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ON THE CONVERGENCE OF QUASI-HERMITE-FEJÉR INTERPOLATION

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The present paper deals with the convergence of quasi-Hermite-Fejér interpolation series $\{S_n(x, f)\}$ satisfying the conditions

$$S_n(1, f) = f(1),\ S_n(x_{n\nu}, f) = f(x_{n\nu})\ 1 \leqq \nu \leqq n,\ S_n(-1, f) = f(-1)$$

and

$$S'_n(x_{n\nu}, f) = \beta_{n\nu} \qquad 1 \leqq \nu \leqq n\ ,$$

where $\beta_{n\nu}$'s are arbitrary numbers; $x_{n0} = 1$, $x_{n,n+1} = -1$ and $\{x_{n\nu}\}$ are the zeros of orthogonal polynomial system $\{p_n(x)\}$ belonging to the weight function $(1-x^2)^p |x|^q$, $0 < p \leqq \frac{1}{2}$, $0 < q < 1$ (which actually vanishes at a point in the interval $[-1, +1]$). Further it has been proved that quasi-conjugate pointsystem $\{X_{n\nu}\}$ (similar to Fejér conjugate pointsystem) belonging to the fundamental pointsystem $\{x_{n\nu}\}$ lie everywhere thickly in the interval $[-1, +1]$.

Let there be given a point system

$$1 = x_{n0} > x_{n1} > x_{n2} > \cdots > x_{nn} > x_{n,n+1} = -1 \qquad (n = 1, 2, \cdots) \tag{1.1}$$

on the real axis and arbitrary real numbers

$$\begin{gathered} y_{n0}, y_{n1}, y_{n2}, \cdots, y_{nn}, y_{n,n+1}\ , \\ y^*_{n1}, y^*_{n2}, \cdots, y^*_{nn}\ . \end{gathered} \tag{1.2}$$

Then setting

$$\omega_n(x) = c_n(x - x_{n1})(x - x_{n2}) \cdots (x - x_{nn}) \qquad (c_n \neq 0) \tag{1.3}$$

and

$$l_{n\nu}(x) = \frac{\omega_n(x)}{\omega'_n(x_{n\nu})(x - x_{n\nu})}\ (\nu = 1, 2, \cdots, n)\ , \tag{1.4}$$

the quasi-Hermite-Fejér interpolation polynomial $S_n(x)$ [6] is given by

$$S_n(x) = \sum_{\nu=0}^{n+1} y_{n\nu} r_{n\nu}(x) + \sum_{\nu=1}^{n} y^*_{n\nu} \rho_{n\nu}(x) \tag{1.5}$$

where $r_{n\nu}(x)$ and $\rho_{n\nu}(x)$ are called the fundamental polynomials of the 1st and the second kind of quasi-Hermite-Fejér interpolation.

For the fundamental polynomials of the 1st kind we have

$$
\begin{aligned}
r_{n0}(x) &= \frac{1+x}{2} \cdot \frac{\omega_n(x)^2}{\omega_n(1)^2}\,, \\
r_{n,n+1}(x) &= \frac{1-x}{2} \cdot \frac{\omega_n(x)^2}{\omega_n(-1)^2}\,, \qquad (1.6)\\
r_{n\nu}(x) &= \frac{1-x^2}{1-x_{n\nu}^2}\, v_{n\nu}(x) l_{n\nu}(x)^2\,, \qquad (\nu = 1, 2, \cdots n)
\end{aligned}
$$

where

$$v_{n\nu}(x) = 1 + c_{n\nu}(x - x_{n\nu})\,, \tag{1.7}$$

$$c_{n\nu} = \frac{2x_{n\nu}}{1-x_{n\nu}^2} - \frac{\omega_n''(x_{n\nu})}{w_n'(x_{n\nu})} \qquad (\nu = 1, 2, \cdots, n) \tag{1.8}$$

and those of second kind

$$\rho_{n\nu}(x) = \frac{1-x^2}{1-x_{n\nu}^2}(x - x_{n\nu}) l_{n\nu}(x)^2 \quad (\nu = 1, 2, \cdots, n)\,. \tag{1.9}$$

The polynomials $S_n(x)$ are the unique polynomials of degree $\leqq 2n+1$ which satisfy the requirements:

$$
\begin{aligned}
S_n(x_{n\nu}) &= y_{n\nu} \qquad \nu = 0, 1, 2, \cdots, n+1\,, \\
S_n'(x_{n\nu}) &= y_{n\nu}^* \qquad \nu = 1, 2, \cdots, n\,.
\end{aligned} \tag{1.10}
$$

From the unicity of the polynomials $S_n(x)$ it follows that for each polynomial $\Pi(x)$ of degree $\leqq 2n$

$$\Pi(x) = \sum_{\nu=0}^{n+1} \Pi(x_{n\nu}) r_{n\nu}(x) + \sum_{\nu=1}^{n} \Pi'(x_{n\nu}) \rho_{n\nu}(x) \tag{1.11}$$

holds. For the special case $\Pi(x) \equiv 1$, we have

$$\sum_{\nu=0}^{n+1} r_{n\nu}(x) \equiv 1\,. \tag{1.12}$$

2. Let $f(x)$ be continuous in $-1 \leqq x \leqq 1$ and $f(x_{n\nu})$ its values at the points $x_{n\nu} (\nu = 0, 1, 2, \cdots, n+1)$. Further let $y_{n\nu}^* (\nu = 1, 2, \cdots n)$ be arbitrary real numbers then the polynomials $S_n(x)$ in (1.5) written as

$$S_n(x, f) = \sum_{\nu=0}^{n+1} f(x_{n\nu}) r_{n\nu}(x) + \sum_{\nu=0}^{n} y_{n\nu}^* \rho_{n\nu}(x) \tag{2.1}$$

