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ON w^*-SEQUENTIAL CONVERGENCE AND QUASI-REFLEXIVITY

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This paper characterizes quasi-reflexive Banach spaces in terms of certain properties of the w^*-sequential closure of subspaces. A real Banach space X is quasi-reflexive of order n, where n is a nonnegative integer, if and only if the canonical image $J_X X$ of X has algebraic codimension n in the second dual space X^{**}. The space X will be said to have property P_n if and only if every norm-closed subspace S of X^* has codimension $\leqq n$ in its w^*-sequential closure $K_X(S)$. By use of a theorem of Singer it is proved that X is quasi-reflexive of order $\leqq n$ if and only if every norm-closed separable subspace of X has property P_n. A certain parameter $Q^{(n)}(X)$ is shown to have value 1 if X has property P_n and to be infinite if X does not have P_n. The space X has P_0 if and only if w-sequential convergence and w^*-sequential convergence coincide in X^*. These results generalize a theorem of Fleming, Retherford, and the author.

2. If X is a real Banach space, S a subspace of X^*, and $K_X(S)$ the w^*-sequential closure of S in X^*, then $K_X(S)$ is a Banach space under the norm φ_S defined by

$$\varphi_S(f) = \inf\left\{\sup_{n\in\omega} \|f_n\| : \{f_n\} \subset S, f_n \xrightarrow{w^*} f\right\}$$

for $f \in K_X(S)$ [5]. If $S \subseteqq T \subseteqq K_X(S)$, let

$$C_X(S, T) = \sup\{\varphi_S(f) : f \in T, \|f\| \leqq 1\} .$$

Thus, $K_X(S)$ is norm-closed in $(X^*, \| \ \|)$ if and only if $C_X(S, K_X(S))$ is finite [5]. For each integer $n \geqq 0$ let $\mathscr{T}_n(S)$ be the family of all subspaces T of X^* such that $S \subseteqq T \subseteqq K_X(S)$ and such that $K_X(S)$ is the algebraic direct sum of T and a subspace of dimension $\leqq n$. Let

$$C_X^{(n)}(S) = \inf\{C_X(S, T) : T \in \mathscr{T}_n(S)\} ,$$

and let

$$Q^{(n)}(X) = \sup\{C_X^{(n)}(S) : S \text{ a subspace of } X^*\} .$$

It will be said that X has *property* P_n if and only if $S \in \mathscr{T}_n(S)$ for every norm-closed subspace S of $(X^*, \| \ \|)$.

3. THEOREM 1. *Let X be a real Banach space and n a non-*

negative integer. If X has property P_n, then $Q^{(n)}(X) = 1$. If X does not have property P_n, then $Q^{(n)}(X) = \infty$.

Proof. If X has property P_n and S_1 is a norm-closed subspace of X^*, then $S_1 \in \mathscr{T}_n(S_1)$ and hence $C_X^{(n)}(S_1) = 1$. If S is an arbitrary subspace of X^* and S_1 the norm-closure of S, then $C_X^{(n)}(S) = C_X^{(n)}(S_1)$ and therefore $Q^{(n)}(X) = 1$.

If X does not have property P_n, then X^* has a norm-closed subspace S such that $K_X(S)$ contains an $(n+1)$-dimensional subspace V such that $S \cap V = \{0\}$. Now V has a basis $\{f_1, \cdots, f_{n+1}\}$ of vectors with $\| f_i \| = 1$, and there exist $F_1, \cdots, F_{n+1} \in X^{**}$ such that for each $j \in \{1, \cdots, n+1\}$, $F_j(f) = 0$ for every $f \in S$ and $F_j(f_i) = \delta_{ij}$ for each $i \in \{1, \cdots, n+1\}$ [7, p. 186]. Let $\alpha = \max\{\| F_j \| : 1 \leqq j \leqq n+1\}$. Further, there exist vectors $x_1, \cdots, x_{n+1} \in X$ such that $f_i(x_j) = \delta_{ij}$ for $1 \leqq i, j \leqq n+1$ [7, p. 138].

