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TOPOLOGICAL METHODS FOR NON-LINEAR ELLIPTIC EQUATIONS OF ARBITRARY ORDER

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Consider a strongly elliptic nonlinear partial differential equation (e): $F(x, u, Du, \cdots, D^{2m}u) = 0$, of order $2m$ on a bounded, smoothly bounded subset Ω of R^n. For second-order operators, Leray and Schauder, using the theory of the topological degree for completely continuous displacements of a Banach space, showed that the existence of solutions of the Dirichlet problem for (e) could be proved under the assumption of suitable a-priori bounds for solutions of the type of (e). In the present paper, using precise results on the solutions of linear elliptic differential operators with Hölder continuous coefficients as well as a variant of the Leray-Schauder method, we extend this result to equations of arbitrary even order. We also obtain results on uniqueness in the large under hypotheses of local uniqueness.

Theorem 1 is our general result of Leray-Schauder type for the most general sort of strongly elliptic nonlinear equation. Its proof is based upon Theorems 2 and 3 which concern equations for which one has local uniqueness of solutions. Theorem 2, which extends a result of Schauder [16] for second order equations, asserts the solvability of the equation $F(u) = f$ for f near f_0 with u near u_0 if the solution is locally unique. Under similar hypotheses and an additional a priori bound, Theorem 3 asserts the existence and uniqueness of the solution for all f. Theorem 4 and 5 specialize Theorem 1 with a drastic simplification of hypotheses to quasi-linear equations of order $2m$ and to nonlinear second-order equations. Theorem 4, in particular, gives a simple and very general extension of the Leray-Schauder method as given in [9] for quasi-linear equations of second order.

The writer is indebted to Stephen Smale for a number of conversations which stimulated his interest in giving a systematic treatment of the Leray-Schauder theory for general non-linear elliptic equations.

1. Let Ω be a bounded, smoothly bounded open subset of the Euclidean space R^n, Γ its boundary in R^n, $\bar{\Omega}$ its closure in R^n, $(n \geqq 1)$. We denote the general point of Ω by $x = (x_1, \cdots, x_n)$ and for each n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ of nonnegative integers, we set

$$D^\alpha = \prod_{j=1}^{n} \left(\frac{\partial}{\partial x_j}\right)^{\alpha_j}, \qquad |\alpha| = \sum_{j=1}^{n} \alpha_j .$$

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For a given integer $m \geqq 1$, we consider vectors $p = \{p_\alpha / |\alpha| \leqq 2m\}$ with a real component for each partial derivative of order $\leqq 2m$. They form a real vector space R^M for some integer M, which we shall not give explicitly. We assume that we are given a function

$$F: \bar{\Omega} \times R^M \to R^1$$

which is twice continuously differentiable on $\Omega \times R^M$ and whose derivatives up to second order are uniformly bounded on compact subsets of $\bar{\Omega} \times R^M$. Using this function F and the mapping $p_x: C^{2m}(\Omega) \to R^M$ given by

$$p_x(u) = \{D^\alpha u(x) / |\alpha| \leqq 2m\}$$

we may form the general partial differential operator (not necessarily linear) of order $2m$ by

$$F(u)(x) = F(x, p_x(u)),\ x \in \Omega,\ u \in C^{2m}(\Omega)$$

and consider the partial differential equation

$$(1) \qquad F(u) = 0\ , \qquad u \in C^{2m}(\Omega)\ .$$

For a given real number λ with $0 < \lambda < 1$, and any nonnegative integer j,

$$(2) \qquad \begin{aligned} &C^{j,\lambda}(\Omega) = \{u \mid u \in C^j(\Omega). \quad \text{There exists a constant} \\ &c > 0 \text{ such that } |D^\alpha u(x) - D^\alpha u(y)| \leqq c\,|x - y|^\lambda, \\ &x, y \in \Omega,\ |\alpha| \leqq j\}. \end{aligned}$$

$C^{j,\lambda}(\Omega)$ is a Banach space (and indeed a Banach algebra) with respect to the norm

$$(3) \qquad \begin{aligned} \|u\|_{C^{j,\lambda}} = &\sum_{|\alpha| \leqq j} \sup_{x \in \Omega} |D^\alpha u(x)| \\ &+ \sum_{|\alpha| = j} \sup_{x,y \in \Omega; x \neq y} \{|x - y|^{-\lambda} |D^\alpha u(x) - D^\alpha u(y)|\}\ . \end{aligned}$$

If we assume (as we shall henceforward) that Γ is locally a manifold of class $C^{2m,\lambda}$ then there is an obvious and unequivocal sense that can be given to $D^\alpha u|_\Gamma$ for any u in $C^{2m,\lambda}(\Omega)$ and any α with $|\alpha| \leqq 2m$.

