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MULTIPLIERS OF TYPE (p, q)

G. I. GAUDRY

This paper is mainly concerned with generalisations of Hörmander's results on multipliers from $L^p(R^n)$ to $L^q(R^n)$ (see Hörmander [6]). Our principal results are that Hörmander's Theorem 1.12 and Corollary 1.5 continue to hold for any LCA group with an infinite discrete subgroup. In order to establish and formulate our results, we define the Fourier transform of functions in $L^p(G)$ where $1 \leqq p \leqq \infty$ and G is any LCA group. Here we use the author's work on quasimeasures in "Quasimeasures and operators commuting with convolution" [4].

The notation throughout is the same as that of Gaudry [4]. We shall use without comment the notions of pseudomeasure and quasimeasure and their properties studied in [4]. We define the singular support of a quasimeasure as the complement of the largest open set on which it is a measure. In the sequel we shall abbreviate "locally compact Abelian Hausdorff group" to "LCA group".

Let G be an LCA group with character group X and suppose p, $q \in [1, \infty]$ with $p \leqq q$. We make the following definition.

DEFINITION. If $p \neq \infty$, denote by L_p^q the space of bounded linear operators mapping $L^p(G)$ into $L^q(G)$ and commuting with the translation operators:

$$T\tau_y = \tau_y T \qquad (y \in G) .$$

Denote by L_∞^∞ the space of linear operators, commuting with translation, which map $L^\infty(G)$ into $L^\infty(G)$, and which are continuous for the weak $(\sigma(L^\infty, L^1))$ topology on $L^\infty(G)$.

The elements of L_p^q are called multipliers of type (p, q). If $p \neq \infty$, L_p^q will be given the usual operator norm: in this case L_p^q is a Banach space. Note that in the case where $G = R^n$, our definition of L_p^q is equivalent to Hörmander's definition of L_p^q as given in 1.2 of [6].

Already much work has been done on characterising L_p^q for various values of p and q (see for example Figà-Talamanca [2], Hörmander [6], Gaudry [4], Wendel [9]). Hörmander [6] studies L_p^q for $G = R^n$. We are largely concerned here with generalisations of Hörmander's Theorem 1.12 and Corollary 1.5 and with other related results.

All the interest resides in the case where G is noncompact, so we shall assume this throughout. In order to establish our results, we define the Fourier transform of functions in any L^p – space $(1 \leqq p \leqq \infty)$.

These generalised Fourier transforms are also useful in the study of spans of translates.

Theorem 2.3 establishes the interesting result that for any LCA group G, we have three conditions, one of which is Hörmander's Corollary 1.5, each being equivalent to his Theorem 1.12. Our main results are that for every LCA group G containing an infinite discrete subgroup, Hörmander's Theorem 1.12 and Corollary 1.5 hold, the latter even in the strengthened form that there exist functions in $\bigcap_{\infty \geqq p > 2} L^p(G)$ whose Fourier transforms are not measures. If such a group is second countable, Hörmander's Corollary 1.5 can be strengthened to the result that there exist functions in $\bigcap_{\infty \geqq p > 2} L^p(G)$ whose Fourier transforms have singular support the whole of X.

1. Fourier transforms of functions in $L^p(G)(1 \leqq p \leqq \infty)$. It is well-known that for $1 \leqq p \leqq 2$, the Fourier transform of functions in $L^p(G)$ can be defined. Schwartz [8] has already defined the Fourier transform of functions in $L^p(R^n \times T^m)$ for all p in $[1, \infty]$. We show here how to define the Fourier transform of functions in $L^p(G)$ where G is any LCA group and $1 \leqq p \leqq \infty$. We use the quasimeasures of Gaudry [4] and define the transform via Plancherel's formula.

We assume known the definition and simple properties of the Fourier transform of integrable functions, so we restrict p to the range $1 < p \leqq \infty$; then p', the conjugate index, is in the range $1 \leqq p' < \infty$. Throughout this paper $\hat{f}$ will denote the Fourier transform (resp. inverse Fourier transform) of f whenever $f \in C^G$ (resp. C^X) and $\hat{f}$ is defined.

1.1. For $f \in L^p(G)$, $1 < p \leqq \infty$, we define $\hat{f}$ as the continuous linear form on $D(X)$ (*i.e.*, the quasimeasure on X) given by

$$\hat{f}(g) = f(\hat{g}_\vee) \qquad\qquad (g \in D(X))$$

($D(X)$ is defined in [4]). $\hat{g}_\vee$ is the reflection of the inverse Fourier transform of g ($D(X) \subset L^1(X)$). Moreover, since $D(X) \subset A_c(X)$, we have by the inversion theorem that $(\hat{g}_\vee)^\wedge = g_\vee$. We show that $\hat{f}$ is indeed a quasimeasure.