are called the generalised quasi-Hermite-Fejér interpolation polynomials. For $y_{n\nu}^* = 0$, they are called quasi-step parabolas. In this case for $\omega_n(x) = P_n(x)$, where $P_n(x)$ stands for the nth Legendre polynomial, the interpolatory polynomials

$$
\begin{aligned}
R_n(x) &= f(1)\,\frac{1+x}{2}\,P_n(x)^2 + f(-1)\,\frac{1-x}{2}\,P_n(x)^2 \\
&\quad + \sum_{\nu=1}^{n} f(x_{n\nu})\,\frac{1-x^2}{1-x_{n\nu}^2}\left(\frac{P_n(x)}{P_n'(x_{n\nu})\,(x-x_{n\nu})}\right)^2
\end{aligned}
\tag{2.2}
$$

have been obtained by E. Egerváry and P. Turán [2]. They have shown that if $f(x)$ is a function continuous in the closed interval $[-1, 1]$, then the polynomials in (2.2) converge uniformly to $f(x)$ in $[-1, 1]$. The convergence of the polynomials $S_n(x, f)$ in (2.1) constructed on the roots of $P_n(x)$ has been investigated by P. Szász [6]. He has shown that assuming $f(x)$ to be continuous and $|y_{n\nu}^*| < \Delta$, where Δ is a constant independent of n and ν the series $S_n(x, f)$ in (2.1) converges uniformly to $f(x)$ in $[-1, 1]$.

In this paper we answer the question of P. Turán for the quasi-Hermite-Fejér interpolation polynomials $S_n(x, f)$ which Balázs has answered [1] in the case of Hermite-Fejér interpolation polynomials.

Does there exist in $[-1, 1]$ an orthogonal polynomial system $\{g_n(x)\}$ whose weight function vanishes some where in this interval while the series $\{S_n(x, f)\}$ in (2.1) constructed on the roots of $\{g_n(x)\}$ converges uniformly to the continuous function $f(x)$ in the closed interval $[-1, 1]$ provided $\{y_{n\nu}^*\}$ are bounded?

The answer to this question is explained in our Theorem 1.

3. Similar to the normal and strongly normal point system due to L. Fejér [3, 4], the notion of quasi-normal and strongly quasi-normal point systems have been defined by Szász [6]. Thus an infinite sequence of point system,

$$
x_{n1}, x_{n2}, \cdots, x_{nn}, \quad (n = 1, 2, \cdots) \tag{3.1}
$$

lying in $-1 < x < 1$, is called strongly quasi-normal if by the notation of (1.3) and (1.7)

$$
1 + c_{n\nu}(x - x_{n\nu}) \geqq \rho > 0\,, \quad -1 \leqq x \leqq 1 \qquad (\nu = 1, 2, \cdots, n;\ n = 1, 2, \cdots) \tag{3.2}
$$

where ρ is a positive number independent of x, ν and n.

If $X_{n\nu}$ indicates a zero of $v_{n\nu}(x)$ in (1.7), then

$$
X_{n\nu} = x_{n\nu} + \frac{1}{c_{n\nu}}\,, \quad \nu = 1, 2, \cdots, n\,. \tag{3.3}
$$

These points will be called quasi-conjugate points similar to the conjugate points due to L. Fejér [4]. The quasi-conjugate points lie outside $[-1, 1]$ when the fundamental point system is quasi-strongly normal. In this connection we shall answer another question of P. Turán for the case of quasi-Hermite-Fejér interpolation polynomials which Balázs [1] has answered for the Hermite-Fejér interpolation polynomials.

Is it possible to assume in the interval $[-1, 1]$ a fundamental point system whose quasi-conjugate points (3.3) lie thickly in $[-1, 1]$ and the interpolation series $\{S_n(x, f)\}$ belonging to this fundamental point-system converges uniformly to the continuous function $f(x)$ in $[-1, 1]$ provided the numbers $\{y^*_{n\nu}\}$ are bounded.

In Theorem II we answer this in affirmative.

4. K. V Laščenov [5] has defined orthogonal polynomials

$$p_n^{(p,q)}(x) = \alpha_n x^n + \alpha_{n-2}x^{n-2} + \cdots , \quad \alpha_n \neq 0, \ p > -1, \ q > -1$$

over the interval $[-1, 1]$ with respect to the weight function $(1 - x^2)^p|x|^q$ which are constant multiples of

$$p_n^{(p,q)}(x)^1 = \begin{cases} P_m^{(p,q-1/2)}(2x^2 - 1), & n = 2m \\ xP_m^{(p,q+1/2)}(2x^2 - 1), & n = 2m + 1 \end{cases} \tag{4.1}$$

$P_n^{(\alpha,\beta)}(t)$ being the classical Jacobi polynomial of degree n with parameters α and β satisfying the differential equation

$$(1 - t^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)t]y' + n(n + \alpha + \beta + 1)y = 0 \, . \tag{4.2}$$

The position of the roots of (4.1) is given by

$$-1 < x_{nm+1} < x_{nm+2} < \cdots < x_{nn} < 0 < x_{n1} < \cdots < x_{nm} < 1 \quad \text{for } n = 2m \tag{4.3}$$

and

$$-1 < x_{nm+2} < x_{nm+3} < \cdots < x_{nn} < 0 = x_{nm+1} < x_{n1} < \cdots < x_{nm} < 1 \quad \text{for } n = 2m + 1 \, . \tag{4.4}$$

Since the roots are symmetrical, we have

$$x_{n\nu} + x_{n,n+1-\nu} = 0, \ \nu = 1, 2, \cdots [n/2] \, . \tag{4.5}$$

We shall prove the following:

