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ON THE UNION OF TWO STARSHAPED SETS

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Let S be a compact subset of a topological linear space. We shall say that S has the property φ if there exists a line segment R such that each triple of points x, y and z in S determines at least one point p of R (depending on x, y and z) such that at least two of the segments xp, yp and zp are in S. It is clear that if S is the union of two starshaped sets then S has the property φ, and the problem has been raised by F. A. Valentine [1] as to whether the property φ ensures that S is the union of two starshaped sets. We shall show that this is not so, in general, but we begin by giving a further constraint which ensures the result.

THEOREM. *If a compact set S, of a topological linear space, has the property φ, and, for any point q of S, the set of points of R which can be seen, via S, from q form an interval, then S is the union of two starshaped sets.*

Proof. Consider the collection of sets $\{T_q\}, q \in S$, where T_q denotes the set of points of R which can be seen, via S, from q. If every two intervals of this collection have a nonempty intersection, then it follows from Helly's Theorem that S is starshaped from a point of R. Suppose, therefore, that there exist points q_1, q_2 of S such that $T_{q_1} \cap T_{q_2} = \phi$. We partition the collection $\{T_q\}, q \in S$, into three collections $\{T_q\}_1, \{T_q\}_2, \{T_q\}_{12}$, so that T_q belongs to $\{T_q\}_1$ if T_q meets T_{q_1} but not T_{q_2}, T_q belongs to $\{T_q\}_2$ if T_q meets T_{q_2} but not T_{q_1}, T_q belongs to $\{T_q\}_{12}$ if T_q meets both T_{q_1} and T_{q_2}. If T_q, T_r are two sets of $\{T_q\}_i$ $(i = 1, 2)$ then it follows from φ applied to the points q, r and q_j $(j \neq i)$ that T_q meets T_r. If T_q, T_r are two sets of $\{T_q\}_{12}$, then, since both T_q and T_r span the gap between T_{q_1} and T_{q_2}, it follows that T_q meets T_r. Further, if T_q belongs to $\{T_q\}_{12}$, then it must meet every set of at least one of the collections $\{T_q\}_i$ $(i = 1, 2)$. For, otherwise, there exists sets T_{r_1}, T_{r_2}, belonging to $\{T_q\}_1, \{T_q\}_2$ respectively, which do not meet T_q. However, by property φ applied to r_1, r_2 and q, this implies that T_{r_1} meets T_{r_2} and hence that

$$T_{r_1} \cup T_{r_2}$$

spans the gap between T_{q_1} and T_{q_2}. But this implies that $T_{r_1} \cup T_{r_2}$ meets T_q; contradiction. We now form the collections $\{T_q\}_{12i}$ $(i = 1, 2)$ so that T_q belongs to $\{T_q\}_{12i}$ if either T_q is in the collection $\{T_q\}_i$ or T_q is in $\{T_q\}_{12}$ and meets every member of $\{T_q\}_i$. We note that

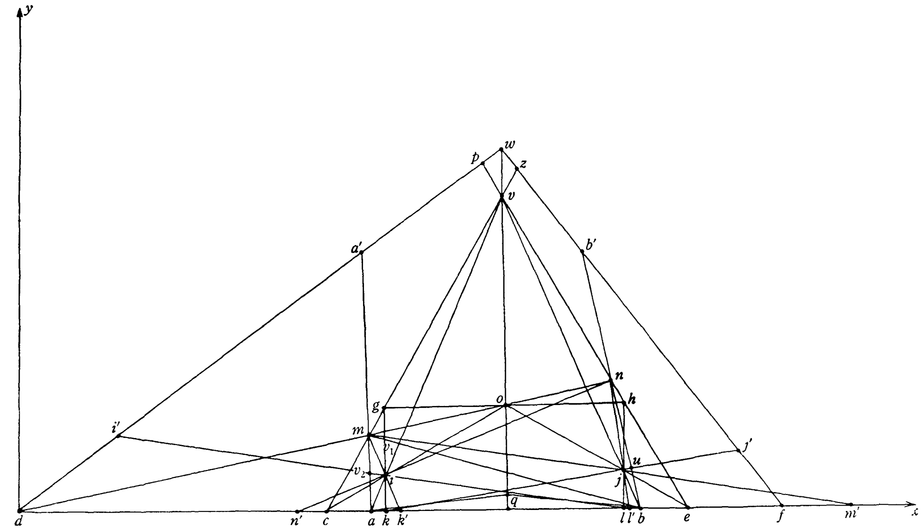

Fig. 1

$$\{T_q\}_{121} \cup \{T_q\}_{122} = \{T_q\},\ q \in S\ ,$$

and combining the results above with Helly's Theorem, we deduce that the intersection U_i of all the members of $\{T_q\}_{12i}$ is a nonempty closed set. Let s_i be a point of U_i and let S_i be the set of points of S which can be seen, via S, from S_i. Then S is the union of S_1 and S_2 which are starshaped from s_1 and s_2 respectively.