The topology of D_K, K a fixed compact subset of X, is stronger than that induced by $L^1(X)$. For if $f = \sum_1^\infty g_i * h_i$ where $g_i, h_i \in C_{c,K}$ and $\sum \| g_i \|_\infty \| h_i \|_\infty < \infty$, then

$$\| f \|_{L^1(X)} \leqq \sum \| g_i \|_{L^1(X)} \| h_i \|_{L^1(X)} \leqq \lambda_K \sum \| g_i \|_\infty \| h_i \|_\infty$$

where λ_K is a constant depending on K. Hence

$$\| f \|_{L^1(X)} \leqq \lambda_K \| f \|_{D_K} .$$

Also, the topology induced on D_K by $A(X)$ is weaker than that of D_K ([4] Theorem 2.5). Now $g \in D(X)$ implies $\hat{g}_\vee \in L^1(G) \cap L^\infty(G) \subset L^{p'}(G)$; and if $g_j \to 0$ in D_K, then $\hat{g}_{j\vee} \to 0$ in $L^1(X)$ and in $L^\infty(X)$ since the topology of D_K is stronger than those induced by $A(X)$ and by $L^1(X)$. This implies that if $g_j \to 0$ in D_K, then $\hat{g}_{j\vee} \to 0$ in $L^{p'}(G)$; hence $\hat{f}(g_j) \to 0$ and, by the definition of the topology of $D(X)$, $\hat{f}$ is a continuous linear form on $D(X)$, i. e., a quasimeasure.

Note that in the case of $L^\infty(G)$, we have $\widehat{L^\infty(G)} \subset P(X)$ and that if $1 < p \leqq 2$, $\widehat{L^p(G)} \subset L^{p'}(X)$

We remark finally that the Fourier transformation $f \to \hat{f}$ from $L^p(G)$ into $D'(X)$ is one-to-one. We need concern ourselves only with the case where $1 < p \leqq \infty$. Suppose then that $f \in L^p(G)$ and that $\hat{f} = 0$, i. e., that $f(\hat{g}_\vee) = 0$ $(g \in D(X))$. In order to show that $f = 0$ a. e., it will suffice to show that $\mathscr{D} = \{\hat{g}: g \in D(X)\}$ is dense in $L^{p'}(G)$. Since $p' \neq \infty$, the set of functions $h \in L^1(G) \cap L^{p'}(G)$ whose Fourier transforms have compact supports is dense in $L^{p'}(G)$. Suppose then that $h \in L^1(G) \cap Lp'(G)$ and that $[\hat{h}]$ is compact. Write (k_i) for an approximate identity in $L^1(X)$ with $k_1 \in C_c(X)$. Then $\hat{k}_i h \in L^1(G) \cap L^{p'}(G)$ and $\hat{k}_i h \to h$ in $L^{p'}(G)$. Further, $\hat{k}_i h \in \mathscr{D}$ since k_i and $\hat{h}$ have compact supports. It follows that $\mathscr{D}$ is dense in $L^{p'}(G)$, so that $f = 0$ a. e.

2. The principal results. With the definition of L_p^q given above, it is easy to see that Hörmander's Theorems 1.3 and 1.4 continue to hold for any noncompact LCA group G. In Theorem 5.1 of Gaudry [4], the generalisation of Hörmander's Theorem 1.2 is established for any LCA group G. The definition of L_p^q adopted in [4] makes it necessary to assume that $p \neq \infty$ in the statement of Theorem 5.1; our present definition of L_p^q allows us to state the generalisation of Hörmander's Theorem 1.2 without any restriction on p. We state the result as Theorem 2.1. The proof is similar to that given for Theorem 5.1 of [4].

THEOREM 2.1. *If $T \in L_p^q$, then there exists a quasimeasure s on G with*

$$Tf = s * f \qquad (f \in C_c(G)) \,. \tag{2.1.1.}$$

Thus L_p^q is isomorphic to a vector subspace of D'. (We shall often identify L_p^q with this vector subspace of D'.)

We show now how to define the Fourier transform of a quasimeasure s on G corresponding to an element T of L_p^q. Since we know from the extended version of Hörmander's Theorem 1.4 that $L_\infty^\infty = L_1^1 = M_{bd}(G)$ and $L_p^\infty = L^{p'}(G)$ if $p < \infty$, and since the Fourier transformations $\mu \to \hat{\mu}$ and $f \to \hat{f}$ are defined and one-to-one if $\mu \in M_{bd}(G)$ and $f \in L^{p'}(G)$, we assume that $p \neq \infty$ and $q \neq \infty$. We define the quasimeasure $\hat{s}$ by defining its local behaviour. Suppose then that Ω is any open relatively compact subset of X, that $f \in C_c(G)$ with $\hat{f}$ nonvanishing on $\bar{\Omega}$, and that $g \in L^1(G)$ with $\hat{f}\hat{g} = 1$ on Ω. We define $\hat{s} \mid \Omega$ by $\hat{s} \mid \Omega = \hat{g} \cdot (s * f)^\wedge \mid \Omega$. (Note that $(s * f)^\wedge$ is already defined since $s * f \in L^q(G)$.)