THEOREM 1. *The quasi-Hermite-Fejér interpolation series* $\{S_n(x, f)\}$, *constructed on the point system*

$$1 = x_{n0}, x_{n1}, \cdots x_{n,n-1}, x_{n,n}, x_{nn+1} = -1 \qquad n = 1, 2, \cdots \tag{4.5}$$

[1] From now onward we shall write $p_n(x)$ to mean $p_n^{(p,q)}(x)$.

where $x_{n\nu}(\nu = 1, 2, \cdots, n)$ *are the zeros of the orthogonal polynomial system belonging to the weight function*[2]

$$(1 - x^2)^p |x|^q \qquad 0 < p \leqq \frac{1}{2}, \; 0 < q < 1,$$

converges uniformly to the continuous function $f(x)$ *in* $[-1, 1]$ *when* $|y^*_{n\nu}| \leqq cn^{\eta}$, $1 > \delta/2 > \eta \geqq 0$ *and* $\delta = \min(2p, q)$.

THEOREM 2. *The quasi-conjugate points* (3.3)

$$x_{n\nu} = x_{n\nu} + \frac{1}{c_{n\nu}} \qquad \nu = 1, 2, \cdots n; \; n = 1, 2, \cdots, \tag{4.6}$$

belonging to the fundamental point system (4.5) *lie thickly in the interval* $[-1, 1]$.

5. Preliminaries. We shall use some well-known facts about Jacobi polynomials. We have

$$P_m^{(\alpha,\beta)}(1) = \binom{m+\alpha}{m} \tag{5.1}$$

$$P_m^{(\alpha,\beta)}(-1) = (-1)^m P_m^{(\alpha,\beta)}(1) = (-1)^m \binom{m+\beta}{m} \tag{5.2}$$

$$P_m^{(\alpha,\beta)}(t) = (-1)^m P_m^{(\beta,\alpha)}(-t). \tag{5.3}$$

Further we have for $-1 < x < 1$

$$P_m^{(\alpha,\beta)}(x) = O(n^{-1/2}), \; \alpha, \beta > -1 \tag{5.4}$$

(5.5)

$$P_m^{(\alpha+1,\beta)}(x) = \frac{2}{(2m + \alpha + \beta + 2)} \frac{(m + \alpha + 1)P_m^{(\alpha,\beta)}(x) - (m+1)P_{m+1}^{(\alpha,\beta)}(x)}{(1-x)}$$

and

$$\frac{d}{dt} P_m^{(\alpha,\beta)}(t) = \frac{1}{2}(m + \alpha + \beta + 1)P_{m-1}^{(\alpha+1,\beta+1)}(t). \tag{5.6}$$

Further let $t_\nu = \cos\theta_\nu$ be the root of the polynomial

$$P_m^{(\alpha,\beta)}(t) = P_m^{(\alpha,\beta)}(\cos\theta)$$

then for $-1/2 \leqq \alpha \leqq 1/2$, $-1/2 \leqq \beta \leqq 1/2$,

$$\frac{2\nu - 1}{2m+1}\pi \leqq \theta_\nu \leqq \frac{2\nu}{2m+1}\pi \qquad (\nu = 1, 2, \cdots, m). \tag{5.7}$$

---

[2] $(1 - x^2)^p |x|^q$ for $0 < p \leqq \frac{1}{2}$, $0 < q < 1$, actually vanishes at $x = 0$.

For $0 < \theta_\nu \leqq \pi/2$ we have

$$P'^{(\alpha,\beta)}_m(\cos\theta_\nu) \geqq c_1 \nu^{-\alpha-3/2} m^{\alpha+2} \tag{5.8}$$

where c_1 is positive numerical constant.

6. In this section we shall obtain certain estimations for the polynomial $p_n(x)$.

We shall first prove:

LEMMA 6.1. *For* $-1 \leqq x \leqq 1$ *we have*

$$(1-x^2)p_n^2(x) = 0(n^{-1}) \, . \tag{6.1}$$

Proof of this lemma follows at once from (4.1) using (5.4).

LEMMA 6.2. *For the roots* $x_{n\nu}$ $(\nu = 1, 2, \cdots, \left[\frac{n}{2}\right],\ n = 1, 2,) \cdots$ *of the polynomial* $p_n(x)$, *we have*

$$x_{n\nu}^2(1-x_{n\nu}^2) \geqq \frac{\nu^2}{4n^2} \, . \tag{6.2}$$

Proof. Let $2x_{n\nu}^2 - 1 = \cos\theta_{n\nu}$, then $(1-x_{n\nu}^2) = \sin^2\theta_{n\nu}/2$, and $x_{n\nu}^2 = \cos^2\theta_{n\nu}/2$. Hence

$$x_{n\nu}^2(1-x_{n\nu}^2) = \frac{4}{4}\cos^2\frac{\theta_{n\nu}}{2}\sin^2\frac{\theta_{n\nu}}{2} = \frac{1}{4}\sin^2\theta_{n\nu} \, .$$

But from (5.7) we have

$$\theta_{n\nu} \geqq \frac{\nu + \frac{1}{2}}{n + \frac{1}{2}}\pi > \frac{\nu\pi}{2n}$$

which gives

$$|\sin\theta_{n\nu}| > \left|\sin\frac{\nu\pi}{2n}\right| > \frac{\nu}{n}.$$

Therefore

$$x_{n\nu}^2(1-x_{n\nu}^2) = \frac{1}{4}\sin^2\theta_{n\nu} > \frac{\nu^2}{4n^2} \, .$$

