PACIFIC JOURNAL OF MATHEMATICS
Vol. 19, No. 2, 1966

ON REPRESENTATIONS OF CERTAIN SEMIGROUPS

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A theory of representations for compact semigroups has been lacking due in large part to the absence of a translation-invariant carrying measure that exists for compact groups. The object in this paper is to show that for a compact, group-extremal affine semigroup there is a sufficient system of representations by linear operators on finite-dimensional complex linear spaces; in the abelian case, a sufficient system of affine semicharacters is obtained. As a result, a compact group-extremal affine semigroup is the inverse limit of compact, finite-dimensional, group-extremal affine semigroups.

A subset S of a locally convex topological linear space X (over the reals or complexes) will be called an affine semigroup if:

(1) S is convex.

(2) There is an associative multiplication defined in S which is jointly continuous in the topology on S inherited from X.

(3) For fixed $x \in S$ the functions $y \rightarrow yx$ and $y \rightarrow xy$ are affine functions of S into S.

In this paper, S will always be compact. By a theorem due to Wendel [2], if S is a compact affine semigroup with identity u, then each point of S with inverse is an extreme point of S. If, conversely, each extreme point has an inverse then the set of extreme points of S is the maximal group of the idempotent u and is, therefore, compact [9]. In this case, we shall say S is *group-extremal*.

Following [2], we will say two affine semigroups S and T are *equivalent* if there exists a bicontinuous isomorphism of S onto T which is also an affine function.

DEFINITION 1. A representation of an affine semigroup S is a function P from S to $B(M)$ the set of bounded linear operator on some finite-dimensional complex linear space M satisfying:

(a) P is continuous (with any locally convex topology on $B(M)$, all of which are equivalent).

(b) P is a homomorphism.

(c) P is affine.

DEFINITION 2. An affine semicharacter on S is any complex-valued continuous affine homomorphism defined on S. We point out that if S is compact and f is any affine semicharacter on S then $|f(x)| \leqq 1$ for each $x \in S$.

In the remainder of this paper, S will be a compact, group-extremal affine semigroup with identity u, and whose extreme points form the compact topological group G.

1. Representations of S. In this section, we shall prove the following:

THEOREM 1. *For $x_0, y_0 \in S$, $x_0 \neq y_0$ there exists a representation P of S in $B(M)$, M a finite-dimensional complex linear space, satisfying*

(1) $P(x_0) \neq P(y_0)$.

(2) $P^*(\sigma) \in P(S)$ *for all* $\sigma \in S$ *(where $P^*(\sigma)$ is the adjoint of the operator $P(\sigma)$).*

Many of the details of the proof are quite similar to those in group representations (cf. [1], [6], [7]) but we shall include them for the sake of completeness. By $C(S)$ $(C(G))$ we mean the collection of all complex-valued continuous functions on $S(G)$. The supremum norm in $C(S)$ is denoted by $\|\cdot\|$ and in $C(G)$ by $\|\cdot\|_*$. $A(S)$ will denote the norm closed subspace of $C(S)$ consisting of all *affine* continuous complex-valued functions. $A(G)$ denotes the set of restrictions to G of elements of $A(S)$.

LEMMA 1.1. (a) *$A(G)$ is a closed subspace of $C(G)$.*

(b) *If $f, g \in A(S)$ and $f(x) = g(x)$ for $x \in G$ then $f(x) = g(x)$ for all $x \in S$.*

(c) *If $f_n \in A(G)$, $g_n \in A(S)$ for $n = 0, 1, 2, \cdots$ if $f_n(x) = g_n(x)$ for $x \in G$, $n = 0, 1, 2, 3, \cdots$ and if $\|f_n - f_0\|_* \to 0$ then $\|g_n - g_0\| \to 0$.*

Proof of (a). Let $f_n \to f$ where $f_n \in A(G)$, $n = 1, 2, 3, \cdots$ and $f \in C(G)$. There exist $g_n \in A(S)$ such that $g_n(x) = f_n(x)$ for $x \in G$. For $\varepsilon > 0$ there exists an N such that if $m, n \geqq N$ and $x \in G$ then $|f_n(x) - f_m(x)| < \varepsilon/2$. If $x_1, \cdots, x_r \in G$, $\lambda_i \geqq 0$, $\sum_{i=1}^r \lambda_i = 1$ and $x = \sum_{i=1}^r \lambda_i x_i$ then

$$|g_n(x) - g_m(x)| = \left|\sum_{i=1}^r \lambda_i[g_n(x_i) - g_m(x_i)]\right| = \left|\sum_{i=1}^r \lambda_i[f_n(x_i) - f_m(x_i)]\right| < \frac{\varepsilon}{2}.$$