For this to be a valid definition, we need to show that $\hat{g} \cdot (s * f)^\wedge \mid \Omega$ is independent of the choice of f and g. Suppose then that $h \in D(X)$ and that $[h] \subset \Omega$. If (k_i) is an approximate identity in $L^1(G)$ with $k_i \in C_c(G)$, then since $f \in C_c(G)$ and s corresponds to a multiplier of type (p, q) with $q < \infty$, $s * (f * k_i) = (s * f) * k_i$ and $s * f = \lim s * f * k_i$ in $L^q(G)$; so

$$\begin{aligned}[\hat{g} \cdot (s * f)^\wedge](h) &= (s * f)(g_\vee * \hat{h}_\vee) = \lim\, (s * f * k_i)(g_\vee * \hat{h}_\vee) \\ &= \lim\, (s * k_i)(f_\vee * g_\vee * \hat{h}_\vee)\ .\end{aligned}$$

Finally, $(f_\vee * g_\vee * \hat{h}_\vee)^\wedge = (\hat{f}\hat{g}h)_\vee = h_\vee$ and it follows that the definition of $\hat{s} \mid \Omega$ is independent of the choice of f and g.

It is now clear that if Ω_1 and Ω_2 are any two open relatively compact subsets of X whose intersection is nonvoid, then $\hat{s} \mid \Omega_1 = \hat{s} \mid \Omega_2$ on $\Omega_1 \cap \Omega_2$. By using Lemma 1.2 of [4] and an argument similar to that used by Schwartz in proving Théorème IV, Chapitre I of [7], we may establish the existence of a unique quasimeasure t on X with the property that $t \mid \Omega = \hat{g} \cdot (s * f)^\wedge \mid \Omega$ for each open relatively compact subset Ω of X and corresponding functions f and g. It is this quasimeasure that we denote by $\hat{s}$.

Finally, we note that the mapping $s \to \hat{s}$ of L_p^q into $D'(X)$ is one-to-one. For suppose that $s \in L_p^q$ and that $\hat{s} = 0$. Let Ω be any open relatively compact subset of X and suppose that $f \in C_c(G)$ with $\hat{f}$ nonvanishing on $\bar{\Omega}$. Then clearly, $(s * f)^\wedge \mid \Omega = 0$. Write (k_i) for an approximate identity in $L^1(G)$ with $k_i \in C_c(G)$; since $\hat{k}_i \to 1$ uniformly on compact subsets of X, there exists i_0 such that if $i \geqq i_0$, $\hat{k}_i$ is nonvanishing on $\bar{\Omega}$. If $g \in C_c(G)$ and $h \in D(X)$ with $[h] \subset \Omega$, then

$$(s * g * k_i)^\wedge(h) = (s * k_i)^\wedge(\hat{g}h) = 0 \text{ if } i \geqq i_0\ .$$

Since $g * k_i \to g$ in $L^p(G)$, it follows that $(s * g)^\wedge \mid \Omega = 0$; but Ω is arbitrary, so $(s * g)^\wedge = 0$ $(g \in C_c(G))$. We noted in 1.1 that the Fourier transformation $L^q(G) \to \widehat{L^q(G)}$ is one-to-one. From this we deduce that $s * g = 0$ $(g \in C_c(G))$. Theorem 3.2 of [4] together with the note following the proof of the theorem, then implies that $s = 0$.

Following Hörmander, we make the following definition.

DEFINITION 2.2. Denote by M_p^q the space of Fourier transforms of quasimeasures defined by elements of L_p^q.

M_p^q is thus isomorphic to L_p^q. We shall often identify M_p^q algebraically and topologically with L_p^q. Thus if $p \neq \infty$, M_p^q is a Banach space.

It is now clear that if $s \in L_p^q$ and $f \in C_c(G)$, then $(s * f)\hat{} = \hat{f}\hat{s}$. Further, if $\hat{s} \in M_p^q$, the multiplier T corresponding to $\hat{s}$ is defined by $Tf = (\hat{f}\hat{s})\hat{}$ $(f \in C_c(G))$ where $(\hat{f}\hat{s})\hat{}$ is the "inverse" Fourier transform of $\hat{f}\hat{s}$.

With definition 2.2 and our definition of L_p^q, it is now easy to see that Theorems 1.5, 1.6, and 1.7 and Corollaries 1.1, 1.2, 1.3 and 1.4 of Hörmander [6] continue to hold for any noncompact LCA group G.