7. We shall need the following lemmas for the estimation of the fundamental polynomials of the first kind.

LEMMA 7.1. *Let $x_{n\nu}$ be a root of $p_n(x)$, then*

$$\text{(i)}\quad \frac{p_n''(x_{n\nu})}{p_n'(x_{n\nu})} = \frac{2}{x_{n\nu}}\left[(p+1)\frac{x_{n\nu}^2}{(1-x_{n\nu}^2)} - \frac{q}{2}\right]$$

except when $n = 2m+1$, and $\nu = m+1$. In this case we have

$$\text{(ii)}\quad \frac{p_n''(x_{n\nu})}{p_n'(x_{n\nu})} = 0\,.$$

Proof. It follows from (4.1) by differentiating with respect to x, for $n = 2m$

$$\frac{p_n''(x_{n\nu})}{p_n'(x_{n\nu})} = 4_{n\nu}\left\{\frac{\dfrac{d^2}{dt^2}P_m^{(p,(q-1/2))}(t)}{\dfrac{d}{dt}P_m^{(p,(q-1/2))}(t)}\right\}_{t=2x_{n\nu}^2-1} + \frac{1}{x_{n\nu}}\,. \tag{7.1}$$

By the substitution $t = 2x^2 - 1$, $\alpha = p$, $\beta = q-1/2$, and $n = m$, the differential equation (4.2) gives

$$\begin{aligned}\left\{\frac{\dfrac{d^2}{dt^2}P_m^{(p,(q-1/2))}(t)}{\dfrac{d}{dt}P_m^{(p,(q-1/2))}(t)}\right\}_{t=2x_{n\nu}^2-1}\\ = \frac{1}{4x_{n\nu}^2(1-x_{n\nu}^2)}[-2(p+1) + (2p+q+3)(1-x_{n\nu}^2)]\,.\end{aligned} \tag{7.2}$$

Substituting (7.2) in (7.1) we get

$$\frac{p_n''(x_{n\nu})}{p_n'(x_{n\nu})} = \frac{2}{x_{n\nu}}\left[(p+1)\frac{x_{n\nu}^2}{1-x_{n\nu}^2} - \frac{q}{2}\right]. \tag{7.3}$$

If however, $n = 2m+1$ and $\nu \neq m+1$, then it follows on account of (4.1) and (4.4) that

$$\frac{p_n''(x_{n\nu})}{p_n'(x_{n\nu})} = 4x_{n\nu}\left\{\frac{\dfrac{d^2}{dt^2}P_m^{(p,(q+1/2))}(t)}{\dfrac{d}{dt}P_m^{(p,(q+1/2))}(t)}\right\}_{t=2x_{n\nu}^2-1} + \frac{3}{x_{n\nu}}\,. \tag{7.4}$$

But from (4.2) by putting $t = 2x^2 - 1$, $\alpha = p$, $\beta = q + 1/2$ and $n = m$ we get

$$\left\{\frac{\dfrac{d^2}{dt^2}P_m^{(p,(q+1/2))}(t)}{\dfrac{d}{dt}P_m^{(p,(q+1/2))}(t)}\right\}_{t=2x_{n\nu}^2-1} \tag{7.5}$$
$$= -\frac{1}{4x_{n\nu}^2(1-x_{n\nu}^2)}[-2(p+1)+(2p+q+5)(1-x_{n\nu}^2)]$$

substituting (7.5) in (7.4) we get

$$\frac{p_n''(x_{n\nu})}{p_n'(x_{n\nu})} = \frac{2}{x_{n\nu}}\left[(p+1)\frac{x_{n\nu}^2}{1-x_{n\nu}^2} - \frac{q}{2}\right]. \tag{7.6}$$

In case $n = 2m+1$ and $\nu = m+1$, $x_{n\nu} = 0$ on account of (4.4). But the polynomial $p_n(x)$ is an odd function of x, therefore $p_n''(x_{n\nu}) = 0$ and in this case

$$\frac{p_n''(x_{n\nu})}{p_n'(x_{n\nu})} = 0 . \tag{7.7}$$

8. Estimation of the fundamental polynomials of the first kind.

LEMMA 8.1. *For* $-1 \leqq x \leqq 1$, *we have*

$$\sum_{\nu=0}^{n+1} | r_{n\nu}(x) | = O(1) . \tag{8.1}$$

Proof. From (1.7), (1.8) and Lemma 7.1 we get for $1 \leqq \nu \leqq n$

$$v_{n\nu}(x) = 1 - \frac{2}{x_{n\nu}}\left\{\frac{px_{n\nu}^2}{(1-x_{n\nu}^2)} - \frac{q}{2}\right\}(x - x_{n\nu}) . \tag{8.2}$$

From the representation (4.4) of $x_{n\nu}$'s it is clear that for $n = 2m+1$, and $\nu = m+1$, $x_{nm+1} = 0$. Whence from Lemma 7.1 (ii) and (1.7) it follows that

$$v_{nm+1}(x) \equiv 1 . \tag{8.3}$$

For $x = 0$ it follows from (8.2) on account of $0 < q < 1$ and $0 < p \leqq \dfrac{1}{2}$ that

$$v_{n\nu}(0) = 1 + \frac{2px_{n\nu}^2}{(1-x_{n\nu}^2)} - q \geqq 1 - q > 0 . \tag{8.4}$$