Since $f_1, \cdots, f_{n+1} \in K_X(S)$, the restrictions of $J_X x_1, \cdots, J_X x_{n+1}$ to S must be linearly independent on S, and hence for each

$$i \in \{1, \cdots, n+1\}$$

there exists $g_i \in S$ such that $g_i(x_j) = \delta_{ij}$ for each j [7, p. 138]. Now for each $i = 1, \cdots, n+1$ there is a sequence $\{p_{ih}\} \subset S$ such that $p_{ih} \xrightarrow[h]{w^*} f_i$. The sequence $\{p_{ih}\}$ may be chosen so that

$$| p_{ih}(x_j) - \delta_{ij} | < \frac{2^{-h}}{(n+1) \| g_j \|}$$

for each j. If we let $f_{ih} = p_{ih} + \sum_{j=1}^{n+1} [\delta_{ij} - p_{ih}(x_j)] g_j$, then $f_{ih}(x_j) = \delta_{ij}$ for all i, h, j, and $\| f_{ih} - p_{ih} \| < 2^{-h}$, so that $f_{ih} \xrightarrow[h]{w^*} f_i$; clearly $\{f_{ih}\} \subset S$.

For each $i \in \{1, \cdots, n+1\}$ and $h \in \omega$, let $g_{ih} = f_{ih} - f_i$. Thus $g_{ih}(x_j) = 0$ and $F_j(g_{ih}) = -\delta_{ij}$ for all i, h, j, and $g_{ih} \xrightarrow[h]{w^*} 0$ for each i. Generalizing a method of Fleming [3], for each positive number N we let R_N be the linear span and S_N the norm-closed linear span of $\{f_{ih} + N g_{ih} : 1 \leqq i \leqq n+1; h \in \omega\}$. Note that for each

$$i \in \{1, \cdots, n+1\}, f_{ih} + N g_{ih} \xrightarrow[h]{w^*} f_i ;$$

thus $V \subseteqq K_X(R_N)$. Now let f be a nonzero element of V and $\{v_m\}$ a sequence in R_N such that $v_m \xrightarrow{w^*} f$. Clearly f has the form

$$f = \sum_{i=1}^{n+1} \alpha_i f_i$$

and each v_m has the form

$$v_m = \sum_{i=1}^{n+1} \sum_{h=1}^{h_{mi}} \alpha_{mih}(f_{ih} + Ng_{ih}) .$$

For every $j \in \{1, \cdots, n+1\}$,

$$\alpha_j = f(x_j) = \lim_m v_m(x_j) = \lim_m \sum_{h=1}^{h_{mj}} \alpha_{mjh} ,$$

and since $F_j(f_{ih} + Ng_{ih}) = -N\delta_{ij}$, it follows that

$$F_j(v_m) = -N \sum_{h=1}^{h_{mj}} \alpha_{mjh} .$$

Thus $\lim_m F_j(v_m)$ exists and is equal to $-N\alpha_j$. Now

$$\| v_m \| \geqq \frac{| F_j(v_m) |}{\| F_j \|} ,$$

and hence $\liminf_m \| v_m \| \geqq N | \alpha_j |/\| F_j \| \geqq N | \alpha_j |/\alpha$. Since j is arbitrary, $\liminf_m \| v_m \| \geqq (N/\alpha) \max | \alpha_j |$. From the definition of φ_{S_N}, it follows that $\varphi_{R_N}(f) = \varphi_{R_N}(f) \geqq N/\alpha \max_j | \alpha_j | \geqq N \| f \|/\alpha(n+1)$. If $T \in \mathscr{T}_n(S_N)$, then T must contain some nonzero $f \in V$ since V is $(n+1)$-dimensional, and hence $C_X(S_N, T) \geqq N/\alpha(n+1)$. Therefore $C_X^{(n)}(S_N) \geqq N/\alpha(n+1)$. Since N is arbitrary and $\alpha(n+1)$ is independent of N, it follows that $Q^{(n)}(X) = +\infty$.

THEOREM 2. *Let X be a real Banach space and n a nonnegative integer. If X is quasi-reflexive of order $\leqq n$, then X has property P_n. If X is separable and has property P_n, then X is quasi-reflexive of order $\leqq n$.*

Proof. If X is quasi-reflexive of order $m \leqq n$ and S is a norm-closed subspace of X^*, then it can be seen from the proofs of Theorems 5 and 6 of [4] that $K_X(S)$ is the direct sum of S with a subspace of X^* of dimension $\leqq m$. Hence $S \in \mathscr{T}_n(S)$, and consequently X has property P_n.