Throughout the remainder of section 2, we shall assume for simplicity, that $p \neq \infty$.

For $G = R^n$, Hörmander has established the existence of functions in $L^p(G)$ whose transform are not measures ([6], Corollary 1.5) and has shown that if $F \geqq 0$ is a nonnegligible measurable function with the property that f measurable, $|f| \leqq F$ implies $f \in M_p^q$, then $p \leqq 2 \leqq q$ (Theorem 1.12). We proceed to show, using the Fourier transform defined in 1.1 and 2.1 that in any LCA group G, Hörmander's Theorem 1.12 and Corollary 1.5 are each equivalent to each of two other conditions. We assume $q > 1$. In the case $q = 1$, the characterisation of L_p^q is already well-known so there is no loss in excluding this case.

THEOREM 2.3. *The following four conditions are equivalent for any LCA group G with character group X:*

(i) *Given $p > 2$, there exists a compact subset K of X and a function $\varphi \in L^\infty(X)$ vanishing outside K with φ not the Fourier transform of a function in $L^{p'}(G)$.*

(ii) *If there exists $F \geqq 0$ which is a nonnegligible measurable function on X such that $|f| \leqq F$, f measurable, implies $f \in M_p^q$, then $p \leqq 2 \leqq q$.*

(iii) *Given $p > 2$, there exist functions in $L^p(G)$ whose transforms are not measures.*

(iv) *Given $p > 2$, there exist sequences in $C_c(G)$, bounded in $L^p(G)$, whose transforms, restricted to K, are unbounded in L_K^1 for some compact subset K of X.*

Proof. (i) $\Rightarrow$ (ii). For, under the hypothesis $F \geqq 0$, F nonnegligible and measurable, $|f| \leqq F$, f measurable, implies $f \in M_p^q$, we have, as in the proof of Hörmander's Theorem 1.12, that for any compact $K \subset X$,

$$\int_K |\hat{u}(\chi)|\, d\chi \leqq \text{const.}\, \|u\|_p \qquad (u \in C_c) .$$

For any compact $K \subset X$, and $\varphi \in L^\infty(X)$, $\varphi(K') = 0$, we then have

$$\int_X |\varphi(\chi)\hat{u}(\chi)|\, d\chi \leqq \text{const.}\, \|u\|_p \qquad (u \in C_c)\,.$$

Write $\varphi = \hat{\sigma}$, $\sigma \in P(G)$, Then $\hat{\sigma}\hat{u} \in L^1(X)$, $\sigma * u \in L^\infty(G)$ and

$$\|\sigma * u\|_\infty \leqq \|\hat{\sigma}\hat{u}\|_1 \leqq \text{const.}\, \|u\|_p \qquad (u \in C_c).$$

This means $\sigma \in L_p^\infty = L^{p'}$ since $p < \infty$ and $\hat{\sigma} = \varphi \in \widehat{L^{p'}}$. So from (i) we get $p \leqq 2$. Since $M_q^p = M_{q'}^{p'}$, we have similarly $q' \leqq 2$ (note $q' < \infty$). Thus $p \leqq 2 \leqq q$.

(iii) $\Rightarrow$ (iv). Suppose $f \in L^p(G)$ to be chosen so that $\hat{f}$ is not a measure. Then there exists an open relatively compact subset Ω of X with $\hat{f}|\Omega$ not a measure. Firstly, since $p < \infty$, $C_c(G)$ is dense in $L^p(G)$ and $f = \lim g_n$ in $L^p(G)$ $(g_n \in C_c)$ where of course $(g_n)_1^\infty$ is L^p bounded. Since $\hat{g}_n \to \hat{f}$ weakly in D', we have $(\hat{g}_n | \bar{\Omega})$ must be unbounded in $L^1_{\bar{\Omega}}$. For if $(\hat{g} | \bar{\Omega})$ is bounded in $L^1_{\bar{\Omega}}$, the sequence has a vague limiting point in $M(\bar{\Omega})$ which must coincide on Ω with $\hat{f}$ since $\hat{g}_n \to \hat{f}$ weakly in $D'(X)$.