This inequality is also applicable on account of (8.3) when $n = 2m+1$, and $\nu = m+1$. For $-1 < x \leqq 0$ and $x_{n\nu} \leqq 0$ we have on account of $v_{n\nu}(x_{n\nu}) = 1$ and (8.4)

$$v_{n\nu}(x) \geqq 1 - q > 0 \qquad (0 < q < 1)\,. \tag{8.5}$$

Since $v_{n\nu}(x)$ is a linear function in the interval $0 \leqq x < 1$ it follows from $v_{n\nu}(x_{n\nu}) \equiv 1$ and $x_{n\nu} \geqq 0$ that

$$v_{n\nu}(x) \geqq 1 - q > 0 \qquad \text{since } 0 < q < 1\,. \tag{8.6}$$

We shall now prove the inequality (8.1) in the interval $-1 < x \leqq 0$. In this interval $r_{n\nu}(x) \geqq 0$ for $x_{n\nu} \leqq 0$. Also $r_{n0}(x)$ and $r_{n,n+1}(x)$ are positive. Hence from (1.12)

$$\begin{aligned}\sum_{\nu=0}^{n+1} | r_{n\nu}(x) | &= \sum_{x_{n\nu} \leqq 0} | r_{n\nu}(x) | + \sum_{x_{n\nu} > 0} | r_{n\nu}(x) | \\ &= \sum_{x_{n\nu} \leqq 0} r_{n\nu}(x) + \sum_{x_{n\nu} > 0} | r_{n\nu}(x) | \\ &= 1 - \sum_{x_{n\nu} > 0} r_{n\nu}(x) + \sum_{x_{n\nu} > 0} | r_{n\nu}(x) | \\ &\leqq 1 + 2 \sum_{x_{n\nu} > 0} | r_{n\nu}(x) |\,.\end{aligned} \tag{8.7}$$

On account of (8.2), (1.6) and (1.4) we obtain

$$\begin{aligned}\sum_{x_{n\nu}>0} | r_{n\nu}(x) | &= \sum_{x_{n\nu}>0} \frac{1 - x^2}{1 - x_{n\nu}^2} \left| 1 - \frac{2}{x_{n\nu}} \left\{ \frac{p x_{n\nu}^2}{1 - x_{n\nu}^2} - \frac{q}{2} \right\} (x - x_{n\nu}) \right| \\ &\quad \times \frac{p_n^2(x)}{p_n'^2(x_{n\nu})\,(x - x_{n\nu})^2} \\ &\leqq \sum_{x_{n\nu}>0} \frac{1 - x^2}{1 - x_{n\nu}^2} \cdot \frac{p_n^2(x)}{p_n'^2(x_{n\nu})\,(x - x_{n\nu})^2} \\ &\quad + 2p \sum_{x_{n\nu}>0} \frac{(1 - x^2) p_n^2(x)}{| x_{n\nu} |\,(1 - x_{n\nu}^2)^2\, p_n'^2(x_{n\nu})\,(x - x_{n\nu})} \\ &\quad + (2p + q) \sum_{x_{n\nu}>0} \frac{1 - x^2}{(1 - x_{n\nu}^2)}\, \frac{1}{| x_{n\nu} |} \cdot \frac{p_n^2(x)}{p_n'^2(x_{n\nu})\,| x - x_{n\nu} |}\,.\end{aligned} \tag{8.8}$$

Since $-1 < x \leqq 0$ and $0 < x_{n\nu} < 1$, therefore $| x - x_{n\nu} | > | x_{n\nu} |$. Hence from (8.8),

$$\begin{aligned}\sum_{x_{n\nu}>0} | r_{n\nu}(x) | &\leqq (1 + 2p + q) \sum_{x_{n\nu}>0} \frac{1 - x^2}{1 - x_{n\nu}^2} \cdot \frac{1}{x_{n\nu}^2} \cdot \frac{p_n^2(x)}{p_n'^2(x_{n\nu})} \\ &\quad + 2 | p | \sum_{x_{n\nu}>0} \frac{(1 - x^2)}{(1 - x_{n\nu}^2)^2 x_{n\nu}^2}\, \frac{p_n^2(x)}{p_n'^2(x_{n\nu})}\,.\end{aligned} \tag{8.9}$$

Owing to (4.1) we have

$$p_n'(x_{n\nu}) = \begin{cases} 4x_{n\nu} P_m'^{(p,2(q-1/2))}(2x_{n\nu} - 1) & \text{for } n = 2m \\ 4x_{n\nu}^2 P_m'^{(p,2(q+1/2))}(2x_{n\nu} - 1) & \text{for } n = 2m + 1\,. \end{cases} \tag{8.10}$$

Thus for $n = 2m$, using (8.9) and (8.10); for n odd using (8.8), (8.10) and $x^2 < (x - x_{n\nu})^2$, we have

$$\sum_{x_{n\nu}>0} | r_{n\nu}(x) |$$

$$(8.11)\quad \leqq \begin{cases} \dfrac{1}{16}(1+4p+q) \displaystyle\sum_{x_{n\nu}>0} \dfrac{(1-x^2)[P_m^{(p,(q-1/2))}(2x^2-1)]^2}{x_{n\nu}^4(1-x_{n\nu}^2)^2\left[\dfrac{d}{dt}P_m^{(p,(q-1/2))}(t)\right]^2_{t=2x_{n\nu}^2-1}} & \text{for } n = 2m \\ \dfrac{1}{16}(1+4p+q) \displaystyle\sum_{x_{n\nu}>0} \dfrac{(1-x^2)}{(1-x_{n\nu}^2)^2 x_{n\nu}^5} \cdot \dfrac{[P_m^{(p,(q+1/2))}(2x^2-1)]^2}{\left[\dfrac{d}{dt}P_m^{(p,(q+1/2))}(t)\right]^2_{t=2x_{n\nu}^2-1}} & \text{for } n = 2m+1 . \end{cases}$$