We may therefore form the closed subspace $C_0^{2m,\lambda}(\Omega)$ of $C^{2m,\lambda}(\Omega)$ which consists of all u in $C^{2m,\lambda}(\Omega)$ which satisfy the homogeneous Derichlet boundary condition of order m on Γ, i.e.

$$(4) \qquad D^\beta u|_\Gamma = 0\ , \qquad |\beta| \leqq m - 1\ .$$

For u in $C^{2m,\lambda}(\Omega)$, $F(u)$ lies in $C^{0,\lambda}(\Omega)$ as follows by a routine argument using the fact that $F(x, p)$ satisfies a Hölder condition with exponent λ in all arguments on compact subsets of $\bar{\Omega} \times R^M$. Moreover, the mapping $u \longrightarrow F(u)$ is continuous from $C^{2m,\lambda}(\Omega)$ to $C^{0,\lambda}(\Omega)$, and indeed continuously Frechet differentiable with Frechet derivative at u in $C^{2m,\lambda}(\Omega)$ given by

$$F'(u)(\eta) = \sum_{|\alpha| \leqq 2m} F_\alpha(x, p(u)) D^\alpha \eta \tag{5}$$

where $F_\alpha = \partial F / \partial p_\alpha$.

We say that the nonlinear differential operator $F(u)$ is *strongly elliptic* if for each $p \in R^M$, the linear differential operator A_p given

$$A_p(\eta) = \sum_{|\alpha| = 2m} F_\alpha(x, p) D^\alpha \eta \tag{6}$$

is uniformly strongly elliptic on Ω.

Let $C_0^{2m-1,\lambda}(\Omega) = \{u \mid u \in C^{2m-1,\lambda}(\Omega); D^\beta u|_\Gamma = 0 \text{ for } |\beta| \leqq m-1\}$.

THEOREM 1. *Let F be a C^2-function $\bar{\Omega} \times R^M$ which defines a strongly elliptic nonlinear partial differential operator $F(u)$ of order $2m$ on Ω. Let $0 < \lambda < 1$, and for $0 \leqq t \leqq 1$, let*

$$F(x, p, t) = tF(x, P) + (1-t)\Sigma_{|\alpha|} |p_\alpha|^2,$$

$F_t(u)$ the corresponding partial differential operator of order $2m$. Suppose that all of the following conditions are satisfied:

(1) *For each $R > 0$, there exists a constant μ with $0 < \mu < \lambda < 1$ and a differential operator H of order $\leqq 2m-1$ (possibly nonlinear) on Ω such that for each $u \in C_0^{2m-1,\lambda}(\Omega)$, $v \in C_0^{2m,\mu}(\Omega)$ with $\|u\|_{C^{2m-1,\lambda}} \leqq R$, and*

$$\{p_x(u, v)\}_\alpha = \begin{cases} D^\alpha u(x), & |\alpha| < m \\ D^\alpha v(x), & |\alpha| = m \end{cases}$$

the linear equation

$$\sum_{|\alpha| = 2m} F_{t\alpha}(x, p_x(u, v)) D^\alpha \eta + \sum_{|\alpha| \leqq 2m-1} tH_\alpha(x, p_x(u)) D^\alpha \eta = 0$$

has only $\eta = 0$ as a solution in $C_0^{2m,\mu}(\Omega)$, $0 \leqq t \leqq 1$.

(2) *For given $R > 0$ and the corresponding function H of condition (1), there exists a function $R_1(s)$ such that for u in $C_0^{2m-1,\lambda}(\Omega)$ with $\|u\|_{C^{2m-1,\lambda}(\Omega)} \leqq R$ and any v in $C_0^{2m,\lambda}(\Omega)$ such that*

$$F_t(p(u, v)) + tH(p(u, v)) = f$$

for some t in $[0, 1]$ and $f \in C^{0,\lambda}(\Omega)$ with $\|f\|_{C^{0,\lambda}(\Omega)} \leqq s$, we have

$$\|v\|_{C^{2m,\lambda}} \leqq R_1(s)$$

(3) *There exists a constant $R_0 > 0$ such that for any t in $[0, 1]$ and $v \in C_0^{2m,\lambda}(\Omega)$, if*

$$F_t(v) = 0 ,$$

we have

$$\| v \|_{C^{2m-1,\lambda}(\Omega)} < R_0 .$$

Then the equation

$$F(u) = 0$$

has a solution u in $C_0^{2m,\lambda}(\Omega)$.

THEOREM 2. *Let F be a* C^2-*function on* $\bar{\Omega} \times R^M$ *such that the corresponding partial differential operator* $F(u)$ *of order* $2m$ *is strongly elliptic. Suppose that for a given* u_0 *in* $C_0^{2m,\lambda}(\Omega)$, *the mapping* $u \rightarrow F(u)$ *of* $C_0^{2m,\lambda}(\Omega)$ *into* $C^{0,\lambda}(\Omega)$ *is one-to-one on some neighborhood of* u_0 *in* $C_0^{2m,\lambda}(\Omega)$.

Then F is an open mapping on some neighborhood of u_0.

THEOREM 3. *Let F be a* C^2-*function on* $\bar{\Omega} \times R^M$ *with* $F(u)$ *strongly elliptic. Suppose that both of the following conditions are satisfied:*

(i) *There exists* μ *with* $0 < \mu < \lambda < 1$ *such that for each u in* $C_0^{2m,\mu}(\Omega)$, *the linear equation*

$$\sum_{|\alpha| \leqq 2m} F_\alpha(x, p_x(u)) D^\alpha \eta = 0 , \qquad in \qquad \Omega$$

has only $\eta = 0$ *as a solution in* $C_0^{2m,\mu}(\Omega)$.

