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ON INDECOMPOSABLE MODULES OVER RINGS WITH MINIMUM CONDITION

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Let A be an associative ring with left minimum condition and identity. Let $g(d)$ denote the number of nonisomorphic indecomposable A-modules which have composition length d, d a nonnegative integer. If, for each integer n, there exists an integer $d > n$, such that $g(d) = \infty$, A is said to be of strongly unbounded module type.

Assume that the center of the endomorphism ring of each simple (left) A-module is infinite. The following results concerning the structure of rings of strongly unbounded type are obtained.

I. If the ideal lattice of A is infinite, then A is of strongly unbounded module type.

II. If A is commutative, then A has only a finite number of (nonisomorphic) finitely generated indecomposable modules if and only if the ideal lattice of A is distributive. Otherwise, A is of strongly unbounded module type.

III. If the ideal lattice of A contains a vertex V of order greater than three such that, for some primitive idempotent $e \in A$, the image Ve of V is a vertex of order greater than three in the submodule lattice of Ae, then A is of strongly unbounded module type.

These results are generalizations of earlier ones obtained by J. P. Jans for finite dimensional algebras over algebraically closed fields.

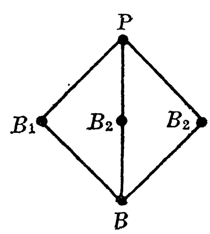
Let A be an associative ring with left minimum condition and identity. The *length*, $c(M)$, of a (left) A-module M with composition series is the number of composition factors of M. Let $g(d)$ denote the number of nonisomorphic indecomposable A-modules which have length d, d a nonnegative integer. If $\sum_d g(d) < \infty$, A is said to be of *finite module type*. If there exists an integer n such that $g(d) = 0$ for all $d > n$, A is of *bounded module type*. If not of bounded module type, A is of *unbounded module type*. If for each integer n, there exists $d > n$ such that $g(d) = \infty$, A is of *strongly unbounded module type*. R. Brauer, J. P. Jans, and R. M. Thrall have conjectured that infinite algebras of unbounded type are of strongly unbounded type, and that algebras of bounded type are of finite type [4]. A discussion of the state of these conjectures may be found in [2] and [4].

J. P. Jans has given sufficient conditions that a finite dimensional algebra over an algebraically closed field be of strongly unbounded type [4]. Through extension and modification of the techniques used by Jans and by H. Tachikawa [6], some of these results can be obtained for arbitrary rings with minimum condition, provided that the endomorphism rings of the simple A-modules have infinite centers.

2. Rings with infinite ideal lattices. Let A be a ring with left minimum condition with the property that the lattice of ideals of A is infinite. H. Tachikawa showed that A is of unbounded type [6]. If A is also a finite dimensional algebra over an algebraically closed field, A is of strongly unbounded type [4]. The following theorem generalizes these results.

THEOREM. *If the center of the endomorphism ring of each simple (irreducible) A-module is infinite and if the ideal lattice of A is infinite, then A is of strongly unbounded module type.*

Proof. Since the ideal lattice of A is infinite, the lattice contains a projective root [1].



Since A/B-modules are A-modules, we can assume that $B=0$. Also, there exists an $A-A$ isomorphism $\psi\colon B_1 \cong B_2$. Let N denote the radical of A and define $M = l(N) \cap r(N)$. Since B_1 and B_2 are simple ideals we have $B_1 + B_2 = B_1 \oplus B_2 \subsetneqq M$. There exist primitive idempotents $e, f \in A$ such that $fMe \supseteqq fB_1e \oplus fB_2e \supset (0)$. Choose $u = fue \neq 0$ in fB_1e and let $v = \psi(u)$. Let $\Lambda \subset fAf$ be a set of representatives for the nonzero distinct cosets of the center of fAf/fNf. Evidently, Λ is infinite. For $\lambda \in \Lambda$, define $s(\lambda) = \lambda v - u$. Since fAu, $fAv, fAs(\lambda)$, are all nonzero and $u, v, s(\lambda) \in M$, we have $Af/Nf \cong Au \cong Av \cong As(\lambda)$.