(iv) $\Rightarrow$ (iii). If $\widehat{L^p(G)} \subset M(X)$ and Ω is any open relatively compact set in X, the map $f \to \hat{f}|\Omega$ is continuous (by the Closed Graph Theorem) from $L^p(G)$ into $M_{bd}(\Omega)$:

$$\|\hat{f}|\Omega\|_{M_{bd}(\Omega)} \leqq \text{const.}\, \|f\|_p \qquad (f \in L^p).$$

Thus (iv) is not true since $\|\hat{f}|\Omega\|_1 = \|\hat{f}|\Omega\|_{M_{bd}}$ for $f \in C_c(G)$. Hence (iv) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (i). If (ii) is true and $p > 2$, then, for any nonnegative nonnegligible measurable function F, there exists a measurable function φ with $|\varphi| \leqq F$ and $\varphi \notin M_1^{p'}$. Choose any nonnegligible compact subset $K \subset X$ and $F \geqq 0$ with $F \in L^\infty(X)$, $F(K') = 0$ and F nonnegligible. Then there exists $\varphi \in L^\infty(X)$ with $|\varphi| \leqq F$ and $\varphi \notin M_1^{p'}$, i.e., $\varphi \notin \widehat{L^{p'}}$, which establishes (i).

We now have equivalence of (i) and (ii) and of (iii) and (iv). To complete the proof, we show that (iv) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (ii) is trivial since under the hypothesis $F \geqq 0$, F measurable, $|f| \leqq F$, f measurable $\Rightarrow f \in M_p^q$, we have, for any compact $K \subset X$,

$$\int_K |\hat{u}(\chi)|\, d\chi \leqq \text{const.}\, \|u\|_p \qquad (u \in C_c);$$

and if (iv) holds, we must have $p \leqq 2$. Similarly, $q \geqq 2$, since $M_p^q = M_{q'}^{p'}$.

(i) $\Rightarrow$ (iv). If (i) holds, then there exist a compact subset K of X and a pseudomeasure $\sigma \in P(G)$ with $\hat{\sigma} = 0$ on K', $\hat{\sigma} \notin M_p^\infty$. This means there exists a sequence (g_n) in $C_c(G)$, bounded in $L^p(G)$, with $(\sigma * g_n)$ unbounded in L^∞. A fortiori, therefore, the transforms

$(\sigma * g_n)^\wedge = \hat{\sigma}\hat{g}_n$ are unbounded in L^1_K. This implies that $(\hat{g}_n \mid K)$ is unbounded in L^1_K. (For if $(\hat{g}_n \mid K)$ is bounded in L^1_K, so then is $(\hat{\sigma}\hat{g}_n)$ since $\hat{\sigma} \in L^\infty$, $\hat{\sigma}(K') = 0$.)

Having established the equivalence of conditions (i)—(iv), we now show that for a large class of groups, we can establish even more than the truth of condition (iii): we can prove that there exists $f \in \bigcap_{\infty \geqq p > 2} L^p(G)$ with $\hat{f} \notin M(X)$; and, if G is in addition second countable, $\hat{f}$ can be chosen to have singular support equal to X. In order to do this we prove a lemma which establishes the connexion between Fourier transforms of functions over an LCA group G and of functions over a closed subgroup of G.

LEMMA 2.4. *Let G be an LCA group and X its character group. Suppose H is a closed subgroup of G and that λ_H is the Haar measure on H. If $C_c(G)$ is mapped onto $C_c(G/H)$ by the mapping $f \to f'$ where f' is the function $xH \to \int_H {}_x f d\lambda_H$ (Hewitt and Ross [5], (15.21)) then $\hat{f}$ coincides on H°, the annihilator of H, with $\hat{f}'$.*

Proof. Suppose $f \in C_c(G)$, $\chi \in H^\circ$. Then

$$\hat{f}(\chi) = \int_G f(x)\chi^{-1}(x)dx = \int_{G/H} (f\chi^{-1})' d\lambda_{G/H}$$

where $\lambda_{G/H}$ is Haar measure on G/H. (This follows from the first part of the proof of (15.22) of Hewitt and Ross [5].) But

$$\begin{aligned}(f\chi^{-1})'(xH) &= \int_H f(xy)\chi^{-1}(xy)dy = \int_H f(xy)\chi(y^{-1}x^{-1})dy \\ &= \chi^{-1}(x)\int_H f(xy)\chi^{-1}(y)dy = \chi^{-1}(xH)\int_H f(xy)dy\end{aligned}$$

since $\chi \in H^\circ$ and $\chi(y) = 1$ for $y \in H$. So

$$(f\chi^{-1})'(xH) = \chi^{-1}(xH) f(xH)$$

and

$$\hat{f}(\chi) = \int_{G/H} \chi^{-1}(xH) f'(xH) d\lambda_{G/H} = \hat{f}'(\chi)\,.$$

The existence theorem now follows:

THEOREM 2.5. *Suppose G is an LCA group containing an infinite discrete subgroup. Then there exists $f \in \bigcap_{\infty \geqq p > 2} L^p(G)$ with $\hat{f}$ not a measure; further, if G is second countable, there exists $f \in \bigcap_{\infty \geqq p > 2} L^p(G)$ with the singular support of $\hat{f}$ the whole of X.*

REMARK. If G is an LCA group whose component of 0 is noncompact then G has an infinite discrete subgroup. The same is true if G contains a noncompact, compactly generated subgroup (Hewitt and Ross [5], (9.8)).