Now Lemmas 6.1 and 6.2, with (5.8) give

$$(8.12)\quad \sum_{x_{n\nu}>0} | r_{n\nu}(x) | = \begin{cases} \left[\displaystyle\sum_{\nu=1}^{m} O(n^{-1}) \frac{n^4}{\nu^4} \cdot \frac{\nu^{2p+3}}{n^{2p+4}} + \sum_{\nu=1}^{m} O(n^{-1}) \frac{n^4}{\nu^4} \cdot \frac{\nu^{q+2}}{n^{q+3}}\right] & \text{for } n = 2m \\ \left[\displaystyle\sum_{\nu=1}^{m} O(n^{-1}) \frac{n^5}{\nu^5} \cdot \frac{\nu^{2p+3}}{n^{2p+4}} + \sum_{\nu=1}^{m} O(n^{-1}) \frac{n^5}{\nu^5} \cdot \frac{\nu^{q+4}}{n^{q+5}}\right] & \text{for } n = 2m+1 \end{cases}$$

and since $0 < p \leqq \dfrac{1}{2}$, $0 < q < 1$, (8.12) gives

$$(8.13)\qquad \sum_{x_{n\nu}>0} | r_{n\nu}(x) | = O(1) .$$

By a similar reasoning we can obtain for the interval $0 \leqq x < 1$ and $x_{n\nu} \geqq 0$, that

$$(8.14)\qquad \sum_{x_{n\nu}<0} | r_{n\nu}(x) | = O(1) .$$

Hence from (8.13) and (8.14) we get the lemma for $1 \leqq \nu \leqq n$, and $-1 < x < 1$. For $\nu = 0$ and $n+1$ it is easy to see from (1.6) with $\omega_n(x) = p_n(x)$ and (5.4) that

$$r_{n0}(x) = O(1) \qquad \text{and} \qquad r_{n,n+1}(x) = O(1) .$$

At $x = \pm 1$, the lemma is trivial.

9. Estimation of the fundamental polynomials of the second kind. In this section we shall estimate the quantity

$$\sum_{\nu=1}^{n} | \rho_{n\nu}(x) | .$$

We shall prove the following:

LEMMA 9.1. *For $-1 \leqq x \leqq 1$ and $n = 1, 2, \cdots$ we have*

$$(9.1) \qquad \sum_{\nu=1}^{n} |\rho_{n\nu}(x)| = O(n^{-\delta/2}) \,, \quad \text{where} \quad \delta = \min(2p, q) > 0 \,.$$

Proof. From (1.9) and (1.4) with $\omega_n(x) = p_n(x)$

$$(9.2) \qquad \sum_{\nu=1}^{n} \rho_{n\nu}(x) = \sum_{\nu=1}^{n} (x - x_{n\nu}) \frac{1 - x^2}{1 - x_{n\nu}^2} \frac{p_n^2(x)}{p_n'^2(x_{n\nu})\,(x - x_{n\nu})^2} \,.$$

Now setting

$$(9.3) \qquad \sum_{\nu=1}^{n} |\rho_{n\nu}(x)| = \sum_{x_{n\nu} \leqq 0} |\rho_{n\nu}(x)| + \sum_{x_{n\nu} > 0} |\rho_{n\nu}(x)|$$

and considering the interval $-1 < x \leqq 0$, we have for $x_{n\nu}$'s > 0,

$$|x - x_{n\nu}| > |x_{n\nu}| \,.$$

Thus from (9.2) and (8.10)

$$\sum_{x_{n\nu}>0} |\rho_{n\nu}(x)| \leqq \begin{cases} \dfrac{1}{16} \displaystyle\sum_{x_{n\nu}>0} \frac{1}{|x_{n\nu}|^3} \frac{(1 - x^2)}{(1 - x_{n\nu}^2)^{3/2}} \frac{[P_m^{(p,(q-1/2))}(2x^2 - 1)]^2}{\left[\dfrac{d}{dt} P_m^{(p,(q-1/2))}(t)\right]^2_{t=2x_{n\nu}^2-1}} & \text{for } n = 2m \\[2ex] \dfrac{1}{16} \displaystyle\sum_{x_{n\nu}>0} \frac{1}{x_{n\nu}^4} \frac{(1 - x^2)}{(1 - x_{n\nu}^2)^2} \frac{[P_m^{(p,(q+1/2))}(2x^2 - 1)]^2}{\left[\dfrac{d}{dt} P_m^{(p,(q+1/2))}(t)\right]^2_{t=2x_{n\nu}^2-1}} & \text{for } n = 2m + 1 \end{cases}$$

which on account of (5.8) and the Lemmas 6.1 and 6.2, gives,

$$(9.4) \qquad \sum_{x_{n\nu}>0} |\rho_{n\nu}(x)| \leqq \begin{cases} \displaystyle\sum_{\nu=1}^{m} O(n^{-1}) \frac{n^3}{\nu^3} \cdot \frac{\nu^{2p+3}}{n^{2p+4}} + \sum_{\nu=1}^{m} O(n^{-1}) \frac{n^3}{\nu^3} \cdot \frac{\nu^{q+2}}{n^{q+3}} & \text{for } n = 2m \\[2ex] \displaystyle\sum_{\nu=1}^{m} O(n^{-1}) \frac{n^4}{\nu^4} \cdot \frac{\nu^{2p+3}}{n^{2p+4}} + \sum_{\nu=1}^{m} O(n^{-1}) \frac{n^4}{\nu^4} \cdot \frac{\nu^{q+4}}{n^{q+5}} & \text{for } n = 2m + 1 \end{cases}$$

Since $0 < p \leqq \dfrac{1}{2}$ and $0 < q < 1$, it follows from (9.4) that

$$(9.5) \qquad \sum_{x_{n\nu}>0} |\rho_{n\nu}(x)| \leqq O(n^{-\delta}) \qquad -1 < x \leqq 0$$

where $\delta = min(2p, q) > 0$.

