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DERIVATIONS AND INTEGRAL CLOSURE

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Let $\mathcal{O}$ be an integral domain containing the rational numbers, Σ its quotient field, D a derivation of Σ, and $\mathcal{O}'$ the ring of elements in Σ quasi-integral over $\mathcal{O}$. It is shown that if $D\mathcal{O} \subset \mathcal{O}$, then $D\mathcal{O}' \subset \mathcal{O}'$.

According to a lemma of Posner [4], which is also used by him in a subsequent paper [5], if $\mathcal{O}$ is a finite integral domain over a ground field F of characteristic 0 and D is a derivation over F sending $\mathcal{O}$ into itself, then D also sends the integral closure of $\mathcal{O}$ into itself. The proof of this in [4] is wrong, but the statement itself is correct and a proof is here supplied. More generally it is proved that if $\mathcal{O}$ is any integral domain containing the rational numbers and D is a derivation such that $D\mathcal{O} \subset \mathcal{O}$, then $D\mathcal{O}' \subset \mathcal{O}'$, where $\mathcal{O}'$ is the ring of elements in the quotient field Σ of $\mathcal{O}$ that are quasi-integral over $\mathcal{O}$. The theorem is not true for characteristic $p \neq 0$, but if one uses the Hasse-Schmidt differentiations instead of derivations, one gets the corresponding theorem for a completely arbitrary integral domain $\mathcal{O}$.

Let $\mathcal{O}$ be an arbitrary integral domain containing the rational numbers, and let $\bar{\mathcal{O}}$ be the integral closure of $\mathcal{O}$. The question whether $D\mathcal{O} \subset \mathcal{O}$ implies $D\bar{\mathcal{O}} \subset \bar{\mathcal{O}}$ is related to the question whether the ring of formal power series $\bar{\mathcal{O}}[[t]]$ is integrally closed. Thus consider the statements:

A. *For every* $\mathcal{O}$, $D\mathcal{O} \subset \mathcal{O}$ *implies* $D\bar{\mathcal{O}} \subset \bar{\mathcal{O}}$, *and*

B. *For every* $\mathcal{O}$, $\bar{\mathcal{O}}[[t]]$ *is integrally closed.* We show that A and B are equivalent statements. (We also show: C. If $\bar{\mathcal{O}}[[t]]$ *is integrally closed, then* $D\mathcal{O} \subset \mathcal{O}$ *implies* $D\bar{\mathcal{O}} \subset \bar{\mathcal{O}}$.) Now according to the last exercise in Nagata's book *Local Rings*, [3; p. 202, Ex. 5], B is a true statement, but we give a counter-example, which also leads to a counter-example for A.

2. Criticism of Posner's proof. Posner purports to prove that if P is a place of the quotient field Σ of $\mathcal{O}$ that has F as residue field and is finite on $\mathcal{O}$ and if $g \in \Sigma$ is finite at P, then Dg is finite at P. This is not so, as the following example shows. Let $\mathcal{O} = F[X, Y]$ be polynomial ring in two indeterminates over F. Let $D = \partial/\partial X$. Let P_1 be the place of $F(X, Y)$ over $F(Y/X)$ obtained by mapping X into 0, let P_2 be the place of $F(Y/X)/F$ obtained by

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mapping Y/X into any element of F, and let P be the composite place. Then X, Y, Y/X are finite at P, but $\partial(Y/X)/\partial X = -Y/X^2$ is not.[1]

One reason that Posner's proof fails is that there are no parameters such as those of which he speaks, except in the case that the degree of transcendency of $\mathcal{O}/F$ is 1. In that case, Posner's argument yields a proof.

3. A generalization. Let $\mathcal{O}$ be an arbitrary domain, with quotient field Σ. An element $\alpha \in \Sigma$ is said to be quasi-integral over $\mathcal{O}$ if all powers of α are contained in a finite $\mathcal{O}$-module contained in Σ, or, what comes to the same, if there is a $d \in \mathcal{O}$, $d \neq 0$, such that $d\alpha^\rho \in \mathcal{O}$, $\rho = 0, 1, \cdots,$; (see [2]). If $\mathcal{O}$ is a Noetherian domain, then the concepts of integral dependence and quasi-integral dependence (for elements in Σ) become the same; but it is the concept of quasi-integral dependence, rather than that of integral dependence, which is adapted to our considerations. The elements in Σ that are quasi-integral over $\mathcal{O}$ form a ring $\mathcal{O}'$, which in the case $\mathcal{O}$ is Noetherian is the integral closure $\bar{\mathcal{O}}$ of $\mathcal{O}$. The base field F plays little role, and it will be sufficient to assume that $\mathcal{O}$ contains the rational numbers.