Proof. Suppose then that A is an infinite discrete subgroup of G and that the mapping $f \to \hat{f}$ maps $\bigcap_{\infty \geqq p > 2} L^p(G)$ into $M(X)$. Define on $V = \bigcap_{\infty \geqq p > 2} L^p$ the following countable family of norms. Let (r_n) be a sequence of real numbers, $r_n > 2$, $r_n \to 2$ as $n \to \infty$; define

$$N_{r_n}(f) = \sup\{ \|f\|_p : r_n \leqq p \leqq \infty \} \qquad (f \in V) .$$

We have

$$N_{r_n}(f) \leqq \sup\{ \|f\|^{1-r_n/p} \cdot \|f\|_{r_n}^{r_n/p} \} < \infty .$$

Further, under the topology defined by this family of norms, V is complete. This is evident from the completeness of L^p under the usual norm. Thus, V becomes a Fréchet space.

If Ω is any open relatively compact subset of X, then by the Closed Graph Theorem, we have that the mapping $f \to \hat{f} \mid \Omega$ is continuous from V to $M_{bd}(\Omega)$. This means that there exists $r > 2$ and a constant $\lambda > 0$ such that

$$\int_\Omega d\,|\hat{f}| \leqq \lambda N_r(f) \qquad (f \in V) . \tag{2.5.1}$$

The subgroup A is discrete; so there exists a symmetric neighbourhood U of 0 in G such that the sets $a + 2U$, $(a \in A)$, are pairwise disjoint. Suppose $(a_n)_1^\infty$ is any countably infinite subset of A and that $(c_n)_1^\infty \in l^r$. Choose $f_0 \in V \cap L^1(G)$ with $[f_0] \subset U$ and $|\hat{f}_0(\chi)| \geqq 1$ on Ω. We then have

$$\| \textstyle\sum_1^N c_n \tau_{a_n} f_0 \|_p^p = \|f_0\|_p^p \textstyle\sum_1^N |c_n|^p \qquad (r \leqq p < \infty)$$

and

$$\| \textstyle\sum_1^N c_n \tau_{a_n} f_0 \|_\infty = \|f_0\|_\infty \cdot \sup_{1 \leqq n \leqq N} |c_n| .$$

Hence, by (2.5.1), $\sum c_n \chi(a_n) \hat{f}_0(\chi) \mid \Omega$ converges in L_Ω^1 and

$$\int_\Omega |\textstyle\sum c_n \chi(a_n) \hat{f}_0(\chi)|\, d\chi \leqq \lambda N_r(f_0) (\textstyle\sum |c_n|^r)^{1/r} .$$

But $|\hat{f}_0(\chi)| \geqq 1$ for $\chi \in \Omega$, so it follows that $\sum c_n \chi(a_n) \mid \Omega$ converges in L_Ω^1 and

$$\int_\Omega |\textstyle\sum c_n \chi(a_n)|\, d\chi \leqq \lambda' (\textstyle\sum |c_n|^r)^{1/r} .$$

Hence if $g \in L^\infty(X)$ and $g = 0$ on Ω', it follows that

$$|\sum c_n \hat{g}(a_n)| \leqq \lambda'(\sum |c_n|^r)^{1/r} \|g\|_\infty .$$

This means that $\hat{g} | A \in \ell^{r'}(A)$ whenever $g \in L^\infty_\Omega(X)$; in particular, if $g \in C_c(X)$ and $[g] \subset \Omega$, then $\hat{g} | A \in \ell^{r'}(A)$. The same is true of any translate of g. Hence, since any function $h \in C_c(X)$ may be expressed as a finite sum of translates of functions in $C_{c,\Omega}$, we have that $\hat{h} | A \in \ell^q(A)$ for all $h \in C_c(X)$ and some $q < 2$. By Lemma 2.4, this implies that $\hat{h}' \in \ell^q(A)$ for all $h' \in C(X/A^\circ)$. Since A is infinite, there exists $\varphi \in \ell^{q'}(A)$ with $\varphi \notin \ell^2(A)$. [e. g. if (a_n) is any countable infinite subset of A, φ defined by

$$\varphi(a) = \begin{cases} 0 & a \neq a_n \\ 1/n^{1/2} & a = a_n \end{cases}$$

has the desired properties.] Therefore $\hat{g}'\varphi \in \ell^1(A)$ for all $g' \in C(X/A^\circ)$, which, by Theorem 2.1(a) of Edwards [1], would imply that $\varphi \in \ell^2(A)$, a contradiction. (Observe that X/A° is a compact group.) We have thus established the existence of $f \in V$ with $\hat{f}$ not a measure.