(ii) *For each* f_0 *in* $C^\lambda(\Omega)$, *there exists constants* $k(f_0)$, $\varepsilon(f_0) > 0$ *such that for every solution u of* $F(u) = f$ *with* $u \in C_0^{2m,\lambda}(\Omega)$ *and* $\| f - f_0 \|_{C^{0,\lambda}(\Omega)} < \varepsilon(f_0)$,

$$\| u \|_{C^{2m,\lambda}(\Omega)} \leqq k(f_0) .$$

Then the equation $F(u) = f$ *has one and only one solution u in* $C_0^{2m,\lambda}(\Omega)$ *for each f in* $C^{0,\lambda}(\Omega)$.

We shall prove Theorems 2, 3, and 1 in that order. The proofs depend upon precise results on the Dirichlet problem for strongly elliptic linear operators which are discussed in detail in § 2, combined with topological arguments concerning nonlinear mappings of Banach spaces.

2. Let

$$A = \sum_{|\alpha| \leqq 2m} a_\alpha(x) D^\alpha$$

be a linear elliptic differential operator of order $2m$ on Ω with real coefficients in $C^{0,\lambda}(\Omega)$. A is said to be uniformly strongly elliptic on Ω if there exists a positive constant $c(c>0)$ such that

$$\sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \geqq c_0 \,|\, \xi \,|^{2m}$$

for all $x \in \Omega$, $\xi \in R^n$, (where $\xi^\alpha = \prod_{j=1}^{n} \xi_j^{\alpha_j}$).

The first basic fact that we shall employ about the Dirichlet problem for the strongly elliptic operator A is the estimate of Schauder type ([1], Theorem 7.3) given in the following Lemma:

LEMMA 1. *There exists a constant $C>0$ depending on C_0, Ω and the $C^{0,\lambda}(\Omega)$-norms of the functions a_α, such that for all u in $C^{2m,1}(\Omega)$*

$$\| u \|_{C^{2m,\lambda}(\Omega)} \leqq \{\| Au \|_{C^{0,\lambda}(\Omega)} + \| u \|_{C^{0,\lambda}(\Omega)}\} .$$

The second fact that we shall employ concerns the spectrum and resolvent of the operator A under null Dirichlet boundary conditions:

LEMMA 2. *Let μ be any number with $0<\mu<\lambda<1$. There exists a constant k_0 depending only on Ω, c_0, and the $C^{0,\mu}(\Omega)$-norms of the coefficients a_α such that:*

(a) *For $k \geqq k_0$, and any f in $C^{0,\lambda}(\Omega)$, the equation*

$$Au + ku = f$$

has a solution u in $C_0^{2m,\lambda}(\Omega)$.

(b) *The solution u of the equation*

$$Au + ku = f$$

with $u \in C_0^{2m,\lambda}(\Omega)$, $f \in C^{0,\lambda}(\Omega)$, satisfies an inequality of the form

$$\| u \|_{L^2(\Omega)} \leqq c_1 \| Au + ku \|_{L^2(\Omega)}$$

with c_1 dependent only on Ω, c_0, and the $C^{0,\mu}(\Omega)$-norms of the a_μ.

Proof of Lemma 2. By Theorem 15 of [2], (p. 75) there exists k_0 depending only on Ω, c_0, and $C^{0,\mu}(\Omega)$ coefficients of the a_α such that for $k \geqq k_0$ and all f in $L^2(\Omega)$, there exists u in the space $W^{2m,2}(\Omega)$ where

$$W^{2m,2}(\Omega) = \{u \mid D^\alpha u \in L^2(\Omega) \text{ for } |\alpha| \leqq 2m\}$$

with

$$Au + ku = f$$

and u lying in the domain $D(A_2)$ of the realization of A in $L^2(\Omega)$ under null Dirichlet boundary conditions (in the sense of § 2 of [2]). For such u and k, we have the inequality

$$\| u \|_{L^2(\Omega)} \leqq \| Au + ku \|_{L^2(\Omega)}$$

of assertion (b). Since $C_0^{2m,\lambda}(\Omega) \subset D(A_2)$, the validity of assertion (b) follows.