LEMMA 1. *If $\lambda \neq \mu \in \Lambda$, $a, b \in A$, and $s(\lambda)a = bs(\mu)$, then $s(\lambda)a = bs(\mu) = 0$.*

Proof. We may assume that $a \in eAe$, $b \in fAf$. Since $B_1 \cap B_2 = 0$, we have $\lambda va = b\mu v$ and $ua = bu$. Since $v = \psi(u)$, $va = bv$ so that $\lambda bv = b\mu v$. Thus, since fAf/fNf is a division ring, $\lambda b = b\mu \pmod{fNf} = \mu b \pmod{fNf}$. Since $\lambda \neq \mu \pmod{fNf}$, $b = 0 \pmod{fNf}$. Since $v \in M$, the lemma follows.

LEMMA 2. *If* $a, b, c, d \in A$ *and* $s(\lambda)a + vb = cs(\lambda) + dv$, *then* $va = cv$, $ua = cu$, *and* $vb = dv$.

Proof. Since $B_1 \cap B_2 = 0$ and $v = \psi(u)$, we have $cu = ua$, $cv = va$, and $\lambda va + vb - c\lambda v - dv = 0$. Hence, since $\lambda c = c\lambda \pmod{fNf}$, $vb = dv$.

For each positive integer n, let X^n be the direct sum of n copies of Ae,

$$X^n = \bigoplus \sum_{i=1}^{n} \varepsilon_i(Ae) ,$$

and let Y^n denote the socle of X^n. For $\lambda \in \Lambda$, define

$$T_\lambda^n = \left\{\sum_{i=1}^{n} \varepsilon_i(a_{i-1}v + a_i s(\lambda)) \colon a_0 = 0,\ a_i \in A\right\} .$$

Let $H_\lambda^n = X^n/T_\lambda^n$ and $S_\lambda^n = Y^n/T_\lambda^n$. Since the length of T_λ^n is n, the length of $S_\lambda^n \geqq 2n - n = n$.

We proceed to show that H_λ^n and H_μ^n are not isomorphic, provided $\lambda \neq \mu \in \Lambda$. Suppose $\theta \colon H_\lambda^n \cong H_\mu^n$. Since X^n is projective [3], there exists $\bar\theta \colon X^n \to X^n$ such that $\theta\pi_\lambda = \pi_\mu\bar\theta$, where π_λ, π_μ are the natural projections of X^n onto H_λ^n, H_μ^n, respectively. There exist $x_1, \cdots, x_n \in eAe$, such that

$$\bar\theta\varepsilon_n(e) = \sum_{i=1}^{n} \varepsilon_i(x_i) .$$

Since $\pi_\lambda\varepsilon_n s(\lambda) = 0$, and $\theta\pi_\lambda = \pi_\mu\bar\theta$, we have $\pi_\mu\bar\theta\varepsilon_n s(\lambda) = 0$ and hence $\bar\theta\varepsilon_n s(\lambda) \in T_\mu^n$. Thus,

$$\sum_{i=1}^{n} \varepsilon_i(s(\lambda)x_i) = \bar\theta\varepsilon_n s(\lambda) \in T_\mu^n .$$

According to the definition of T_μ^n, there exist $a_0 = 0, a_1, \cdots, a_n \in A$, such that

$$s(\lambda)x_i = a_{i-1}v + a_i s(\mu) , \qquad i = 1, \cdots, n .$$

Using an induction and Lemma 1, we conclude that $x_1, \cdots, x_n \in eNe$, and hence

$$\theta\pi_\lambda\varepsilon_n(v) = \pi_\mu \sum_{i=1}^{n} \varepsilon_i(vx_i) = 0 .$$

This contradicts the assumption that θ is an isomorphism.

Next, suppose that H_λ^n decomposes. Let η be the idempotent endomorphism of H_λ^n associated with an indecomposable direct summand of H_λ^n such that $\eta\pi_\lambda\varepsilon_n(v) \neq 0$.