Since $g_n - g_m$ is continuous on S, and the elements x of the above form are dense in S [4], we have $|g_n(x) - g_m(x)| < \varepsilon$ for $x \in S$. Thus, $\{g_n\}_{n=1}^\infty$ is a Cauchy sequence in $C(S)$ and, hence, converges to $g \in C(S)$. Since $A(S)$ is clearly closed, $g \in A(S)$. Now for $x \in G$, $f_n(x) \to f(x)$ but $f_n(x) = g_n(x) \to g(x)$ so that $f(x) = g(x)$ and $f \in A(G)$.

Proof of (b). An application of the Krein-Milman Theorem.

Proof of (c). By an argument similar to the proof of (a) $\|g_n - h\| \to 0$ for some $h \in A(S)$. But $f_n(x) = g_n(x)$ for all $x \in G$ so that $h(x) = f_0(x) = g_0(x)$ for $x \in G$. By (b), $h(x) = g_0(x)$ for all $x \in S$.

Proof of theorem. By $L^2(G)$, we mean the Hilbert space of all functions on G which are square-integrable with respect to Haar measure on G, where the inner product is defined as usual. $\left(\text{i.e. } (f, g) = \int f\bar{g}dx\right)$. We denote the norm of an element $f \in L^2(G)$ by $\|f\|_2 = \left(\int |f|^2 dx\right)^{1/2}$.

We now fix $x_0, y_0 \in S$ where $x_0 \neq y_0$. There exists a set U which is open in G, $u \in U$, and $\langle U \rangle x_0 \cap \langle U \rangle y_0 = \varnothing$. ($\langle U \rangle$ denotes the closed convex hull of U). This follows from $ux_0 \neq uy_0$, the continuity of multiplication in S, and the local convexity of the containing space X.

There exists a real-valued function $f_0 \in A(S)$ satisfying:

$$\min_{z \in \langle U \rangle x_0} \{f_0(z)\} > \max_{z \in \langle U \rangle y_0} \{f_0(z)\}$$

[3]. Choose $h \in C(G)$, $h(u) = 1$, $h = 0$ in $G \backslash U$, and $0 \leqq h \leqq 1$. For $z \in G$, let

$$k(z) = \frac{h(z) + h(z^{-1})}{2}$$

then $k \in C(G)$, $0 \leqq k \leqq 1$, $k(u) = 1$, $k = 0$ in $G \backslash U$ and $k(z) = k(z^{-1})$. We then have

$$\int k(z^{-1})f_0(zx_0)dz = \int_U k(z^{-1})f_0(zx_0)dz > \int_U k(z^{-1})f_0(zy_0)dz$$
$$= \int k(z^{-1})f_0(zy_0)dz \, .$$

Hence,

(1) $\int k(z^{-1})f_0(zx_0)dz \neq \int k(z^{-1})f_0(zy_0)dz$.

The operator in $L^2(G)$ defined by

(2) $Tf(x) = \int k(z^{-1})f(zx)dz$ for $f \in L^2(G), x \in G$ takes $L^2(G)$ into $C(G)$ and is a completely continuous, symmetric bounded linear operator in $L^2(G)$ [8; p. 242]. Further, $\|Tf\|_* \leqq \|k\|_2 \cdot \|f\|_2$ so that $f \to Tf$ is continuous in the norm topology on $C(G)$. If $f \in A(G)$ then there is a $g \in A(S)$ such that $g(x) = f(x)$ for $x \in G$. If we define:

(3) $g'(x) = \int k(z^{-1})g(zx)dz$ then $g' \in A(S)$ and for

$$x \in G, \, g'(x) = \int k(z^{-1})g(zx)dz = \int k(z^{-1})f(zx)dz = Tf(x) \, .$$

Thus, if $f \in A(G)$, then $Tf \in A(G)$. If we let H denote the closure of

$A(G)$ in $L^2(G)$, then H is a closed invariant subspace of T. In fact, if $f \in H$, there exists a sequence $f_n \in A(G)$ such that $\|f_n - f\|_2 \to 0$. But then $\|Tf_n - Tf\|_* \to 0$ and since $Tf_n \in A(G)$, which is norm closed in $C(G)$, we have $Tf \in A(G)$. Hence, T takes H into $A(G)$. Using T again to denote the restriction of T to H, we have again that T is a completely-continuous, symmetric bounded linear operator in H. By a well-known theorem (cf. [8; p. 233]) there exists a sequence $\{\psi_i\}_{i=1}^{\infty}$ where