THEOREM. *Let $\mathcal{O}$ be an arbitrary integral domain containing the rational numbers, let $\mathcal{O}'$ be the ring of elements in the quotient field Σ of $\mathcal{O}$ quasi-integral over $\mathcal{O}$, and let D be a derivation of Σ. Then: if $D\mathcal{O} \subset \mathcal{O}$, then $D\mathcal{O}' \subset \mathcal{O}'$.*

Proof. Let $\Sigma[[t]]$ be the ring of formal power series in a letter t over Σ and let $\Sigma((t))$ be its quotient field. The mapping $\Sigma c_i t^i \longrightarrow \Sigma(Dc_i)t^i$, $i \geqq 0$, $c_i \in \Sigma$, is a derivation of $\Sigma[[t]]$ into itself and extends D; it has a unique extension to $\Sigma((t))$, which will also be denoted D. Let E be the expression $1 + tD + (t^2/2!)D^2 + \cdots (= e^{tD})$. Then $\alpha + tD\alpha + (t^2/2!)D^2\alpha + \cdots$, to be denoted $E\alpha$, has a meaning for every $\alpha \in \Sigma[[t]]$, i.e., the partial sums converge in the topology defined by powers of (t); and the mapping $\alpha \longrightarrow E\alpha$ is an isomorphism of $\Sigma[[t]]$ into itself, as one easily verifies.[2] Its unique extension to $\Sigma((t))$ will

[1] Far from all, or even infinitely many, valuation rings $\mathfrak{B}$ centered at (X, Y) being sent into themselves by $D = \partial/\partial X$, there is one and only one. In fact, restricting oneself to valuation rings $\mathfrak{B}$ centered at (X, Y), if $D\mathfrak{B} \subset \mathfrak{B}$, then $X/Y \notin \mathfrak{B}$, since $D(X/Y) = 1/Y \notin \mathfrak{B}$. Hence $Y/X \in \mathfrak{B}$, and therefore $D(Y/X)$, $D^2(Y/X)$, etc. are also in $\mathfrak{B}$. Since $D^{n-1}(Y/X) = c_n Y/X^n$ $(c_n \in K)$, $v(Y) \geqq n\, v(X)$ for $n = 1, 2, \cdots$, where v is the valuation corresponding to $\mathfrak{B}$. Thus $\mathfrak{B}$ could not be other than the ring of the valuation in which $v(X)$ is infinitely small with respect to $v(Y)$; and for that ring one checks that $D\mathfrak{B} \subset \mathfrak{B}$.

[2] We only use that $\alpha \rightarrow E\alpha$ is a monomorphism, but it is actually onto $\Sigma[[t]]$ as one sees from the identity $e^{tD}(e^{-tD}\alpha) = \alpha$.

also be denoted E. Since $D\mathscr{O} \subset \mathscr{O}$, one has $D\mathscr{O}[[t]] \subset \mathscr{O}[[t]]$, and since $\mathscr{O}$ contains the rationals, $E\mathscr{O}[[t]] \subset \mathscr{O}[[t]]$.

Let α be quasi-integral over $\mathscr{O}$, and let $d \in \mathscr{O}$ be such that $d\alpha^\rho \in \mathscr{O}$, $\rho = 0, 1, \cdots$. Then $E(d\alpha^\rho) = Ed(E\alpha)^\rho \in \mathscr{O}[[t]]$, $\rho = 0, 1, \cdots$. Hence $dEd(E\alpha - \alpha)^\rho \in \mathscr{O}[[t]]$, $\rho = 0, 1, \cdots$; here we use that d and Ed are in $\mathscr{O}[[t]]$. The coefficient of t^ρ in $dEd(E\alpha - \alpha)^\rho$, i.e., the leading coefficient, is $d^2(D\alpha)^\rho$; and this coefficient, as well as all the others, are in $\mathscr{O}$. Hence $D\alpha$ is quasi-integral over $\mathscr{O}$.

