ANOTHER PROOF OF A THEOREM ON RATIONAL CROSS SECTIONS

MAXWELL ROSENLICHT

The extant proofs of the existence of a rational cross section for a transformation space for a connected solvable linear algebraic group either use a certain amount of algebraic curve theory or restrict themselves to the case of a principal space, where the question is one of galois cohomology, the result being equivalent to the statement that $H^1(G, k) = 0$ for G a k-solvable linear algebraic group. The present proof of the general result may be considered more elementary in that it depends only on the standard facts on fields of rationality of algebraic sets.

The result in question says that *if G is a k-solvable linear algebraic group and V a transformation space for G, all rational over a field k, then there exists a G-invariant dense k-open subset V' of V such that V'/G exists and is rational over k and a cross section k-morphism $V'/G \to V'$ exists.* This statement appears to be somewhat stronger than the original statement [1, Th. 10], but is exactly equivalent to it (except for the purely technical matter of the possible reducibility of V) once one accepts the result that any algebraic transformation space admits a quotient space, provided one restricts to a dense open subset [2].

If V happens to be G-homogeneous the theorem says nothing more than that V has a point rational over k. This special case also implies the general theorem, and that without much labor (in fact the detailed argument in [1] can be much shortened by use of the result quoted at the end of the paragraph above). As for proving the special case just quoted, if one assumes the case $\dim G = 1$ there is a straightforward induction argument on $\dim G$ (again refer for details to [1]) so the crux of the proof is that *a k-homogeneous space for G_a or G_m has a point rational over k.* This note concentrates on the proof of this last statement.

The proof given in [1] that a k-homogeneous space for G_a or G_m has a rational point over k uses some technical information from the theory of algebraic curves, considerably out of the spirit of the present subject. If the homogeneous space happens to be principal homogeneous the question becomes an easy one of galois cohomology [3, pp. 170–171], but some generality has been lost. The present proof is accomplished by starting with a somewhat stronger version of the fact to be proved (Lemma 3 below, which can be taken as the substitute for the Lemma

to Theorem 10 in [1]), which is easy to prove when k is the universal domain, at the same time retaining some nontrivial content in this extreme case. Lemmas 1 and 2 below are well known, but convenient to have formally stated at this point.

LEMMA 1. *The factor group of G_a (G_m) by a finite subgroup is isomorphic to G_a (G_m).*

G_a (G_m) is the additive group of the universal domain Ω (multiplicative group of nonzero elements of Ω) with its usual structure as an algebraic set over the prime field. Since G_a and G_m are commutative we can use induction on the order of the finite subgroup to reduce the proof to the case where this subgroup is cyclic. In the case where the group is G_a and Ω has characteristic $p \neq 0$ the cyclic group generated by a nonzero $a \in \Omega$ is the set of zeros of the polynomial $X^p - a^{p-1}X$ and the map sending any $x \in \Omega$ into $x^p - a^{p-1}x$ is a separable homomorphism of G_a onto itself whose kernel is the cyclic group generated by a, proving the lemma in this case. In the remaining cases we do not even need the reduction to the cyclic subgroup: if the group is G_a and the field characteristic zero, then $\{0\}$ is the only finite subgroup of G_a and there is nothing to prove, while in the case G_m a finite subgroup of order n consists of the nth roots of unity, hence n is prime to the characteristic, and here the map sending each $x \in \Omega$ into x^n is a separable homomorphism of G_m onto itself with the correct kernel, completing the proof.

LEMMA 2. *If V_1, V_2 are homogeneous spaces for the algebraic group G and $\tau: V_1 \to V_2$ is a rational map such that $\tau(gv_1) = g\tau(v_1)$ (in the sense of rational maps of $G \times V_1$ into V_2) then τ is a morphism.*

The proof for this is the usual homogeneity argument: the domain of definition of τ is a dense open subset of V_1 that is also G-invariant, hence V_1 itself.

LEMMA 3. *Let G be the algebraic group G_a or G_m and let V be a homogeneous space for G of dimension greater than zero, V and the morphism $G \times V \to V$ being rational over k. Then there exists an algebraic group G', rational over k and k-isomorphic to G, a surjective k-homomorphism $\varphi: G \to G'$ and a k-isomorphism of algebraic sets $\psi: G' \to V$ such that for any $g \in G$ and $g' \in G'$ we have $g\psi(g') = \psi(\varphi(g)g')$.*

We first prove this in the special case where k is the universal domain Ω. The dimension of V is one, and since G is commutative

each point of V has the same finite isotropy subgroup $H \subset G$. The composition $G \times V \rightarrow V$, which sends (g, v) into gv, factors (a priori in a merely set-theoretic sense) into

$$G \times V \rightarrow (G/H) \times V \rightarrow V$$

via the natural map $G \rightarrow G/H$. Since the H-invariant elements of the function field $\Omega(G \times V)$ can be identified with the elements of $\Omega((G/H) \times V)$, the above map $(G/H) \times V \rightarrow V$ is actually rational. Application of Lemma 2 to the group $G \times G$ operating on $(G/H) \times V$ and V in the appropriate manners shows the map $(G/H) \times V \rightarrow V$ to be a morphism, so that V is homogeneous for G/H. By Lemma 1, G/H is isomorphic to G, so we are reduced to proving the result when $H = \{e\}$, i.e. all isotropy groups are trivial. Fix a point $v_0 \in V$. Then the morphism $G \rightarrow V$ which sends any $g \in G$ into gv_0 is bijective. Identifying the function field $\Omega(V)$ with a subfield of $\Omega(G)$ via this morphism, we see that $\Omega(G)$ is a purely inseparable extension of $\Omega(V)$. If x is the natural coordinate function on G (so that the map sending any $g \in G$ into $x(g)$ gives the identification of G with Ω or $\Omega - (0)$) then $\Omega(G) = \Omega(x)$ and there exists a positive integer ν such that $x^{p^\nu} \in \Omega(V)$, p being the characteristic of Ω, or 1 if the latter is zero. If ν is taken minimal then the minimal polynomials of x over $\Omega(V)$ and over $\Omega(x^{p^\nu})$ will coincide, giving