LEMMA 3. *The restriction of η to S_λ^n is a monomorphism.*

Proof. Since X^n is projective, η may be lifted to an endomorphism $\bar{\eta}$ of X^n. There exist $y_{ij} \in eAe$ such that

$$\bar{\eta}\varepsilon_j(e) = \sum_{i=1}^{n} \varepsilon_i(y_{ij}) , \qquad j = 1, \cdots, n .$$

From the definition of T_λ^n, we have that

$$\bar{\eta}(\varepsilon_{j-1}(s(\lambda)) + \varepsilon_j(v)) \in T_\lambda^n , \qquad j = 2, \cdots, n .$$

and $\bar{\eta}\varepsilon_n(s(\lambda)) \in T_\lambda^n$. Thus,

$$\bar{\eta}(\varepsilon_{j-1}(s(\lambda)) + \varepsilon_j(v)) = \sum_{i=1}^{n} \varepsilon_i(s(\lambda)y_{i,j-1} + vy_{ij}) \in T_\lambda^n ,$$

for $j = 2, \cdots, n$, and

$$\bar{\eta}\varepsilon_n s(\lambda) = \sum_{i=1}^{n} \varepsilon_i(s(\lambda)y_{in}) \in T_\lambda^n .$$

Hence, there exist $a_{ij} \in fAf$ such that

$$\begin{aligned} s_\lambda y_{1,j-1} + vy_{1j} &= a_{1,j-1}s_\lambda , \\ s_\lambda y_{i,j-1} + vy_{ij} &= a_{i,j-1}s_\lambda + a_{i-1,j-1}v , \\ s_\lambda y_{1n} &= a_{1n}s_\lambda , \end{aligned}$$

and

$$s_\lambda y_{in} = a_{in}s_\lambda + a_{i-1,n}v , \qquad \text{for } i, j = 2, 3, \cdots, n .$$

Since $fs_\lambda e = s_\lambda$ and $fve = v$, we may assume that $a_{ij} \in fAf$, $i, j = 1, 2, \cdots, n$. Applying Lemma 2, we obtain,

$$uy_{ij} = a_{ij}u ,$$

and

$$\begin{aligned} vy_{ij} &= a_{ij}v , & i, j &= 1, 2, \cdots, n ; \\ vy_{ij} &= a_{i-1,j-1}v , & i, j &= 2, 3, \cdots, n ; \end{aligned}$$

and

$$y_{i-1,n} = 0 \ (\mathrm{mod}\ eNe) , \qquad i = 2, 3, \cdots, n .$$

Suppose $i < j$. Then we have

$$vy_{ij} = a_{ij}v = vy_{i+1,j+1} = \cdots = vy_{i+n-j,n} = 0 .$$

Therefore, $y_{ij} = 0 \ (\mathrm{mod}\ eNe)$. Suppose $i > j$. Then

$$vy_{ij} = a_{i-1,j-1}v = vy_{i-1,j-1} = \cdots = vy_{i-j+1,1}\ .$$

Also,

$$vy_{kk} = vy_{nn}\ , \qquad k = 1, 2, \cdots, n\ .$$

Since $\eta\pi_\lambda(\varepsilon_n(v)) = \pi_\lambda\varepsilon_n(vy_{nn}) \neq 0$, we have $y_{nn} \neq 0 \pmod{eNe}$. From these equations and the idempotence of η it follows that

$$y_{ij} = \begin{cases} e \pmod{eNe}, & \text{if } i = j\ . \\ 0 \pmod{eNe}, & \text{if } i < j\ . \\ y_{i-j+1,1} \pmod{eNe}, & \text{if } i > j\ . \end{cases}$$

Next assume that $x \in Y^n$ and $\eta\pi_\lambda(x) = 0$. Then $\bar{\eta}(x) \in T^n_\lambda$. There exist elements r_j of the socle of Ae such that $x = \sum_{j=1}^n \varepsilon_j r_j$, from which the equation

$$\bar{\eta}(x) = \sum_{i=1}^n \varepsilon_i\Big(r_i + \sum_{j=1}^{i-1} r_j y_{i-j+1,1}\Big)$$

follows. Since $\bar{\eta}(x) \in T^n_\lambda$, there exist $b_0 = 0, b_1, \cdots, b_n \in Ae$ such that

$$\sum_{j=1}^{i-1} r_j y_{i-j+1,1} + r_i = b_i s(\lambda) + b_{i-1}v\ , \qquad i = 2, \cdots, n\ .$$