Again let $x_{n\nu} \leqq 0$, $-1 < x \leqq 0$ and

$$(9.6) \quad \sum_{x_{n\nu}\leqq 0} |\rho_{n\nu}(x)| = \sum_{\substack{x_{n\nu}\leqq 0 \\ |x - x_{n\nu}| \leqq n^{-\delta/2}}} |\rho_{n\nu}(x)| + \sum_{\substack{x_{n\nu}\leqq 0 \\ |x - x_{n\nu}| > n^{-\delta/2}}} |\rho_{n\nu}(x)| = \Sigma' + \Sigma'' \,.$$

On account of (9.2) the following holds in the interval $-1 < x \leqq 0$.

$$(9.7)\qquad \Sigma' | \rho_{n\nu}(x) | \leqq n^{-\delta/2} \Sigma' \frac{(1-x^2)p_n^2(x)}{(1-x_{n\nu}^2)p_n'^2(x_{n\nu})(x-x_{n\nu})^2}$$

$$\leqq \frac{n^{-\delta/2}}{|1-q|} \Sigma' \frac{(1-x^2)}{(1-x_{n\nu}^2)} v_{n\nu}(x) \frac{p_n^2(x)}{p_n'^2(x_{n\nu})(x-x_{n\nu})^2}$$

$$\leqq \frac{n^{-\delta/2}}{|1-q|} \Sigma' r_{n\nu}(x) \leqq \frac{n^{-\delta/2}}{|1-q|} .$$

From (9.6) we have

$$\Sigma'' | \rho_{n\nu}(x) | \leqq n^{\delta/2} \Sigma'' \frac{(1-x^2)}{(1-x_{n\nu}^2)} \frac{p_n^2(x)}{p_n'^2(x_{n\nu})} .$$

But owing to (8.10), we have

$$\Sigma'' | \rho_{n\nu}(x) | \leqq \begin{cases} \dfrac{n^{\delta/2}}{16} \Sigma'' \dfrac{1}{x_{n\nu}^2} \dfrac{(1-x^2)}{(1-x_{n\nu}^2)} \dfrac{p_n^2(x)}{\left[\dfrac{d}{dt} P_{m(t)}^{(p,(q-1/2))}\right]_{t=2x_{n\nu}^2-1}} & \text{for } n = 2m \\[2ex] \dfrac{n^{\delta/2}}{16} \Sigma'' \dfrac{(1-x^2)}{x_{n\nu}^4(1-x_{n\nu}^2)^2} \dfrac{p_n^2(x)}{\left[\dfrac{d}{dt} P_m^{(p,(q+1/2))}(t)\right]_{t=2x_{n\nu}^2-1}^2} & \text{for } n = 2m+1 \end{cases}$$

which by (5.8), and Lemmas 6.1 and 6.2 gives

$$(9.8)\qquad \Sigma'' | \rho_{n\nu}(x) |$$
$$\leqq \begin{cases} n^{\delta/2}\left[\sum_{\nu=1}^{m} O(n^{-1}) \dfrac{n^2}{\nu^2} \dfrac{\nu^{2p+3}}{n^{2p+4}} + \sum_{\nu=1}^{m} O(n^{-1}) \dfrac{n^2}{\nu^2} \dfrac{\nu^{q+2}}{n^{q+3}}\right] & \text{for } n = 2m \\[2ex] n^{\delta/2}\left[\dfrac{(1-x^2)p_n^2(x)}{p_n'^2(0)} + \sum_{\nu=1}^{m} O(n^{-1}) \dfrac{n^4}{\nu^4} \dfrac{\nu^{2p+3}}{n^{2p+4}} + \sum_{\nu=1}^{m} O(n^{-1}) \dfrac{n^4}{\nu^4} \dfrac{\nu^{q+4}}{n^{q+5}}\right] & \text{for } n = 2m+1 . \end{cases}$$

For $n = 2m + 1$ we obtain by using (6.2)

$$\frac{(1-x^2)p_n^2(x)}{p_n'^2(0)} = \frac{(1-x^2)x^2 P_{m(2x^2-1)}^{2(p,(q+1/2))}}{[P_{m(-1)}^{(p,(q+1/2))}]^2} = \frac{O(n^{-1})}{\binom{m+\frac{q+1}{2}}{m}^2} .$$

From this as well as from (9.8) we see that in the interval $-1 < x \leqq 0$

$$(9.9)\qquad \sum_{\nu=1}^{n} | \rho_{n\nu}(x) | \leqq O(n^{-\delta/2}) .$$

Similarly it follows in the interval $0 \leqq x < 1$ that

$$\sum_{\nu=1}^{n} | \rho_{n\nu}(x) | \leqq O(n^{-\delta/2}) .$$

At $x = \pm 1$, the lemma obviously holds.