In order to establish the existence of $f \in V$ with the singular support of $\hat{f}$ equal to X, in the case where G is second countable, we use the Category Theorem. Second countability of G is equivalent to second countability of X. Thus X has a countable base of open relatively compact subsets, say $(\Omega_n)_1^\infty$. Then for each ordered pair (m, n) of positive integers, write

$$S_{m,n} = \{f \in V : \hat{f} | \Omega_m \in M_{bd}(\Omega_m), \|\hat{f} | \Omega_m\|_{M_{bd}} \leqq n\} .$$

Suppose that there does not exist $f \in V$ with the singular support of $\hat{f}$ equal to X. Then $\bigcup_{m,n} S_{m,n} = V$. Further, each $S_{m,n}$ is a closed subset of V: this follows immediately from the weak relative compactness of bounded sets of measures and the continuity of the map $f \to \hat{f}$ from V into $D'(X)$ (weak topology). By the Category Theorem, there exists an ordered pair (m_0, n_0) with S_{m_0, n_0} having an interior point. This is clearly impossible since it would imply that there exists an open relatively compact subset Ω of X with $\hat{V} | \Omega \subset M(\Omega)$. However, by the first part of the present theorem, there exists $f_0 \in V$ with $\hat{f}_0$ not a measure, so that a suitable translate of $\hat{f}_0$ will be such that its restriction to Ω is not a measure. This contradiction completes the proof.

Already, from 2.3 and 2.5, we have Hörmander's Theorem 1.12 established. However, starting from 2.5, R. E. Edwards has proved a theorem which implies both Hörmander's Theorem 1.12 and a similar result due to Figà-Talamanca. In a recent paper [3], Figà-Talamanca has proved the following result, and noted that it is valid, with essentially the same proof, when R is replaced by any connected, locally compact, noncompact, Hausdorff, Abelian group.

2.6. Let $f \in L^p(R)(1 \leqq p < 2)$ and $\hat{f} \in L^{p'}(R)$ be its Fourier transform. Suppose that for every continuous function h, there exists $g \in L^p(R)$ such that $\hat{f}h = \hat{g}$ a.e.; then $f = 0$ a.e.

Figà-Talamanca notes that this implies Hörmander's result.

We now present the statement and proof of Edwards' result.

THEOREM 2.7. *Suppose G is an LCA group which contains an infinite discrete subgroup, that F is a function on X with the property that $\varphi F \in \bigcup_{1 \leqq p < 2} \widehat{L^p(G)}$ for each $\varphi \in C_0(X)$. Then $F = 0$ l.a.e.*

Proof. Suppose F is not locally negligible. Then there exists F_0 with compact support, F_0 not locally negligible and with F_0 having the same property as F. So we shall assume F has compact support.

Now $F \in L^{p'}(X)$ for some $p' > 2$: for if $\varphi \in C_0(X)$, $\varphi(\chi) = 1$ on $[F]$, we have $\varphi F \in L^{p'}(X)$ for some $p' > 2$. But F has compact support; so $F \in L^2(X)$ also. Then for any $\varphi \in C_0(X)$, $\varphi F = \hat{f}$ with $f \in L^p(G)$ where $1 \leqq p < 2$. Also $\varphi F = \hat{f}'$ with $f' \in L^2(G)$. So $f = f'$ a.e., and we have that $f \in L^q(G)$ for $p \leqq q < 2$.

Choose a sequence $(p_n)_1^\infty$, $2 > p_n > 1$, $p_n \to 2$. Then $\varphi F \in \bigcup_{n=1}^\infty \widehat{L^{p_n}}$. We now apply the Category Theorem. Let

$$C_{n,k} = \{\varphi \in C_0(X) : \varphi F = \hat{f} \text{ a.e. for some } f \in L^{p_n},\ \|f\|_{p_n} \leqq k\} .$$

Then $C_{n,k}$ is closed in $C_0(X)$ and $C_0(X) = \bigcup_{n,k=1}^\infty C_{n,k}$. So by the Category Theorem, there exist n_0, k_0 such that C_{n_0, k_0} is a neighbourhood of 0. Write $p = p_{n_0}$, so that $1 < p < 2$. Then we have $\varphi F \in \widehat{L^p}$ for all $\varphi \in C_0(X)$.

Consider the mapping T from $C_0(X)$ into $L^p(G)$ defined by

$$(T\varphi)^\wedge = F\varphi \text{ a.e.} \qquad (\varphi \in C_0(X)) \tag{2.7.1}$$

T is linear; and since F is evidently finite a.e., T has a closed graph, hence is continuous.