Defining

$$\begin{aligned} \alpha_0 &= 0\ , \\ \alpha_1 &= b_1\ , \\ \alpha_k &= b_k - \sum_{j=1}^{k-1} \alpha_j a_{k-j+1,1}\ , \qquad k = 2, \cdots, n\ . \end{aligned}$$

it follows that

$$r_k = \alpha_k s(\lambda) + \alpha_{k-1}v\ , \qquad k = 1, \cdots, n\ .$$

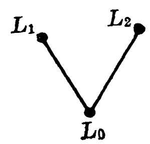
Thus, $x \in T^n_\lambda$ and $\pi_\lambda x = 0$. This proves Lemma 3.

From Lemma 3, we conclude that S^n_λ is contained in an idecomposable direct summand V_λ of H^n_λ. Calculation of $H^n_\lambda/S^n_\lambda \cong X^n/Y^n$ shows that every direct summand of H^n_λ not equal to V_λ is isomorphic to $Ae/S(Ae)$, $S(Ae)$ the socle of Ae. Thus, $V_\lambda \cong V_\mu$ if and only if $H^n_\lambda \cong H^n_\mu$ and hence $V_\lambda \not\cong V_\mu$ if $\lambda \neq \mu \in \Lambda$. This completes the proof of the theorem.

3. Commutative rings.

THEOREM. *If A is commutative, then A is of finite type if and only if the ideal lattice of A is distributive. Otherwise, A is of* *unbounded type, strongly so if the endomorphism ring of each simple A-module is infinite.*

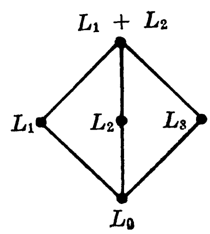
Proof. It is sufficient to show that, if the ideal lattice of A is distributive, A is generalized uni-serial (see [5]). Let e be a primitive idempotent in A and consider the lattice of submodules of Ae. Since A is commutative, these submodules are ideals in A. Suppose the lattice contains a vertex



where we assume, without loss of generality, that the lattice from (0) to L_0 is a chain. Then $L_0 = N^{k+1}e$ for some k, and $L_1 + L_2 \subseteqq N^k e$. Choose $\alpha_1 \in L_i - L_0$, $i = 1, 2$, and define

$$L_3 = Ae(\alpha_1 + \alpha_2) + L_0 .$$

The mapping $ae \longrightarrow ae(\alpha_1 + \alpha_2) + L_0$ induces an isomorphism $L_3/L_0 \cong Ae/Ne$ so that we have $L_0 \subset L_3 \subset L_1 + L_2$. Since $L_1 \cap L_2 = L_0$, it follows directly that $L_3 \cap L_1 = L_3 \cap L_2 = L_0$. Clearly $L_1 + L_2 = L_1 + L_3 = L_2 + L_3$. Hence the ideal lattice of A contains the projective root



which contradicts the assumption that the lattice is distributive. Thus, A is generalized uni-serial and of finite type.

4. Lattices with vertex of order four. In this section we assume that the center of the endomorphism ring of each simple A-module is infinite.

THEOREM. *If the ideal lattice of A contains a vertex V of order greater than three such that for some primitive idempotent $e \in A$, the image Ve of V is a vertex of order greater than three in the submodule lattice of Ae, then A is of strongly unbounded module type.*

Proof. There exists an ideal $B_0 \subsetneqq A$ with distinct covers B_1, B_2, B_3, B_4 such that $B_i e \supset B_0 e$, $i = 1, 2, 3, 4$. Since A/B_0 modules are A-modules we can assume that $B_0 = 0$. Because of the theorem of § 1, we assume that the ideal lattice of A is distributive and hence that