COROLLARY. *If $d \in \mathscr{O}$ and $\alpha \in \Sigma$ are such that $d\alpha^i \in \mathscr{O}, i = 0,1,\cdots,\rho$, then $d^2(D\alpha)^i \in \mathscr{O}$, $i = 0, 1, \cdots, \rho$.*

Let $\mathfrak{C} = \{c \mid c \in \mathscr{O}, c\mathscr{O}' \subset \mathscr{O}'\}$; then $\mathfrak{C}$ is an ideal, which in the case $\mathscr{O}'$ is the integral closure $\bar{\mathscr{O}}$ of $\mathscr{O}$ is called the conductor of $\mathscr{O}$.

COROLLARY. *If $D\mathscr{O} \subset \mathscr{O}$, then $D\mathfrak{C} \subset \mathfrak{C}$. In other words, $\mathfrak{C}$ is a differential ideal for any derivation (or any family of derivations) sending $\mathscr{O}$ into itself.*

Proof. If $c \in \mathfrak{C}$ and $\alpha \in \mathscr{O}'$, then $(Dc)\alpha = D(c\alpha) - cD\alpha \in \mathscr{O}$, so that also $(Dc)\mathscr{O}' \subset \mathscr{O}'$.

The last corollary can sometimes be used to prove that a given integral domain $\mathscr{O}$ is integrally closed (see [4]). We first restrict ourselves to a class of integral domains $\mathscr{O}$ such that $\bar{\mathscr{O}} = \mathscr{O}'$, for example, the class of Noetherian domains. Then we restrict ourselves further to a class $\mathscr{C}$ of domains $\mathscr{O}$ such that $\mathscr{O}$ has a conductor $\mathscr{O} : \bar{\mathscr{O}} \neq (0)$, or equivalently, such that $\bar{\mathscr{O}}$ is contained in a finite $\mathscr{O}$-module (contained in Σ), for example, the class of finite integral domains (see [7; p. 267]), or quotient rings thereof, or the class of complete local domains (see [3; p. 114]). (For examples of Noetherian domains not having this property, see [3; p. 205 ff]; for an example in characteristic 0, see [1]). Then we can state:

COROLLARY. *Let $\mathscr{O}$ be an integral domain belonging to a class $\mathscr{C}$ defined just above, let $\mathscr{O}$ contain the rational numbers, and let $\{D\}$ be a (finite or infinite) family of derivations of $\mathscr{O}$ into itself. Then, if $\mathscr{O}$ is differentiably simple under $\{D\}$ (i.e., has no differential ideal other than (0) or (1)), then $\mathscr{O}$ is integrally closed.*

4. Extension of D to $\bar{\mathscr{O}}$. The above is a simplification of our original proof for a finite integral domain. The idea was that since E sends $\mathscr{O}[[t]]$ into itself, it also sends the integral closure of $\mathscr{O}[[t]]$

into itself. It was then sufficient to prove that $\bar{\mathcal{O}}[[t]]$ is integrally closed; in fact, we have the following theorem *for any integral domain* $\mathcal{O}$ *containing* the rational numbers.

THEOREM C. *If* $\bar{\mathcal{O}}[[t]]$ *is integrally closed and* $D\mathcal{O} \subset \mathcal{O}$, *then* $D\bar{\mathcal{O}} \subset \bar{\mathcal{O}}$. *(Here* $\bar{\mathcal{O}}$ *is the integral closure of* $\mathcal{O}$.*)*

Proof. If $\alpha \in \Sigma$, $\alpha = c/d$, $c, d \in \mathcal{O}$, then $E\alpha = Ec/Ed$, so $E\alpha$ is in the quotient field of $\mathcal{O}[[t]]$. If α is integral over $\mathcal{O}$, then $E\alpha = \alpha + tD\alpha + \cdots$ is integral over $\mathcal{O}[[t]]$, hence in $\bar{\mathcal{O}}[[t]]$, whence $D\alpha \in \bar{\mathcal{O}}$.

Our proof that $\bar{\mathcal{O}}[[t]]$ was integrally closed for $\mathcal{O}$ a finite integral domain depended on the following observation, *which holds for an arbitrary domain* $\mathcal{O}$.

THEOREM. *If* $\mathcal{O}$ *is completely integrally closed (i.e., if* $\mathcal{O}' = \mathcal{O}$*), then so is* $\mathcal{O}[[t]]$. *More generally, for any* $\mathcal{O}$, $(\mathcal{O}[[t]])' \subset \mathcal{O}'[[t]]$.