We shall now show that, without loss of generality, we may assume F is continuous. From (2.7.1), if χ, $\chi_0 \in X$,

$$\varphi(\chi - \chi_0)F(\chi - \chi_0) = (\chi_0 T\varphi)^\wedge(\chi) \text{ a.e.}$$

and

$$\varphi(\chi)F(\chi - \chi_0) = (\chi_0 T\varphi\chi_0)^\wedge(\chi) \text{ a. e.}$$

where $\varphi\chi_0$ is the χ_0 – translate of φ. If $\rho \in C_c(X)$,

$$\varphi(\chi)\int_X F(\chi - \chi_0)\rho(\chi_0)d\chi_0 = \left(\int_X \chi_0 T\varphi_{\chi_0}\rho(\chi_0)\,d\chi_0\right)^\wedge(\chi)$$

where the function $x \longrightarrow \int_X \chi_0(x)(T\varphi_{\chi_0})(x)\rho(\chi_0)d\chi_0$ is in $L^p(G)$ by virtue of the compact support of ρ and the continuity of the map $\chi_0 \longrightarrow T\varphi_{\chi_0}$ from X into $L^p(G)$. So $F * \rho$ has the same property as F; and moreover, since $F \in L_c^{p'}(X)$, $F * \rho \in C_c(X)$. If F is nonnegligible, ρ can be chosen so that $F * \rho \not\equiv 0$. It therefore suffices to show that if $F \in C_c(X)$ satisfies the hypotheses of the theorem, then $F \equiv 0$.

Consider the adjoint map $T' : L^{p'}(G) \longrightarrow M_{bd}(X)$. By the definition of T', we have

$$(T\varphi) * g(0) = \varphi * (T'g)(0) \qquad (\varphi \in C_0(X),\ g \in L^{p'}(G))\,.$$

If $\varphi \in C_c$, then since $F \in C_c(X)$, we have $\varphi F \in L^2(X)$ and so $T\varphi \in L^2(G)$ by Plancherel's Theorem. If $g \in L^2(G)\cap L^{p'}(G)$, Parseval's formula gives

$$\begin{aligned}\int_G T\varphi g_\vee dx &= \int_X (T\varphi)^\wedge(\chi)\hat{g}(\chi)d\chi = \int_X (\varphi F)(\chi)\hat{g}(\chi)d\chi \\ &= \int_X \varphi(-\chi)F(-\chi)\hat{g}(-\chi)d\chi \\ &= \int_X \varphi(-\chi)d(T'g)(\chi)\end{aligned}$$

and we deduce that

$$d(T'g)(\chi) = F(-\chi)\hat{g}(-\chi)d\chi \qquad (g \in L^2(G)\cap L^{p'}(G))$$

and by the continuity of T' that

$$\|F\hat{g}\|_1 \leqq \text{const.}\ \|g\|_{p'}\,. \tag{2.7.2}$$

If $g \in L^{p'}(G)$, g can be approximated in $L^{p'}(G)$ by a sequence $(g_n)_1^\infty$ of functions in $C_c(G) \subset L^2(G)\cap L^{p'}(G)$ with $(\|g_n\|_{p'})_1^\infty$ bounded of necessity. Then the measures $(F\hat{g}_n)$ are bounded in $M_{bd}(X)$, by virtue of (2.7.2). If $F \not\equiv 0$, then for some open relatively compact subset Ω of X, $F(\chi) \neq 0$ on $\bar{\Omega}$, and so $|F|$ is bounded away from 0 on $\bar{\Omega}$. This means that $(\hat{g}_n \mid \bar{\Omega})_1^\infty$ is bounded in $M_{bd}(\bar{\Omega})$, and so must have a weak limiting point in $M_{bd}(\bar{\Omega})$. But since $g_n \longrightarrow g$ in $L^{p'}(G)$, $\hat{g}_n \mid \Omega \longrightarrow \hat{g} \mid \Omega$ weakly in D'. We should have then that $\hat{g} \mid \Omega$ is a measure. But by Theorem 2.5, for any open relatively compact subset Ω of X, there exists $g \in L^{p'}(G)$ with $\hat{g} \mid \Omega$ not a measure, since $p' > 2$. Hence we must have $F \equiv 0$.

REMARK. If G is compact, the situation is known to be entirely different. In this case a function F on X has the property

(a) $\varphi F \in M(G)^{\wedge} \qquad (\varphi \in c_0(X))$

if and only if

(b) $F \in \ell^2(X)$,

in which case, of course, it is evident that

(c) $\varphi F \in L^2(G)^{\wedge} \qquad (\varphi \in \ell^{\infty}(X))$.

Indeed, it is easy to deduce from (a) that $\varphi F \in M(G)^{\wedge}$ $(\varphi \in \ell^{\infty}(X))$, which is known to imply (b) (see e. g. Edwards [1], Theorem (1.1)).

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INSTITUTE OF ADVANCED STUDIES
THE AUSTRALIAN NATIONAL UNIVERSITY
CANBERRA, AUSTRALIA