$$\sum_{i=1}^{4} B_i = \bigoplus \sum_{i=1}^{4} B_i .$$

There exist primitive idempotents $f_i \in A$ such that $f_i B_i e \neq 0$, $i = 1, 2, 3, 4$. Let $\Lambda \subset eAe$ be a set of representatives for the nonzero cosets of the center of eAe/eNe. Choose $u_i = f_i u_i e \neq 0 \in B_i e$, $i = 1, 2, 3, 4$. For $\lambda \in \Lambda$ we have $Af_i/Nf_i \cong Au_i \cong Au_i\lambda$, $i = 1, 2, 3, 4$. For each positive integer n define

$$X^n = \bigoplus \sum_{i=1}^{2n} \varepsilon_i(Ae)$$

and denote the socle of X^n by Y^n. Define

$$T_\lambda^n = \left\{\sum_{i=1}^{n} \varepsilon_i(a_i u_1 + c_i u_3 + d_i u_4 \lambda + d_{i-1} u_4) + \varepsilon_{i+n}(b_i u_2 + c_i u_3 + d_i u_4): \right.$$
$$\left. d_0 = 0,\ a_i,\ b_i,\ c_i,\ d_i \in A,\ i = 1, \cdots, n. \right\},$$

$$H_\lambda^n = X^n / T_\lambda^n ,$$

and

$$S_\lambda^n = Y^n / T_\lambda^n .$$

Since the composition length of T_λ^n is equal to $4n$ and the composition length of Y^n is greater than or equal to $8n$, the composition length of S_λ^n increases without bound as n increases.

Let $\lambda \neq \mu$ be elements of Λ. We next prove that H_λ^n and H_μ^n are not isomorphic. Suppose θ is an isomorphism from H_λ^n onto H_μ^n. Since X^n is projective, θ can be lifted to a endomorphism $\bar{\theta}$ of X^n. There exist $x_1, \cdots, x_{2n}, y_1, \cdots, y_{2n}$ in eAe such that

$$\bar{\theta}\varepsilon_{2n}(e) = \sum_{i=1}^{2n} \varepsilon_i(x_i)$$

and

$$\bar{\theta}\varepsilon_n(e) = \sum_{i=1}^{2n} \varepsilon_i(y_i) .$$

Since, $\theta\pi_\lambda\varepsilon_n(u_4) \neq 0$, we have

$$\pi_\mu\Big(\sum_{i=1}^{2n} \varepsilon_i(u_4 y_i)\Big) = \theta\pi_\lambda\varepsilon_n(u_4) \neq 0 .$$

Thus, since $u_4 \in M$, there exists k, $1 \leqq k \leqq 2n$, such that

$$y_k \notin eNe .$$

Since $u_1 y_i \in Au_2 + Au_3 + Au_4$ for $i > n$, we have $u_1 y_i = 0$ for $i > n$, and hence, since eAe/eNe is a division ring, $y_i \in eNe$, for $i > n$. Similarly, $\bar{\theta}\varepsilon_{2n}(u_2) \in T^n_\mu$ implies $x_i \in eNe$, for $i \leqq n$. It follows that

$$\bar{\theta}(\varepsilon_n u_3 + \varepsilon_{2n} u_3) = \sum_{i=1}^{n} \varepsilon_i(u_3 y_i) + \sum_{i=n+1}^{2n} \varepsilon_i(u_3 x_i) \in T^n_\mu .$$

Therefore, $u_3 y_i = u_3 x_{i+n}$ for $i = 1, \cdots, n$, and hence,

$$y_i = x_{i+n}, \ (\operatorname{mod} eNe) \qquad i = 1, \cdots, n .$$

From this we obtain

$$\bar{\theta}(\varepsilon_n(u_4\lambda) + \varepsilon_{2n}(u_4)) = \sum_{i=1}^{n} \varepsilon_i(u_4 \lambda y_i) + \sum_{i=n+1}^{2n} \varepsilon_i(u_4 y_{i-n}) \in T^n_\mu .$$

Hence, using the definition of T^n_μ there exist $d_1, \cdots, d_n \in A$ such that

$$u_4 \lambda y_1 = d_1 u_4 \mu ,$$
$$u_4 \lambda y_j = d_j u_4 \mu + d_{j-1} u_4 , \qquad j = 2, \cdots, n ,$$

and

$$u_4 y_j = d_j u_4 , \qquad j = 1, \cdots, n .$$

Replacing $d_j u_4$ by $u_4 y_j$ in these equations, we have

$$u_4 \lambda y_1 = u_4 y_1 \mu$$

and

$$u_4 \lambda y_j = u_4 y_j \mu + u_4 y_{j-1} , \qquad j = 2, \cdots, n .$$

Since $u_4 \in M$, a simple induction shows that

$$y_i \in eNe , \qquad i = 1, \cdots, n .$$

We conclude that H^n_λ and H^n_μ are not isomorphic.