If $R(G)$ is the restriction to Ω of the family of C^∞ functions with compact support in R^n, $R(G)$ is dense in $C^{0,\lambda}(\Omega)$. For every f in $R(G)$, the solution u in $D(A_2)$ of the equation $Au + ku = f$ lies in $C_0^{2m,\lambda}(\Omega)$ by Theorem 8 of [2] (page 65). For this solution, we have by Lemma 1,

$$\begin{aligned} \| u \|_{C^{2m,\lambda}(\Omega)} &\leqq c\{\| Au \|_{C^{0,\lambda}(\Omega)}\} + \| u \|_{C^{0,\lambda}(\Omega)} \\ &\leqq c\{\| Au + ku \|_{C^{0,\lambda}(\Omega)} + (k+1) \| u \|_{C^{0,\lambda}(\Omega)}\} . \end{aligned}$$

Moreover, since the injection maps of $C^{2m,\lambda}(\Omega)$ into $C^{0,\lambda}(\Omega)$ and $L^2(\Omega)$ are compact, for each $\varepsilon > 0$, there exists $k(\varepsilon)$ such that

$$\| u \|_{C^{0,\lambda}(\Omega)} \leqq \varepsilon \| u \|_{C^{2m,\lambda}(\Omega)} + k(\varepsilon) \| u \|_{L^2(\Omega)} .$$

Hence

$$\begin{aligned} \| u \|_{C^{2m,\lambda}(\Omega)} \leqq c \| Au + ku \|_{C^{0,\lambda}(\Omega)} & \\ + c(k+1)\varepsilon \| u \|_{C^{2m,\lambda}(\Omega)} &+ c(k+1)k(\varepsilon) \| u \|_{L^2(\Omega)} . \end{aligned}$$

Choosing $\varepsilon > 0$ so small that $c(k+1)\varepsilon < 1/2$, we have

$$\begin{aligned} \| u \|_{C^{2m,\lambda}(\Omega)} &\leqq 2c \| Au + ku \|_{C^{0,\lambda}(\Omega)} + 2c(k+1)k(\varepsilon) \| u \|_{L^2(\Omega)} \\ &\leqq c_2 \| Au + ku \|_{C^{0,\lambda}(\Omega)} + c_3 \| Au + ku \|_{L^2(\Omega)} \\ &\leqq c_4 \| Au + ku \|_{C^{0,\lambda}(\Omega)} \end{aligned}$$

i.e.

$$\| u \|_{C^{2m,\lambda}(\Omega)} \leqq c_4 \| f \|_{C^{0,\lambda}(\Omega)}$$

for such f and u. Hence the mapping $f \to u$ from the dense subset $R(G)$ of $C^{0,\lambda}(\Omega)$ into $C_0^{2m,\lambda}(\Omega)$ can be extended by continuity to a bounded linear map S of $C^{0,\lambda}(\Omega)$ into $C^{2m,\lambda}(\Omega)$ such that

$$(A + kI)Sf = f , \qquad f \in C^{0,\lambda}(\Omega) .$$

Hence assertion (a) is proved.

The proof of Lemma 2 also established the following:

LEMMA 3. *The solution u in Lemma 2 of the equation $Au + ku = f$, $u \in C^{2m,\lambda}(\Omega)$, for $k \geqq k_0$, satisfies an inequality of the form*

$$\| u \|_{C^{2m,\lambda}(\Omega)} \leqq c_4 \| f \|_{C^{0,\lambda}(\Omega)}$$

where c_4 depends only on $|k|$, Ω, c_0, and the $C^{0,\lambda}$-norms of the coefficients a_α.

LEMMA 4. *A is a Fredholm operator of index zero from $C^{2m,\lambda}(\Omega)$ to $C^{0,\lambda}(\Omega)$. In particular if A is one-to-one, A is an isomorphism.*

Proof of Lemma 4. Let S be the inverse of $(A+kI)$ as above. Then for $f \in C^{0,\lambda}(\Omega)$,

$$\begin{aligned} ASf &= (A+kI)Sf - hSf \\ &= (I-kS)f \end{aligned}$$

i.e.

$$A = (I-kS)(A+kI) .$$

Since $(I-kS)$ is a Fredholm operator of index zero from $C^{0,\lambda}(\Omega)$ to $C^{0,\lambda}(\Omega)$ and $(A+kI)$ is an isomorphism of $C_0^{2m,\lambda}(\Omega)$ onto $C^{0,\lambda}(\Omega)$, A is a Fredholm operator of index zero from $C_0^{2m,\lambda}(\Omega)$ to $C^{0,\lambda}(\Omega)$.

3. Proof of Theorem 2. Let $u_0 \in C_0^{2m,\lambda}(\Omega)$ and suppose that F is one-to-one on an open neighborhood N of u_0 in $C_0^{2m,\lambda}(\Omega)$. We may take N of the form

$$N = \{u \mid u \in C_0^{2m,\lambda}(\Omega),\ \|u-u_0\|_{C^{2m,\lambda}} < \varepsilon\} .$$

Let A be the linear differential operator on Ω given by

$$A\eta = \sum_{|\alpha| \leqq 2m} F_\alpha(p(u_0))D^\alpha \eta .$$

Since F satisfies condition (a) of Theorem 1, A is uniformly strongly elliptic on Ω. Since the F_α are C^1 on $\bar{\Omega} \times R^M$ and therefore satisfy a Hölder condition with expondent λ on compact subsets of $\bar{\Omega} \times R^M$, the coefficients of A lie in $C^{0,\lambda}(\Omega)$. Hence by Lemmas 2 and 3, there exists $k > 0$ such that the linear mapping

$$u \longrightarrow Au + ku$$

of $C_0^{2m,\lambda}(\Omega)$ into $C^{0,\lambda}(\Omega)$ is an isomorphism of the two spaces. Let S be the inverse mapping, so that S is a bounded linear mapping of $C^{0,\lambda}(\Omega)$ into $C^{2m,\lambda}(\Omega)$.