10. The proof of the Theorem 1. We now apply the usual argument. We have $S_n(x, f)$ our interpolating polynomial and $\Pi(x)$ an arbitrary polynomial of degree $2n$ at most. Then there holds

$$S_n(x, f) - f(x) = S_n(x, f - \Pi) + (\Pi(x) - f(x)) . \tag{10.1}$$

From (2.1) and (1.11) we get

(10.2)

$$S_n(x, f) - f(x) = \sum_{\nu=0}^{n+1} \{f(x_{n\nu}) - \Pi(x_{n\nu})\} r_{n\nu}(x) + \sum_{\nu=0}^{n} (y^*_{n\nu} - \Pi'(x_{n\nu})\rho_{n\nu}(x) .$$

Now by Weistrass approximation theorem for $-1 \leqq x \leqq 1$

$$\Pi(x) - f(x) = o(1) . \tag{10.3}$$

Now

$$\begin{aligned} &\left| \sum_{\nu=0}^{n+1} \{f(x_{n\nu}) - \Pi(x_{n\nu})\} r_{n\nu}(x) \right| \\ &\leqq \max_{-1 \leqq x \leqq 1} | f(x) - \Pi(x) | \sum_{\nu=0}^{\eta+1} | r_{n\nu}(x) | = o(1) \end{aligned} \tag{10.4}$$

owing to (10.3) and Lemma 8.1

If $M = \max. \Pi'(x)$ then in the interval $-1 \leqq x \leqq 1$

$$\left| \sum_{\nu=1}^{n} (y^*_{n\nu} - \pi'(x_{n\nu}))\rho_{n\nu}(x) \right| \leqq (cn^{\eta} + M) \sum_{\nu=1}^{n} | \rho_{n\nu}(x) | = o(1) \tag{10.5}$$

in consequence of Lemma 9.1 and $| \beta_{n\nu} | \leqq cn^{\eta}$, where $0 \leqq \eta < \dfrac{\delta}{2} < 1$ and $\delta = (2p, q) > 0$.

Thus (10.2), (10.3), (10.4) and (10.5) complete the proof of our Theorem 1.

11. Proof of Theorem 2. The conjugate points belonging to our point-system owing to (4.6), (1.8) and Lemma 7.1 (i) are given by

$$\begin{aligned} X_{n\nu} &= x_{n\nu} + \frac{x_{n\nu}}{2\left\{\dfrac{px^2_{n\nu}}{1 - x^2_{n\nu}} - \dfrac{q}{2}\right\}} \\ &= x_{n\nu}\left[\frac{2p + (1 - 2p - q)(1 - x^2_{n\nu})}{2p - (2p + q)(1 - x^2_{n\nu})}\right] \qquad x_{n\nu} \neq 0 . \end{aligned} \tag{11.1}$$

If however $x_{n\nu} = 0$ i.e., in the case when $n = 2m + 1$ and $\nu = m + 1$, then it follows from (4.6), (1.8) and Lemma 7.1(ii) that

$$X_{2m+1,m+1} = \infty .$$

Now we shall make use of the following statements in the proof of Theorem 2.

Let (α, β) be a fixed part of the interval $[-1, 1]$ but as small as we please. Consider the fundamental point system (4.3) or (4.4). We prove that for any value of n sufficiently large at least one member of the series of triangular matrix of the fundamental point-system lies within the interval (α, β). Let

$$f(x) = \begin{cases} 0 \text{ for } -1 \leqq x < \alpha \\ (x - \alpha)(\beta - x) \text{ for } \alpha \leqq x \leqq \beta \\ 0 \text{ for } \beta < x \leqq 1 . \end{cases}$$

Then $f(x)$ is apparently continuous in the interval $-1 \leqq x \leqq 1$. Let us assume that it is not so then there is a series $n_1 < n_2 < n_3 \cdots < n_i \cdots$ such that no member of the point group belonging to these indices $x_{ni,1}, x_{ni,2}, \cdots, x_{nini}$ $(i = 1, 2, \cdots)$ lie in the interval (α, β). Therefore in the interval $-1 \leqq x \leqq 1$ $\lim_{i\to\infty} S_{ni}(f, x) = 0$ holds. On the other-hand according to Theorem 1 in place of $x = \alpha + \beta/2$

$$\lim_{i\to\infty} S_{ni}(f, x) = f\left(\frac{\alpha + \beta}{2}\right) = \left(\frac{\alpha - \beta}{2}\right)^2 \neq 0$$

contradicts the foregoing inference, i.e., point-system (4.3) or (4.4) lie thickly in the interval $-1 \leqq x \leqq 1$. It can also be proved that the conjugate point-system belonging to (4.3) or (4.4) thickly cover the interval $-1 \leqq x \leqq 1$.

The conjugate points belonging to points $x_{n\nu} \neq 0$ can according to (11.1) be obtained from the function

$$g(x) = x\left[\frac{1 - q - (1 - 2p - q)x^2}{(2p + q)x^2 - q}\right]$$

in the places $x_{n\nu}$. In the interval $-1 \leqq x \leqq 1$, $g'(x) < 0$. Therefore the function $g(x)$ in the interval $(-\sqrt{q/2p + q}, \sqrt{q/2p + q})$ which on account of $0 < p \leqq \frac{1}{2}$ and $0 < q < 1$ forms a part interval of $[-1, 1]$ diminishes continuously, is continuous and its value includes all values from $+\infty$ to $-\infty$. There must also be two points a_1 and b_1 different from each other within the interval $[-\sqrt{q/2p + q}, \sqrt{q/2p + q}]$ so that $g(a_1) = -1$ and $g(b_1) = 1$. Since $g'(x) < 0$ it follows that $-1 \leqq g(x) \leqq 1$ holds in the interval $b_1 \leqq x \leqq a_1$. Let a_2 and b_2 be again two different real values for which $-1 < a_2 < b_2 < 1$ holds.

Then there must obviously lie in the interval (a_1, b_1) two different points a_3 and b_3 such that $g(a_3) = a_2$ and $g(b_3) = b_2$. Since we have already proved that at least one point of each series of the point-system (4.3) or (4.4) must belong to the index n within the interval (a_3, b_3). Therefore it follows that the conjugate points belonging to the fundamental points lying within the interval (α, β) must owing to monotony of $g(x)$ from this index onwards lie within the interval (a_2, b_2), a_2 and b_2 can lie as near to each other as we please. Thus Theorem 2 is proved.

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