Next, suppose that H^n_λ decomposes and let η be an idempotent endomorphism of H^n_λ such that $\eta\pi_\lambda(\varepsilon_n(u_3)) \neq 0$. Since X is projective, η can be lifted to an endomorphism $\bar{\eta}$ of X^n. There exist $y_{ij} \in eAe$ such that $\bar{\eta}(\varepsilon_j(e)) = \sum_{i=1}^{2n} \varepsilon_i(y_{ij})$. If $j \leqq n$, we have

$$\bar{\eta}(\varepsilon_j u_1) = \sum_{i=1}^{2n} \varepsilon_i(u_1 y_{ij}) \in T^n_\lambda$$

and hence

$$y_{ij} = 0 , \quad (\operatorname{mod} eNe) \qquad 1 \leqq i \leqq n, n + 1 \leqq j \leqq 2n .$$

For $j \leqq n$, we have,

$$\bar{\eta}(\varepsilon_j(u_3) + \varepsilon_{j+n}(u_3)) = \sum_{i=1}^{n} \varepsilon_i(u_3 y_{ij}) + \sum_{i=n+1}^{2n} \varepsilon_i(u_3 y_{i,j+n}) \in T^n_\lambda .$$

Thus, by the definition of T^n_λ,

$$y_{ij} = y_{i+n,j+n} \pmod{eNe} , \qquad 1 \leqq i, j \leqq n .$$

We infer that

$$\bar{\eta}(\varepsilon_n(u_4\lambda) + \varepsilon_{2n}(u_4)) = \sum_{i=1}^{n} \varepsilon_i(u_4\lambda y_{in}) + \sum_{i=n+1}^{2n} \varepsilon_i(u_4 y_{i-n,n}) \in T^n_\lambda .$$

Hence, there exist $d_0, \cdots, d_n \in A$, $d_0 = 0$, such that

$$u_4\lambda y_{jn} = d_j u_4\lambda + d_{j-1}u_4 ,$$

and

$$u_4 y_{jn} = d_j u_4 , \qquad j = 1, \cdots, n .$$

Replacing $d_j u_4$ by $u_4 y_{jn}$, we have

$$u_4\lambda y_{1n} = u_4 y_{1n}\lambda ,$$

and

$$u_4\lambda y_{jn} = u_4 y_{jn}\lambda + u_4 y_{j-1,n} , \qquad j = 2, \cdots, n .$$

Hence, for $i < n$ we obtain $y_{in} = 0 \pmod{eNe}$. And, since η is idempotent and eAe/eNe is a division ring, $y_{nn} = e \pmod{eNe}$. Now suppose $k < n$. Then

$$\begin{aligned} &\bar{\eta}(\varepsilon_k(u_4\lambda) + \varepsilon_{k+1}(u_4) + \varepsilon_{k+n}(u_4)) \\ &\quad = \sum_{i=1}^{n} \varepsilon_i(u_4\lambda y_{ik} + u_4 y_{i,k+1}) + \sum_{i=n+1}^{2n} \varepsilon_i(u_4 y_{i-n,k}) \in T^n_\lambda . \end{aligned}$$