(4) $\psi_i \in H$ for $i = 1, 2, \cdots$

(5) $T\psi_i = \lambda_i \psi_i$ for some real number $\lambda_i \neq 0$

(6) $(\psi_i, \psi_j) = \delta_{ij}$ (δ_{ij} is the Kronecker delta function)

(7) $Tf = \sum_{i=1}^{\infty} (Tf, \psi_i)\, \psi_i$ for each $f \in H$ and where the series converges in $L^2(G)$ norm.

(8) For each $\lambda \neq 0$, $M_\lambda = \{f \in H\colon Tf = \lambda \cdot f\}$ is finite-dimensional.

Note that $\psi_i = T((1/\lambda_i)\psi_i)$ and since $(1/\lambda_i)\psi_i \in H$, it follows that $\psi_i \in A(G)$. Also, using a computation that can be found in [1; p. 209] the series in (7) converges to Tf in the supremum norm on $C(G)$.

Now since $\psi_i \in A(G)$ for each i, there exists $\hat{\psi}_i \in A(S)$ such that $\hat{\psi}_i(x) = \psi_i(x)$ for $x \in G$. Further, if $g \in A(S)$ and f denotes the restriction of g to G then $f \in A(G)$ so that $Tf = \sum_{i=1}^{\infty} (Tf, \psi_i)\psi_i$ where the series converges in supremum norm on $C(G)$. As in (3), if $g'(x) = \int k(z^{-1})g(zx)dz$ for $x \in S$ then $g' \in A(S)$ and for $x \in G$, $g'(x) = Tf(x)$. Also for $x \in G$, $n \geqq 1$, $\sum_{i=1}^{n} (Tf, \psi_i)\hat{\psi}_i(x) = \sum_{i=1}^{n} (Tf, \psi_i)\psi_i(x)$ and, hence, Lemma 1.1(c) implies that $g' = \sum_{i=1}^{\infty} (Tf, \psi_i)\hat{\psi}_i$ where the series converges in $A(S)$. In particular, if f_0 is our original function (1) and g_0 is the restriction to G of f_0 then $f_0' = \sum_{i=1}^{\infty} (Tg_0, \psi_i)\hat{\psi}_i$. But by (1), $f_0'(x_0) \neq f_0'(y_0)$ so that for some i, $\hat{\psi}_i(x_0) \neq \hat{\psi}_i(y_0)$.

For $\lambda = \lambda_i$, $M_\lambda = \{f \in H, Tf = \lambda \cdot f\}$ is a finite-dimensional subspace of H; hence, by Lemma 1.1(b) $N_\lambda = \{f \in A(S)\colon f' = \lambda f\}$ is a finite-dimensional subspace of $A(S)$, and there exists $\hat{\psi}_i \in N_\lambda$ for which $\hat{\psi}_i(x_0) \neq \hat{\psi}_i(y_0)$. N_λ is easily seen to be a finite-dimensional Hilbert space with inner product again $(f, g) = \int f\bar{g}dx$. In fact, if $f \in A(S)$ and $(f, f) = 0$ then $\int |f|^2 dx = 0$ so that $f(x) = 0$ for $x \in G$. By Lemma 1.1(b), $f(x) = 0$ for all $x \in S$. For $f \in N_\lambda$, it is easily seen $(|\lambda|/\|k\|_2)\, \|f\| \leqq (f, f)^{1/2} \leqq \|f\|$ so that N_λ is complete with respect to this inner product. For $\sigma \in S$, we define the linear operator $P(\sigma)$ in N_λ by:

(9) $[P(\sigma)f](x) = f(x\sigma)$ where $f \in N_\lambda$, $x \in S$. We have

$$[P(\sigma)f]'(x) = \int k(z^{-1})P(\sigma)f(zx)dz = \int k(z^{-1})f(zx\sigma)dz$$
$$= f'(x\sigma) = \lambda f(x\sigma) = \lambda[P(\sigma)f](x)\ .$$