Let u and v lie in N. Then

$$\begin{aligned} F(u) - F(v) &= \int_0^1 F'(\lambda u + (1-\lambda)v)(u-v)d\lambda \\ &= F'(u_0)(u-v) + R(u_0, u, v) \end{aligned}$$

where

$$R(u_0, u; v) = \int\{F'(\lambda u + (1-\lambda)v)(u-v) - F'(u_0)(u-v)\}d\lambda .$$

Given $\delta > 0$, we can choose $\varepsilon > 0$ so small that for $u, v \in N$, $0 \leqq \lambda \leqq 1$,

$$\| F'(\lambda u + (1-\lambda)v) - F'(u_0) \| < \delta .$$

Then

$$\| R(u_0, u, v) \|_{C^{0,\lambda}(\Omega)} \leqq \delta \| u - v \|_{C^{2m,\lambda}(\Omega)} .$$

Let

$$R_{u_0}(u) = F(u) - F'(u_0)u , \qquad u \in N .$$

Then

$$\begin{aligned} \| R_{u_0}(u) - R_{u_0}(v) \|_{C^{0,\lambda}(\Omega)} &= \| R(u_0, u, v) \|_{C^{0,\lambda}(\Omega)} \\ &\leqq \delta \| u' - v \|_{C^{2m,\lambda}(\Omega)} \end{aligned}$$

We form a mapping G of $C^{0,\lambda}(\Omega)$ into $C^{0,\lambda}(\Omega)$ by setting

$$G(f) = F(Su + u_0) .$$

There exists a neighborhood N_1 of 0 in $C^{0,\lambda}(\Omega)$ mapped by $u \to Su + u_0$ into N. To show that F is open at u_0, it suffices to show that G is open at zero. However

$$\begin{aligned} G(f) &= (F'(u_0) + R_{u_0})(Sf + u_0) \\ &= (A + kI)(Sf) - kSf + R_{u_0}(Sf + u_0) \\ &= f - kSf + R_{u_0}(Sf + u_0) . \end{aligned}$$

Here kS like S itself is a continuous linear map of $C^{0,\lambda}(\Omega)$ with $C^{2m,\lambda}(\Omega)$ and since the injection map of $C^{2m,\lambda}(\Omega)$ into $C^{0,\lambda}(\Omega)$ is compact, kS is a compact linear mapping of $C^{0,\lambda}(\Omega)$ into $C^{0,\lambda}(\Omega)$.

For f and f_1 in N_1, we know that

$$\begin{aligned} &\| R_{u_0}(Sf + u_0) - R_{u_0}(Sf_1 + u_0) \|_{C^{0,\lambda}(\Omega)} \\ &\quad \leqq \delta \| (Sf + u_0) - (Sf_1 + u_0) \|_{C^{2m,\lambda}(\Omega)} \\ &\quad \leqq c_5 \delta \| f - f_1 \|_{C^{0,\lambda}(\Omega)} \end{aligned}$$

If we choose $\delta > 0$ so small that $c_5\delta < 1$, the mapping

$$f \to f - R_{u_0}(Sf + u_0)$$

is a bicontinuous mapping of N_1 on an open neighborhood N_2 of zero in $C^{0,\lambda}(\Omega)$ and is T is the inverse of this mapping, $GT = I - kST$ has the same image on N_2 as G has on N_1. Moreover GT is one-to-one on N_2 since G is one-to-one on N_1.

Finally

$$GT = I - C$$

where C is a compact mapping of N_2 into $C^{0,\lambda}(\Omega)$. Since GT is one-to-one, it has an open image by the Schauder theorem on invariance of domain for compact displacements in Banach spaces (see Schauder [15], Leray [18], Nagumo [12]).

Proof of Theorem 3. By the construction of the proof of Theorem 2, for each u_0 in $C_0^{2m,\lambda}(\Omega)$, there is a homeomorphism h of a neighborhood N of u_0 with an open neighborhood N_2 of the origin in $C^{0,\lambda}(\Omega)$ such that for f in N_2

$$(F \circ h^{-1})f = F(u_0) + f - Cf$$

where C is a compact (possibly nonlinear) map of N_2 into $C^{0,\lambda}(\Omega)$.

Let u be an element of $C_0^{2m,\mu}(\Omega)$. The Frechet differential of F as a map of $C_0^{2m,\mu}(\Omega)$ into $C^{0,\mu}(\Omega)$ is given by the linear operator

$$A_u(\eta) = \sum_{|\alpha| \leqq 2m} F_\alpha(u) D^\alpha \eta \,, \qquad \eta \in C_0^{2m,\mu}(\Omega) \,.$$

By the hypotheses of Theorem 3, A_u is uniformly strongly elliptic on Ω, has coefficients in $C^{0,\mu}(\Omega)$, and is one-to-one on $C_0^{2m,\mu}(\Omega)$. Hence A_u is an isomorphism of $C_0^{2m,\mu}(\Omega)$ onto $C^{0,\mu}(\Omega)$. By the implicit function theorem, F is a local homeomorphism of $C_0^{2m,\mu}(\Omega)$ into $C^{0,\mu}(\Omega)$, i.e., it maps some neighborhood of each point u homeomorphically onto a neighborhood of $F(u)$.