$$[\Omega(G) : \Omega(V)] = [\Omega(G) : \Omega(x^{p^\nu})] ,$$

from which it follows that $\Omega(V) = \Omega(x^{p^\nu})$. We proceed to prove the lemma in the present case (trivial isotropy groups), taking G' to be G and φ to be the p^ν-th power map. Setting $y = x^{p^\nu}$, we have $\Omega(V) = \Omega(y)$ and for $g \in G$ we have $y(gv_0) = (x(g))^{p^\nu}$. If we set $v = g'v_0$, with $g' \in G$, and denote by $\circ$ the operation $+$ or $\times$ according as G is G_a or G_m, we have $y(gv) = y(g(g'v_0)) = y((gg')v_0) = (x(gg'))^{p^\nu} = (x(g))^{p^\nu} \circ (x(g'))^{p^\nu} = (x(g))^{p^\nu} \circ y(v)$. y is defined on a dense open subset of V, hence, by homogeneity, on all of V. If $G = G_a$ then y takes on all finite values and if $G = G_m$ all finite values except zero, so that y gives an isomorphism between the algebraic set V and G_a or G_m. Everything else we want is trivial to verify, which finishes the case when $k = \Omega$. Now consider the general case. By what we have just proved, there exists a function $y \in \Omega[V]$ inducing an isomorphism (of algebraic sets) between V and G_a or G_m and a surjective homomorphism φ from G into G_a or G_m (identified with Ω or $\Omega - (0)$, so that φ can be regarded as a numerical function) such that $y(gv) = \varphi(g) \circ y(v)$ for all $g \in G$, $v \in V$, and we need only show that y and φ may be taken to be rational over k. So express y and φ as linear combinations with coefficients in Ω of elements of $k[V]$ and $k[G]$ respectively. By linear algebra we may write

$$y = \sum_{i=1}^{n} c_i y_i , \qquad \varphi = \sum_{i=1}^{n} c_i \varphi_i ,$$

where $\{c_i\} \subset \Omega$ are linearly independent over k and each $y_i \in k[V]$, each $\varphi_i \in k[G]$. If $G = G_a$, then $y(gv) = \varphi(g) + y(v)$, so

$$\sum_{i=1}^{n} c_i y_i(gv) = \sum_{i=1}^{n} c_i \varphi_i(g) + \sum_{i=1}^{n} c_i y_i(v) ,$$

or

$$\sum_{i=1}^{n} c_i(y_i(gv) - \varphi_i(g) - y_i(v)) = 0 .$$

Since $G \times V$ is rational over k, $k(G \times V)$ and Ω are linearly disjoint over k, so $y_i(gv) = \varphi_i(g) + y_i(v)$ for all i. Each $y_i \in \Omega[y]$ so there exists $f_i(Y) \in \Omega[Y]$ (Y an indeterminate over Ω) such that $y_i = f_i(y)$, giving

$$f_i(\varphi(g) + y(v)) = \varphi_i(g) + f_i(y(v)) .$$

Now if $x \in k[G]$ is the additive coordinate function on G_a then $\varphi_i \in k[G] = k[x]$ has a certain degree ν_i in x and $\varphi = \sum_i c_i \varphi_i$ has degree equal to $\max_i \nu_i > 0$, so the last equation implies that the degree of each f_i is at most one. Not all f_i's are constant, so say f_1 is nonconstant, whence $f_1(Y) = aY + b$, with $a, b \in \Omega$, $a \neq 0$. Then $y_1 = ay + b$. We can now replace y by y_1 as a coordinate function on V (at the same time changing φ appropriately), i.e. we can assume that $y \in k[V]$. Then the equation $\varphi(g) = y(gv) - y(v)$ gives $\varphi(g) = y(g \circ 0) - y(0)$, so $\varphi \in k[G]$, completing the proof for the G_a case. Finally let $G = G_m$. Then $y(gv) = \varphi(g)y(v)$, so

$$\sum_{i=1}^{n} c_i y_i(gv) = \sum_{i=1}^{n} c_i \varphi(g) y_i(v) .$$

But φ is a surjective homomorphism from G_m into itself, therefore given by sending any x into x^ν for some nonzero integer ν, in particular $\varphi \in k[G]$. Thus linear disjointness implies that $y_i(gv) = \varphi(g)y_i(v)$ for all i. In the present case $\Omega[V] = \Omega[y, 1/y]$ so we can write $y_i = \sum_j a_{ij} y^j$, where each $a_{ij} \in \Omega$ and the index j ranges over a finite set of integers, some possibly negative. We therefore have

$$\sum_j a_{ij}(\varphi(g)y(v))^j = \varphi(g) \sum_j a_{ij}(y(v))^j$$

for all $g \in G$, $v \in V$, which implies that $a_{ij} = 0$ if $j \neq 1$. Thus $y_i = a_i y$, with $a_i \in \Omega$. If $a_1 \neq 0$ we can replace y by y_1 as a coordinate function on V so as to get, once more, $y \in k[V]$, and we are done.

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UNIVERSITY OF CALIFORNIA, BERKELEY