Hence, $P(\sigma)$ clearly takes N_λ to N_λ. It is clear that the map $\sigma \to P(\sigma)$ is continuous in the strong operator topology. Further, $[P(\sigma\tau)f](x) = f(x\sigma\tau) = P(\sigma)[P(\tau)f](x)$ so that $P(\sigma\tau) = P(\sigma)P(\tau)$ and $\sigma \to P(\delta)$ is a homomorphism. For $\sigma, \tau \in S$ $0 \leqq \lambda \leqq 1$ and $x \in S$ we have

$$\begin{aligned}[P(\lambda\sigma + (1-\lambda)\tau)f](x) &= f(x[\lambda\sigma + (1-\lambda)\tau]) \\ &= \lambda f(x\sigma) + (1-\lambda)f(x\tau) \\ &= [\lambda P(\sigma) + (1-\lambda)P(\tau)f](x)\end{aligned}$$

and $\sigma \to P(\sigma)$ is now an affine continuous homomorphism of S into the bounded linear operators on the finite-dimensional space N_λ.

Note further that there exists $\hat{\psi}_i \in N_\lambda$ where $\hat{\psi}_i(x_0) \neq \hat{\psi}_i(y_0)$. Then $[P(x_0)\hat{\psi}_i](u) = \hat{\psi}_i(x_0) \neq \hat{\psi}_i(y_0) = [P(y_0)\hat{\psi}_i](u)$ and $P(x_0) \neq P(y_0)$. Finally, for $x \in G$, $f, g \in N_\lambda$

$$\begin{aligned}(P(x)f, g) &= \int [P(x)f](y)\overline{g(y)}dy = \int f(yx)\overline{g(y)}dy \\ &= \int f(y)\overline{g(yx^{-1})}dy = \int f(y)\overline{[P(x^{-1})g](y)}dy = (f, P(x^{-1})g) \,.\end{aligned}$$

Hence, we have for $x \in G$, $P^*(x) = P(x^{-1})$. If $x_1, x_2, \cdots x_n \in G$, $\lambda_i \geqq 0$, $\sum_{i=1}^n \lambda_i = 1$ and $x = \sum_{i=1}^n \lambda_i x_i$ then

$$P^*(x) = \sum_{i=1}^n \lambda_i P^*(x_i) = \sum_{i=1}^n \lambda_i P(x_i^{-1}) = P\Big(\sum_{i=1}^n \lambda_i x_i^{-1}\Big) \in P(S) \,.$$

Since $P(S)$ is compact and convex, it follows by continuity of P and the Krein-Milman Theorem that $P^*(\sigma) \in P(S)$ for each $\sigma \in S$ and the proof is complete.

COROLLARY 1.1. *If G is metrizable, there is a countable number of representations which separate points.*

Proof. In Theorem 1, to separate two points we obtained a neighborhood of the identity, and then constructed a countable number of representations using this neighborhood. It is clear this neighborhood may be taken from a countable basis at the identity, giving rise to a countable number of representations which separate the points of S.

2. Affine semicharacters. In this section, we assume the additional condition that S is abelian; then we have:

THEOREM 2. *If $x_0, y_0 \in S$, $x_0 \neq y_0$ there exists an affine semicharacter p such that $p(x_0) \neq p(y_0)$.*

Proof. By Theorem 1, there exists a representation P of S in the bounded linear operators $B(M)$ on the n-dimensional complex vector space M for which $P(x_0) \neq P(y_0)$ and $P^*(\sigma) \in P(S)$ for each $\sigma \in S$. The space M is then a finite-dimensional space invariant under the abelian family of operators $\{P(\sigma) : \sigma \in S\}$ satisfying $P^*(\sigma) \in P(S)$ for $\sigma \in S$ and, hence, is spanned by one dimensional invariant subspaces. We thus obtain a basis $e_1, \cdots, e_n$ for M where $P(\sigma)e_i = P_i(\sigma)e_i$ for each $i = 1, 2, \cdots, n$ and $p_i(\sigma)$ is a complex number. The functions $p_1, \cdots, p_n$ are easily seen to be affine semicharacters of S. Since $P(x) \neq P(y)$, $p_i(x) \neq p_i(y)$ for some integer i and we are finished. Using the representations of S and the fact that they are affine maps we have:

THEOREM 3. *A group-extremal affine semigroup is equivalent to the inverse limit of finite-dimensional group-extremal affine semigroups.*

The proof of this theorem is completely analogous to the proof of the well-known theorem that a compact group is the inverse limit of compact Lie groups, so we shall omit it.

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Received August 2, 1965. The author would like to express his appreciation to Professor R. J. Koch for his advice in the preparation of this paper. This research was supported in part by National Science Foundation Grant No. GP-1637.

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