Proof. Let $\alpha(t)$ be quasi-integral over $\mathcal{O}[[t]]$. Then there is a $d \in \mathcal{O}[[t]]$, $d = d(t) \neq 0$, such that $d\alpha^\rho \in \mathcal{O}[[t]]$, $\rho = 0, 1, \cdots$. Since $\operatorname{ord} d + \rho \operatorname{ord} \alpha \geqq 0$, $\rho = 0, 1, \cdots$, one first observes that $\alpha \in \Sigma[[t]]$. Let $d = d_s t^s + d_{s+1} t^{s+1} + \cdots$, $d_s \neq 0$, and let $\alpha = \alpha_r t^r + \alpha_{r+1} t^{r+1} + \cdots$. Since the leading coefficient of $d\alpha^\rho$ is in $\mathcal{O}$, we have $d_s \alpha_r^\rho \in \mathcal{O}$, whence α_r is quasi-integral over $\mathcal{O}$. Now $\alpha - \alpha_r t^r$ is quasi-integral over $\mathcal{O}[[t]]$, whence α_{r+1} is quasi-integral over $\mathcal{O}$; and in this way one sees that all the coefficients of α are quasi-integral over $\mathcal{O}$.

If $\mathcal{O}$ is Noetherian, then so is $\mathcal{O}[[t]]$. Hence:

COROLLARY. *If* $\mathcal{O}$ *is an integrally closed Noetherian domain, then so is* $\mathcal{O}[[t]]$.

This is Nagata's (47.6) in [3; p. 200].

Finally, if $\mathcal{O}$ is a finite integral domain, then so is $\bar{\mathcal{O}}$, whence in this case $\bar{\mathcal{O}}[[t]]$ is integrally closed. Recalling that $\bar{\mathcal{O}}$ is a finite $\mathcal{O}$-module (see [7; p. 267]), one sees that $\bar{\mathcal{O}}[[t]]$ is even the integral closure of $\mathcal{O}[[t]]$ in accordance with the following:

THEOREM. *Let* $\mathcal{O}$ *be an integral domain whose integral closure is Noetherian and is a finite* $\mathcal{O}$*-module. Then the integral closure of* $\mathcal{O}[[t]]$ *is* $\bar{\mathcal{O}}[[t]]$.

Proof. Let $\bar{\mathcal{O}} = \mathcal{O}w_1 + \cdots + \mathcal{O}w_s$. Then

$$\mathscr{O}[[t]] = \mathscr{O}[[t]]w_1 + \cdots + \mathscr{O}[[t]]w_s ,$$

whence $\bar{\mathscr{O}}[[t]]$ is a finite $\mathscr{O}[[t]]$-module and thus integral over $\mathscr{O}[[t]]$. Let d be a common denominator of the w_i when written as quotients of elements in $\mathscr{O}$. Then $d\bar{\mathscr{O}}[[t]] \subset \bar{\mathscr{O}}[[t]]$, whence $\mathscr{O}[[t]]$ and $\bar{\mathscr{O}}[[t]]$ have the same quotient field. As we have already seen that $\bar{\mathscr{O}}[[t]]$ is integrally closed, the proof is complete.

Although not necessary for our considerations, we mention the following:

THEOREM. *If $\mathscr{O}$ is a Noetherian domain, then $\bar{\mathscr{O}}[[t]]$ is integrally closed, where t abbreviates a set $t_1, \cdots, t_n$ of n distinct letters.*

Proof. $\bar{\mathscr{O}}$ is a Krull ring (see [3; p. 118]), hence from the definition [3; p. 115], $\bar{\mathscr{O}}_p$ is a Noetherian valuation ring for every minimal prime ideal p of $\bar{\mathscr{O}}$. Moreover $\bar{\mathscr{O}} = \cap \bar{\mathscr{O}}_p$, where the intersection is taken over the minimal prime ideals of $\bar{\mathscr{O}}$ (see [3; p. 116]). Since $\bar{\mathscr{O}}_p[[t]]$ is integrally closed, also $\bar{\mathscr{O}}[[t]] = \cap \bar{\mathscr{O}}_p[[t]]$ is integrally closed.

Now consider the statements A and B mentioned at the beginning. We say that A and B are equivalent. Recall that we are assuming that $\mathscr{O}$ contains the rational numbers.