Hence, there exist $d^k_1, d^k_2, \cdots, d^k_n \in A$, $d^k_0 = 0$, such that

$$u_4\lambda y_{ik} + u_4 y_{i,k+1} = d^k_j u_4\lambda + d^k_{j-1}u_4 ,$$

and

$$u_4 y_{jk} = d^k_j u_4 , \qquad j = 1, \cdots, n .$$

Replacing $d^k_j u_4$ by $u_4 y_{jk}$ we obtain $u_4 y_{1,k+1} = 0$, and $u_4 y_{j,k+1} = u_4 y_{j-1,k}$, $j = 2, \cdots, n$, $k = 1, \cdots, n-1$. It follows from these equations that $y_{1k} = 0$, $\pmod{eNe}$ for $k = 2, \cdots, n$, and $y_{jk} = y_{j+1,k+1} \pmod{eNe}$, $j, k = 1, \cdots, n-1$. If $i < j \leqq n$, then

$$y_{ij} = y_{i-1,j-1} = \cdots = y_{1,j-i+1} = 0 \pmod{eNe} .$$

And, if $n \geqq i \geqq j$,

$$y_{ij} = y_{i-1,j-1} = \cdots = y_{i-j+1,1} \pmod{eNe} .$$

These results imply

$$y_{ij} = \begin{cases} 0 \pmod{eNe} , & \text{if } i < j, \text{ or } j \leqq n < i , \\ e \pmod{eNe} , & \text{if } i = j , \\ y_{i-j+1,1} \pmod{eNe} , & \text{if } j < i \leqq n, \text{ or } n < j < i . \end{cases}$$

We shall now show that the restriction of η to S_λ^n is a monomorphism and that $\eta(S_\lambda^n) = S_\lambda^n$. Suppose that $x \in Y^n$ is such that $\pi_\lambda(s)$ is an element of the kernel of η,

$$x = \sum_{i=1}^{2n} \varepsilon_i(x_i) .$$

We have $\eta\pi_\lambda(x) = \pi_\lambda\bar{\eta}(x) = 0$, and so

$$\begin{aligned} &\bar{\eta}(x) \in T_\lambda^n . \\ &\bar{\eta}(x) = \sum_{j=1}^{2n} \bar{\eta}\varepsilon_j(x_j) \\ &\qquad = \sum_{j=1}^{2n} \sum_{i=1}^{2n} \varepsilon_i(x_j y_{ij}) \\ &\qquad = \sum_{j=1}^{n} \sum_{i=1}^{n} \varepsilon_i(x_j y_{i-j+1,1}) + \sum_{j=n+1}^{2n} \sum_{i=j}^{2n} \varepsilon_i(x_j y_{i-j+1,1}) . \end{aligned}$$

Thus, there exist $a_1, b_i, c_i, d_i, i = 1, \cdots, n$ in A, $d_0 = 0$ such that

$$\sum_{j=1}^{i} x_j y_{ij} = a_i u_1 + c_i u_3 + d_i u_4 \lambda + d_{i-1} u_4 ,$$

and

$$\sum_{j=1}^{i} x_{n+j} y_{ij} = b_i u_2 + c_i u_3 + d_i u_4 , \qquad \text{for } i = 1, 2, \cdots, n .$$

Using the definition of T_λ^n, it follows that

$$x_j = \alpha_j u_1 + \gamma_j u_3 + \delta_j u_4 \lambda + \delta_{j-1} u_4 ,$$

and

$$x_{j+n} = \beta_j u_2 + \gamma_j u_3 + \delta_j u_4 , \qquad j = 1, \cdots, n-1 ,$$

where $\alpha_1 = a_1$, $\beta_1 = b_1$, $\gamma_1 = c_1$, $\delta_0 = 0$, $\delta_1 = d_1$, and

$$\begin{aligned} \alpha_k u_1 &= a_k u_1 - \alpha_{k-1} u_1 y_{21} - \cdots - \alpha_1 u_1 y_{k1} \\ \beta_k u_2 &= b_k u_2 - \beta_{k-1} u_2 y_{21} - \cdots - \beta_1 u_2 y_{k1} \\ \gamma_k u_3 &= c_k u_3 - \gamma_{k-1} u_3 y_{21} - \cdots - \gamma_1 u_3 y_{k1} \\ \delta_k u_1 &= d_k u_4 - \delta_{k-1} u_4 y_{21} - \cdots - \delta_1 u_4 y_{k1} , \qquad \text{for } k > 1 . \end{aligned}$$

Hence, $\pi_\lambda(x) = 0$, and the restriction of η to S_λ^n is a monomorphism.

The proof can now be completed as in § 1.

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