COUNTER-EXAMPLE. There exists a plane compact set S which has the property φ but, nevertheless, cannot be expressed as the union of two starshaped sets.

We assume the existence of a rectangular coordinate system and let c, e, v be the vertices of an equilateral triangle, with c, e on the x-axis, e lying to the right of c, and v lying above the x-axis. Let o be the centroid of the triangle cev and let the line through o, which is parallel to the x-axis, meet cv, ev at g, h respectively. Let the vertical line through g meet co at i and ce *at* k. Let the vertical line through h meet eo at j and ce at ρ. Let vi produced meet ck at a and let vj produced meet ρe at b. So far we have defined six distinct points c, a, k, ρ, b, e, in that order, on the x-axis. Let d be a point on the x-axis which lies to the left of c and let the line od produced meet cg at m and hv at n. Suppose the lines mi produced, nj produced, meet the x-axis at points k', ρ', respectively. Let kg meet mb at v_1 and let ρi produced meet am at v_2. As

$$d \rightarrow -\infty\,,\ \rho' \rightarrow \rho,\ k' \rightarrow k,\ m \rightarrow g,\ v_1 \rightarrow g\ .$$

Hence we can suppose that d has been chosen as to ensure that (i)

k' and ρ' are distinct interior points of ab, with k' lying to the left of ρ', and (ii) the quadrilateral mv_1iv_2 is nondegenerate, and i is closer to the x-axis than is m. We choose a point f on the x-axis and to the right of e, and a point w on the line ov produced and strictly above v. Let ev produced meet dw at p and let cv produced meet wf at z. We also choose a point q on vo produced, which lies strictly above the x-axis but which lies below the line segments aj and bi. Now, by (ii), the interior C_1 of the quadrilateral mv_1iv_2 is nonempty, and, if kj produced meets nb at u, the interior C_2 of the triangle jun is nonempty. We define C_3 to be the interior of the triangle aqb together with the open line segment ab. Finally we take S to be $T - C_1 \cup C_2 \cup C_3$, where T denotes the set within and on wdf. Note that by construction every point of S, other than those within $vzwp$, can see, via S, one of a and b, and one of c, d and e. We first show that S has the property φ, with $R \equiv df$.

Suppose that p_1, p_2, p_3 are points of S for which no two can together be seen from any point of df. As any point within $vzwp$ can see each of c, d and e, we deduce, from above, that none of p_1, p_2, p_3 can lie within $vzwp$. But this implies that each of p_1, p_2, p_3 can see one of a and b; contradiction. Therefore, we conclude that such a triple of points cannot be chosen in S and hence that S has the property φ, with $R \equiv df$.

We now show that S is not the union of two starshaped sets. Suppose, therefore, that p_1, p_2 are points of S and that each point of S can be seen from at least one of p_1, p_2. Let am produced meet dw at a' and let bn produced meet wf at b'. If neither of p_1, p_2 lie within $aa'd$, then neither point can see the interior of the segment mv_2. Hence p_1, say, lies within $aa'd$ and, similarly, p_2 lies within $bb'f$. Let iv_2 produced meet dw at i' and let ju produced meet wf at j'. Then p_1 must lie within dav_2i', for, otherwise, the interior of the line segment v_2i cannot be seen from p_1 or p_2. Similarly p_2 lies within $fbuj'$. Let ni produced meet the x-axis at n' and let mj produced meet the x-axis at m'. As p_2 cannot see the interior of the line segment nj, p_1 must lie within ain'. But then p_1 cannot see the interior of the line segment mv_1 and so p_2 must lie within jbm'. We note that $p_1 = a$, $p_2 = b$ is impossible and that i and j are the same distance from the x-axis. It follows that $p_1\, i$ produced, $p_2\, j$ produced meet at an interior point g' of ijv. But as C_1 and C_2 are nonempty open sets, it follows that there is a nonempty quadrilateral Q, which lies within ijv and has g' as its lowest vertex, whose interior cannot be seen from either p_1 or p_2. As Q lies in S, this is a contradiction, and we conclude that S cannot be expressed as the union of two starshaped sets.

Reference

1. F. A. Valentine, *Convex sets*, McGraw-Hill, 1964.

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