The inverse A_u^{-1} of A_u is a bounded linear mapping of $C^{0,\mu}(\Omega)$ into $C_0^{2m,\mu}(\Omega)$ and its norm $\| A_u^{-1} \|$ between this pair of spaces is bounded for u on compact subset of $C^{2m,\mu}(\Omega)$. Hence so is the norm of A_u^{-1} as a linear map of $C^{0,\lambda}(\Omega)$ into $C^{0,\lambda}(\Omega)$. Let u run through a bounded subset B of $C_0^{2m,\lambda}(\Omega)$. Since B is precompact in $C_0^{2m,\mu}(\Omega)$, it follows that there exists a constant c such that for all u in $B, \eta \in C^{0,\lambda}(\Omega)$

$$\| \eta \|_{C^{0,\lambda}(\mu)} \leqq c \, \| A_u \eta \|_{C^{0,\lambda}(\mu)} \,.$$

If we apply Lemma 1 of § 2, we have

$$\| \eta \|_{C^{2m,\lambda}(\Omega)} \leqq c' \, \| A_u \eta \|_{C^{0,\lambda}(\Omega)}$$

with c' independent of u on B.

However A_u is also the Frechet differential of F as a map of $C_0^{2m,\lambda}(\Omega)$ into $C^{0,\lambda}(\Omega)$ at u, and between this pair of spaces F is a local homeomorphism as before.

We now apply the following theorem of Hadamard and P. Levy [10]: *If F is a local homeomorphism of a Banach space X into a Banach space Y and if no curve of infinite length in X is mapped by F onto a line segment in Y, then F is a homeomorphism of X onto Y.*

Our given mapping F is a local homeomorphism of $C_0^{2m,\lambda}(\Omega)$ into $C^{0,\lambda}(\Omega)$. Let $t \to \varphi(t)$ be a curve in $C^{2m,\lambda}(\Omega)$ covering a line segment in $C^{0,\lambda}(\Omega)$ with respect to F. If the curve is bounded in $C^{2m,\lambda}(\Omega)$ then since $\|(F'_u)^{-1}\|$ is bounded on bounded subsets of $C^{2m,\lambda}(\Omega)$, the length of the curve would be finite. Hence it suffices to prove that the inverse image under F of every closed line segment in $C^{0,\lambda}(\Omega)$ is bounded in $C^{2m,\lambda}(\Omega)$.

By hypothesis, each point f_0 of the line segment L_0 has a neighborhood such that $\|F^{-1}f\|_{C^{2m,(\Omega)}} \leqq k(f_0)$ in this neighborhood. Covering the compact set L_0 by a finite number of neighborhoods, we have $\|F^{-1}(f)\|_{C^{2m,\lambda}(\Omega)} \leqq k$ for $f \in L_0$.

4. Proof of Theorem 1. Let

$$R = R_0\,, \qquad B_R = \{u \mid u \in C_0^{2m-1,\lambda}(\Omega),\ \|u\| \leqq R\}\,.$$

Let H be the function corresponding to R by condition (1) of the hypothesis of Theorem 1. For each u in R_R, we consider the equation

$$(\mathrm{e}): \qquad F_t(p(u, v)) + tH(p(v)) = tH(p(u))$$

for v in $C_0^{2m,\lambda}(\Omega)$. The linearized form of equation (e) is

$$(\mathrm{e})': \qquad \sum_{|\alpha|=2m} F_{t,\alpha}(p(u, v))D^\alpha\eta + \sum_{|\alpha|\leqq 2m-1} tH_\alpha(p(u, v))D^\alpha\eta = 0$$

which by condition (1) has only $\eta = 0$ as a solution in $C_0^{2m,\mu}(\Omega)$ for a fixed μ with $0 < \mu < \lambda < 1$. Moreover by condition (2) of the hypothesis, the solution v of the equation

$$(\mathrm{e})_f \qquad F_t(p(u, v)) + tH(p(u, v)) = f$$

for $\|u\|_{C^{2m-1,\lambda}} \leqq R$ and f in $C^{0,\lambda}(\Omega)$ with $\|f\|_{C^{0,\lambda}} \leqq s$, where v lies in $C_0^{2m,\lambda}(\Omega)$, must satisfy the inequality

$$\|v\|_{C^{2m,\lambda}} \leqq R_1(s)\,.$$

Hence the hypotheses of Theorem 3 are satisfied for the family of equations $(e)_f$ and in particular, equation (e) has one and only one solution v_t for each t in $[0, 1]$.

