that $h^*(q) \leqq h(c)$; thus, $h(b) = h^*(i(b))$. Hence, if S_α is true for every ordinal α such that $\alpha < \beta$, then $S_\beta(i)$ is true.

Next, suppose that b is an element of B such that $i(b) \cap X^{(\beta)} = \{p\}$. Using $S_\beta(i)$, we have

$$\begin{aligned} &h(b)\Pi\{\overline{h^*(q)} : q \in i(b) \sim X^{(\beta)}\} \\ &\quad = [h(bb_p) + h^*(i(b\overline{b_p}))]\Pi\{\overline{h^*(q)} : q \in i(b) \sim X^{(\beta)}\} \\ &\quad = h(bb_p)\Pi\{\overline{h^*(q)} : q \in i(bb_p) \sim X^{(\beta)}\}\Pi\{\overline{h^*(q)} : q \in i(b\overline{b_p})\} \\ &\quad = h(bb_p)\Pi\{\overline{h^*(q)} : q \in i(bb_p) \sim X^{(\beta)}\} . \end{aligned}$$

In the same way, we can show that

$$h(b_p)\Pi\{\overline{h^*(q)} : q \in i(b_p) \sim X^{(\beta)}\} = h(bb_p)\Pi\{\overline{h^*(q)} : q \in i(bb_p) \sim X^{(\beta)}\} .$$

This completes our proof.

Remark. It is easily seen that there is only one complete homomorphism of B^* into C which is an extension of h, since such an extension must have the property that if $p \in X$, then

$$h^*(p) = \Pi\{h(b) : p \in i(b)\} ,$$

and if $S \subset X$, then $h^*(S) = \Sigma\{h^*(p) : p \in S\}$.

It remains only to be shown that a non-superatomic Boolean algebra has no free complete extension.

LEMMA. *If the Boolean algebra B has a free complete extension, then so does every subalgebra of B.*

Proof. Suppose that B has a free complete extension B^*, that i is the associated isomorphism of B into B^*, and that B' is a subalgebra of B. Let B^{**} be the complete subalgebra of B^{**} which is completely generated by $i[B']$. If h' is a homomorphism of B' into the complete Boolean algebra C, then, according to a result of Sikorski ([8], p. 112), h' can be extended to a homomorphism h of B into C. Since B^* is the free complete extension of B, there is a complete homomorphism h^* of B^* into C such that $h^* \circ i = h$. The restriction of h^* to B^{**} is a complete homomorphism of B^{**} into C, and for every $b \in B'$, $h^*(i(b)) = h'(b)$.

THEOREM. *A Boolean algebra has a free complete extension if and only if it is superatomic.*

Proof. It was shown above that every superatomic Boolean algebra has a free complete extension. Now suppose that B is a nonsuperatomic Boolean algebra. According to the theorem of § 2, B must have a subalgebra which is an infinite free Boolean algebra. It follows immediately from the result of Gaifman ([3]) and Hales ([4]) mentioned in our introduction, and the above lemma, that B has no free complete extension.

4. The free α-extensions of superatomic Boolean algebras. F. M. Yaqub showed in [10] that if a Boolean algebra B has a free α-extension B_α which is α-representable (that is, if B_α is the image of an α-field of sets under an α-complete homomorphism), then B_α is isomorphic to an α-field of sets, and if $\alpha \geqq 2^{\aleph_0}$, then B is superatomic. The results of § 3 enable us to prove the following converse of this statement.

THEOREM. *If B is a superatomic Boolean algebra, then the free α-extension of B is isomorphic to an α-field of sets.*

Proof. It was shown in § 3 that a superatomic Boolean algebra B has a free complete extension B^* which is a complete field of sets. For each cardinal α, let β_α denote the α-complete subfield of B^* which is α-generated by B. Suppose that h is a homomorphism of B into an α-complete Boolean algebra C. Let C' denote the normal completion of C, and let h^* be the complete homomorphism of B^* into C' which is an extension of h. h^* carries B_α into C since B α-generates B_α, $h^*[B] \subset C$, and C is an α-complete, α-regular subalgebra of C'. The restriction of h^* to B is thus an α-complete homomorphism of B_α into C, and is an extension of h. Hence B_α is the free α-extension of B.

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