$B \Rightarrow A$. This follows at once from C, the first theorem of this section.

$A \Rightarrow B$. Let α be in the quotient field of $\bar{\mathscr{O}}[[t]]$ and integral over $\bar{\mathscr{O}}[[t]]$. Then $\alpha \in \Sigma[[t]]$, $\alpha = \alpha_0 + \alpha_1 t + \cdots$. From an equation of integral dependence for α on $\bar{\mathscr{O}}[[t]]$, by placing $t = 0$, one sees that $\alpha_0 \in \bar{\mathscr{O}}$. Now apply A to the ring $\bar{\mathscr{O}}[[t]]$ and the derivation $D = \partial/\partial t$. Then $\partial\alpha/\partial t, \partial^2\alpha/\partial t^2, \cdots$ are integral over $\bar{\mathscr{O}}[[t]]$, whence all the coefficients of α are in $\bar{\mathscr{O}}$.

Now according to the last exercise in Nagata's *Local Rings*, B is a true statement; however, we will show that this is incorrect.

THEOREM. *If $\mathscr{O}$ is an (integrally closed) integral domain containing a field and there is a nonunit $b \in \mathscr{O}$ such that $\cap\, (b^\rho) \neq (0)$, then $\mathscr{O}[[t]]$ is not integrally closed.*

Proof. Let p be the characteristic and $n > 1$, an integer such that $n \not\equiv 0(p)$. Then $b^n + b^{n-2}t$ has an nth root $\alpha = b[1 + (t/b^2)]^{1/n} = b[1 + c_1(t/b^2) + c_2(t^2/b^4) + \cdots]$ in $\Sigma[[t]]$, where $c_1, c_2, \cdots$ are in the prime field of Σ and $c_1 \neq 0$. If $a \in \cap\, (b^\rho)$ and $a \neq 0$, then $a\alpha \in \mathscr{O}[[t]]$, so that α is in the quotient field of $\mathcal{O}[[t]]$. Now α is integral over $\mathcal{O}[[t]]$, but is not in $\mathcal{O}[[t]]$. Hence $\mathcal{O}[[t]]$ is not integrally closed.

THEOREM. *Let $\mathfrak{B}$ be a (proper) valuation ring containing a field. Then $\mathfrak{B}[[t]]$ is integrally closed if and only if $\mathfrak{B}$ is of rank 1, i.e., if and only if there is no chain $0 < p_1 < p_0 < \mathfrak{B}$ of prime ideals.*

Proof. If $\mathfrak{B}$ is of rank 1, then it is well-known and can be checked at once, that $\mathfrak{B}$ is completely integrally closed. Hence $\mathfrak{B}[[t]]$ is completely integrally closed, hence integrally closed.

On the other hand, if $\mathfrak{B}$ is of rank > 1 and $0 < p_1 < p_0 < \mathfrak{B}$ is a chain of prime ideals in $\mathfrak{B}$ and $b \in p_0 - p_1$, then $p_1 \subset \cap\ (b^\rho)$, whence $\mathfrak{B}[[t]]$ is not integrally closed.

To get a counter-example to Nagata's last exercise, one has but to take $\mathcal{O}$ to be a valuation ring of rank > 1 that contains a field.[3]

To get an example of a ring $\mathcal{O}$ and derivation D such that $D\mathcal{O} \subset \mathcal{O}$ but $D\bar{\mathcal{O}} \not\subset \bar{\mathcal{O}}$, let $\mathfrak{B}$ be a valuation ring of rank 2 containing the rational numbers, let $\mathcal{O} = \mathfrak{B}[[t]]$ and $D = \partial/\partial t$. Let b be a nonunit in $\mathfrak{B}$ such that $\cap\ (b^\rho) \neq (0)$, and let

$$\alpha = (b^2 + t)^{1/2} = b\left[1 + c_1\frac{t}{b^2} + c_2\frac{t^2}{b^4} + \cdots\right],$$

where $c_1, c_2, \cdots$ are rational numbers. Then α is integral over $\mathcal{O} = \mathfrak{B}[[t]]$ but $D\alpha$ is not.

Concerning the proof spoken of at the beginning of this section, the author is obliged to Professor Mumford for the remark in context that if D is a derivation, then e^D, formally at any rate, is an isomorphism. The introduction of the parameter t on the one hand prevents the computations from collapsing into meaninglessness, and on the other allows one to recover D from e^{tD}.