We set $C_t(u) = v_t$. Then C_t is a well defined mapping on B_R whose range we consider as a subset of $C_0^{2m-1,\lambda}(\Omega)$. Since the map $u \to H(p(u))$ carries bounded sets of $C_0^{2m-1,\lambda}(\Omega)$ into bounded sets of $C^{0,\lambda}(\Omega)$, it follows from the argument of the preceding paragraph that

$$\|C_t(u)\|_{C^{2m,\lambda}(\Omega)} \leqq R_2$$

for all u in B_R and all t in $[0, 1]$ with a fixed constant $R_2 > 0$. Since $C_0^{2m,\lambda}(\Omega)$ has a compact injection into $C^{2m-1,\lambda}(\Omega)$, it follows that

$$\bigcup_{0\leq t\leq 1} C_t(B_R)$$

is precompact in $C_0^{2m-1,\lambda}(\Omega)$.

We wish now to verify that the mapping

$$[t, u] \to C_t(u)$$

is a continuous mapping of $[0,1] \times B_R$ into $C^{2m-1,\lambda}(\Omega)$. Let t_0 be a fixed number in $[0,1]$, u_0 a fixed element of B_R. For t near t_0 and u near u_0, we have

$$\begin{aligned} &F_t(p(u,v)) + tH_t(p(v)) - tH_t(p(u)) \\ &\quad = F_{t_0}(p(u_0,v)) + t_0H_{t_0}(p(v)) + \{F_t(p(u,v) - F_{t_0}(p(u_0,v)\} \\ &\qquad + \{tH_t(p(v)) - t_0H_{t_0}(p(v))\} - t_0H_{t_0}(p(u_0)) \\ &\qquad + \{tH_t(p(u)) - t_0H_{t_0}(p(u))\} \,. \end{aligned}$$

Furthermore

$$F_{t_0}(p(u_0,v)) = F_{t_0}(p(u_0,v_0)) + F'_{t_0}(p(u_0,v_0))(v-v_0) + R(u_0,v_0,t_0,v)$$

where $v_0 = C_{t_0}(v_0)$ and

$$\| R(u_0,v_0,t_0,v)\|_{C^{0,\lambda}} = o(\| v - v_0 \|)$$

as $\| v - v_0 \| \to 0$. (The norm of $v - v_0$ will be taken in $C_0^{2m,\lambda}(\Omega)$ throughout this argument.) Similarly

$$t_0H_{t_0}(p(v)) = t_0H_{t_0}(p(v_0)) + t_0H'_{t_0}(p(v_0))(v-v_0) + R_1(u_0,v_0,t_0,v)$$

where

$$\| R_1(u_0,v_0,t_0,v)\|_{C^{0,\lambda}} = o(\| v - v_0 \|)$$

as

$$\| v - v_0 \|_{C^{2m,\lambda}} \to 0 \,.$$

It follows that for v near v_0 in $C_0^{2m,\lambda}(\Omega)$ we have

$$\begin{aligned} &F_t(p(u,v)) + tH_t(p(v)) - tH_t(p(u)) \\ &\quad = \{F_{t_0}(p(u_0,v_0)) + t_0H_{t_0}(p(v_0)) - t_0H_{t_0}(p(u_0))\} \\ &\qquad + [F'_{t_0}(p(u_0,v_0)) + t_0H'_{t_0}(p(v_0))](v-v_0) \\ &\qquad + R_2(u_0,v_0,t_0,t,u,v) \end{aligned}$$

where

$$\begin{aligned} &\| R_2(u_0,v_0,t_0,t,u,v)\|_{C^{0,\lambda}(\Omega)} \\ &\quad \leqq \sigma(v-v_0)\,\| v-v_0\|_{C^{2m,\lambda}} + \sigma_3(u_0,u,t,t_0) \,, \end{aligned}$$

and

$$\sigma_3(u_0, u, t, t_0) \to 0$$

as

$$\|u - u_0\|_{C^{2m-1,\lambda}} + |t - t_0| \to 0$$

while

$$\sigma(v - v_0) \to 0 \qquad \text{as} \qquad \|v - v_0\|_{C^{2m,\lambda}} \to 0 .$$

The condition that

$$\text{(i)} \qquad F_t(p(u, v)) + tH_t(p(v)) - tH_t(p(u)) = 0$$

can therefore be satisfied if

$$[F'(p(u_0, v_0)) + t_0 H'_{t_0}(p(v_0))](v - v_0) = -R_2(u_0, v_0, t_0, t, u, v) .$$

The operator in square brackets is an isomorphism of $C_0^{2m,\lambda}(\Omega)$ with $C^{0,\lambda}(\Omega)$ by condition (1) of the hypothesis. Hence for $\|u - u_0\| + |t - t_0|$ sufficiently small, we may find a solution v of equation (i) in a prescribed neighborhood of v_0 in $C_0^{2m,\lambda}(\Omega)$ with

$$\|v - v_0\|_{C^{2m,\lambda}} \leqq \rho(\|u - u_0\| + |t - t_0|)$$

where

$$\rho(s) \to 0 \qquad \text{as} \qquad s \to 0 .$$

Since the solution of (i) is unique, $v = C_t(u)$. Hence C_t maps $[0, 1] \times B_R$ continuously into $C_0^{2m,\lambda}(\Omega)$ and afortiori into $C_0^{2m-1,\lambda}(\Omega)$.

We now apply the theory of the Leray-Schauder degree ([9], [12]) to the family of mappings $I - C_t$, $0 \leqq t \leqq 1$. For $t = 0$, $C_t = 0$ since then v is a solution of

$$\sum_{|\alpha|=m} D^\alpha D^\alpha v = 0 .$$

Hence the degree of T_0 over B_R with respect to 0 is equal to $+1$. For each t in $[0, 1]$, C_t is a compact map and the degree of T_t over B_R with respect to 0 is well-defined since for u in B_R with $\|u\|_{C^{2m,\lambda}(\Omega)} = R$, $T_t u = 0$ implies that

$$F_t(p(u)) + tH(p(u)) = tH(p(u))$$

i.e.

$$F_t(p(u)) = 0$$

and for solutions of the latter equation, condition (3) of the hypothesis assures that $\|u\|_{C^{2m-1,\lambda}} < R_0 = R$. The degree of T_t over B_R with respect to 0 is constant in t by the continuity and compactness of C_t

in the pair $[t, u]$. Hence the degree of T_1 over B_r with respect to 0 is equal to $+1$ and there exists a solution u in B_R of $T_1 u = 0$. This is equivalent, however, to

$$F(p(u)) = 0$$

and Theorem 1 is proved.

As an important specialization of Theorem 1, we have the following result for the quasi-linear case.

THEOREM 4. *Suppose F is a C^2-function on $\Omega \times R^M$ such that the corresponding differential operator $F(u)$ of order $2m$ is strongly elliptic and quasi-linear, i.e.*

$$F(u) = \sum_{|\alpha|=2m} A_\alpha(x, u, \cdots, D^{2m-1}u) D^\alpha u = G(x, u, \cdots, D^{2m-1}u) .$$

Suppose that for $F_t(u) = tF + (1-t)\Delta^m$, we know the existence of $R_0 > 0$ such that for u in $C_0^{2m,\lambda}(\Omega)$, we have $\| u \|_{C^{2m-1,\lambda}} < R_0$ if $F_t(p(u)) = 0$ for some t in $[0, 1]$.

Then the equation $F(p(u)) = 0$ has a solution u in $C_0^{2m,\lambda}(\Omega)$.

Proof of Theorem 4. We apply Theorem 1 with

$$H(x, p) = k_R \, | p_0 |^2 .$$

Since for $\| u \|_{C^{2m-1,\lambda}} < R$, the $C^{0,\lambda}(\Omega)$-norms of coefficients in the differential operator

$$\sum_{|\alpha|=2m} A_\alpha(x, u, Du, \cdots, D^{2m-1}u) D^\alpha v$$

are bounded by a function of R, it follows from Lemmas 2 and 3 of Section 2 that the conditions (1) and (2) of Theorem 1 are satisfied. Since condition (3) is part of the hypothesis of Theorem 4, we may apply Theorem 1 and obtain a solution of $F(u) = 0$.

Another interesting specialization is to the case of nonlinear second order equations.

THEOREM 5. *Let $F(u)$ be a nonlinear strongly elliptic differential operator of second order. Suppose that both of the following hypotheses are satisfied:*

(a) *There exists a constant $R_0 > 0$ such that if $F_t(u) = tF(u) + (1-t)\Delta = 0$ for $u \in C^{2m,\lambda}(\Omega)$, then $\| u \|_{C^{2m-1,\lambda}} < R_0$.*

(b) *The equation*

$$F_t(p(u, v)) = f$$

for $\|u\|_{C^{2m-1,\lambda}} \leqq R$, $\|f\|_{C^{0,\lambda}} \leqq s$ *has all its solutions* v *in* $C_0^{2m,\lambda}(\Omega)$ *bounded by*

$$\|v\|_{C^{2m,\lambda}} \leqq R_1(s) .$$

Then the equation $F(u) = 0$ *has a solution* u *in* $C_0^{2m,\lambda}(\Omega)$.

Proof of Theorem 5. We apply Theorem 1 with $H = 0$. The linearized equation

$$\sum_{|\alpha|=2} F_\alpha(p(u, v))D^\alpha \eta = 0$$

has only $\eta = 0$ for a solution in $C_0^{2,\lambda}(\Omega)$.

5. **Historical remarks.** The basic work on the Leray-Schauder degree and its application to elliptic boundary value problems is of course the original paper of Leray and Schauder [9]. The result of the latter were only given for equations of second order because of the need for precise results on linear equations not then established for higher order differential operators. Our treatment of the case of strongly nonlinear rather than quasilinear equations follows somewhat different lines from that given in the second part of [9].

Theorem 2 is a generalization of the result of Schauder [15] for second order equations. A partial generalization is given by Agmon-Douglis-Nirenberg ([1], Theorem 12.6).

Theorem 3 is an application of the ideas of the writer's papers [4] and [5].

Systematic accounts of the Leray-Schauder theory of the degree are given by Nagumo [12], Krasnoselski [7], and Cronin [6]. Complete treatments of applications to second order quasilinear equations in R^2 are given by Nirenberg [13] and Miranda [11].

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