5. The case of characteristic $p \neq 0$. For $p \neq 0$, the theorem of § 3 is not true, even for curves. Thus consider the curve given by $Y^p - X^p - X^{p+1} = 0$. One checks that $Y^p - X^p - X^{p+1}$ is irreducible (over the ground field F). Let (x, y) be a generic point of the curve over F. Let D be a derivation of $F(y)/F$ with $Dy = 1$; since x is separable over $F(y)$, D can be extended uniquely to a derivation, still to be denoted D, of $F(y, x)$. One finds $-(p+1)x^pDx = 0$, hence $Dx = 0$. Let $\mathcal{O} = F[x, y]$. Then $D\mathcal{O} \subset \mathcal{O}$. Now y/x is integral

[3] In reference to the exercise, Nagata [**3**; p. 221] cites Sugaku, Vol. 9, No. 1 (1957), p. 61, which we have not been able to locate; and while he notes that the proof there is not complete, he remarks that "a supplement is expected to appear soon".

over $\mathcal{O}$, since $(y/x)^p = 1 + x$, but $D(y/x) = 1/x$ is not, as otherwise it would be integral over $F[x]$.

However, if one uses the Hasse-Schmidt differentiations [6] instead of derivations, one gets the corresponding theorem.[4] Recall that a differentiation D of a field Σ into itself is a sequence $D = (\delta_0, \delta_1, \delta_2, \cdots)$ of mappings of Σ into itself with $\delta_0 = 1$ and satisfying the properties:

$$\delta_i(x + y) = \delta_i x + \delta_i y$$
$$\delta_\nu xy = \sum_{i+j=\nu} \delta_i x \delta_j y \,.$$

By $D\mathcal{O} \subset \mathcal{O}$ we now mean $\delta_i\mathcal{O} \subset \mathcal{O}$ for every i. Then

$$E = \delta_0 + t\delta_1 + t^2\delta_2 + \cdots$$

still yields an isomorphism and can be used instead of our previous E to get the conclusion $D\mathcal{O}' \subset \mathcal{O}'$. (After obtaining $\delta_1\mathcal{O}' \subset \mathcal{O}'$ as before, we argue that $d^3Ed(E\alpha - \alpha - t\delta_1\alpha)^\rho \in \mathcal{O}[[t]]$, $\rho = 0, 1, \cdots$, whence $d^4(\delta_2\alpha)^\rho \in \mathcal{O}$, $\rho = 0, 1, \cdots$, and $\delta_2\alpha$ is quasi-integral over $\mathcal{O}$, etc.) In the case of characteristic 0, the same argument shows one can drop the assumption that $\mathcal{O}$ contains the rationals (i.e., if one uses differentiations instead of derivations).

The corollaries of the theorem of § 3 also have easily stated generalizations, with similar proofs.

REMARK. Since $(1 + (1 + 4t)^{1/2})/2 \in Z[[t]]$, the last two theorems of §4 hold without the field condition.

REFERENCES

1. Y. Akizuki, *Einige Bemerkungen über primäre Integritätsbereich mit Teilerkettensatz*, Proc. Phys.-Math. Soc. Japan, 3rd ser., **17** (1935), 327-336.
2. W. Krull, *Beiträge zur Arithmetik kommutativer Integritätsbereiche, II*, Math. Zeitschrift, **41** (1936), 665-679.
3. M. Nagata, *Local Rings*. New York, 1962.
4. E. C. Posner, *Integral closure of differential rings*, Pacific J. Math. **10** (1960), 1393-1396.
5. ———, *Integral closure of rings of solutions of linear differential equations*, Pacific J. Math. **12** (1962), 1417-1422.
6. F. K. Schmidt, *Noch eine Begründung der Theorie der höheren Differentialquotienten in einer algebraischen Funktionenkörper einer Unbestimmten. Zusatz bei der Korrektur*, J. für die reine u. angewandte Math. **177** (1937), 223-237.
7. O. Zariski and P. Samuel, *Commutative Algebra*, Vol. 1. New York, 1958.

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[4] For some useful information on differentiations, see K. Okugawa, "Basic properties of differential fields of an arbitrary characteristic and the Picard-Vessiot theory", J. of Math. of Kyoto Univ., Vol. 2 (1963), pp. 295-322.