On the other hand, let X be separable and suppose that X has property P_n. Let $F_1, \cdots, F_{n+1}$ be linearly independent elements of X^{**} and $S = \bigcap_{i=1}^{n+1} \{f \in X^* : F_i(f) = 0\}$. Thus S is a norm-closed subspace of X^* of codimension $n+1$, and hence, by property P_n, $K_X(S)$ has codimension m for some $m \in \{1, \cdots, n+1\}$. There exists a subspace U of X^* of codimension 1 such that $K_X(S) \subseteqq U$. Thus $U = S \oplus V$ for some subspace V of X^* of dimension n. Now $U = K_X(U)$. Indeed, if $\{g_i\} \subset U$ and $g_i \xrightarrow{w^*} g$, and if P is the projection of U onto

V along S, then as in the proof of Theorem 5 of [4], P is bounded and $\{g_i\}$ is bounded, so that $\{Pg_i\}$ is bounded and hence has a subsequence $\{Pg_{ij}\}$ which converges inner m to some v in the finite-dimensional subspace V. It follows that $g_{i_j} - Pg_{i_j} \xrightarrow{w^*} g - v \in K_X(S)$ and hence that $g \in K_x(S) + V = U$.

Since $U = K_X(U)$ and X is separable, it follows, by an argument involving the bw^*-topology of X^* [3], that U is w^*-closed. If $n = 0$, let $F = F_1$. If $n > 0$, there exist linearly independent vectors $f_1, \cdots, f_n$ spanning V, and there exist scalars $\alpha_1, \cdots, \alpha_{n+1}$, not all of which are zero, such that $\sum_{i=1}^{n+1} \alpha_i F_i(f_j) = 0$ for $1 \leqq j \leqq n$; indeed, the $(n+1)$ vectors

$$\begin{bmatrix} F_i(f_1) \\ \vdots \\ F_i(f_n) \end{bmatrix} \qquad (i = 1, \cdots, n+1)$$

in n-dimensional Euclidean space must be linearly dependent. Let $F = \sum_{i=1}^{n+1} \alpha_i F_i$. Thus, for $n \geqq 0$, $F \neq 0$ and $U = \{f \in X^* : F(f) = 0\}$. Since U is w^*-closed, F is w^*-continuous on X^* [7, p. 139], and hence $F \in J_X X$. Thus every $(n+1)$-dimensional subspace of X^{**} contains a nonzero element of $J_X X$, which means that X is quasi-reflexive of order $\leqq n$.

REMARK. Theorems 1 and 2 contain a generalization of Fleming's theorem [3] that if X is a separable Banach space, then X is reflexive if and only if $Q(X) = 1$. The following theorem generalizes a theorem of [3] and [4].

THEOREM 3. *A real Banach space X is quasi-reflexive of order $\leqq n$, where $n \geqq 0$, if and only if every norm-closed separable subspace Y of X has the property P_n.*

Proof. If X is quasi-reflexive of order $\leqq n$ and Y is a closed subspace of X, then Y is also quasi-reflexive of order $\leqq n$ [1] and hence Y has property P_n by Theorem 2. Conversely, if every norm-closed separable subspace Y of X has property P_n, then every such Y is quasireflexive of order $\leqq n$ by Theorem 2, and hence X is quasi-reflexive of order $\leqq n$ by a theorem of Singer [6].

REMARK. In Theorem 3 the word "separable" can be deleted. By virtue of Theorem 1, Theorem 3 is also true if "property P_n" is replaced with "property that $Q^{(n)}(Y) = 1$". Since a space X is quasi-reflexive of order n if and only if X is quasi-reflexive of order $\leqq n$ but not of order $\leqq (n-1)$, Theorem 3 can easily be rewarded in such a way as to give a necessary and sufficient condition that X be quasi-reflexive of order exactly n.

4. THEOREM 4. *If X is a real Banach space, then $Q^{(0)}(X) = 1$ if and only if w-sequential convergence and w^*-sequential convergence coincide in X^*.*

Proof. Suppose the two kinds of sequential convergence coincide and S is a subspace of X^*. If $\{f_i\} \subset S$ and $f_i \xrightarrow{w^*} f$, then $f_i \xrightarrow{w} f$ and hence some sequence of averages far out in $\{f_i\}$ converges in norm to f [2, p. 40]; thus $f \in S_1$, the norm-closure of S, and hence $\varphi_S(f) = \|f\|$. Therefore, $C_X^{(0)}(S) = 1$ and $Q^{(0)}(X) = 1$.

Conversely, suppose there are a sequence $\{f_i\}$ in X^* and an $f_0 \in X^*$ such that $f_i \xrightarrow{w^*} f_0$ but $f_i \overset{w}{\not\to} f_0$. Then there exists an $F \in X^{**}$ such that $F(f_i) \not\to F(f_0)$. The sequence $\{F(f_i)\}$ is bounded and hence contains a subsequence $\{F(f_{i_j})\}$ such that the limit $\alpha = \lim_j F(f_{i_j})$ exists, but $\alpha \neq F(f_0)$. Since $F \neq 0$, there exists $g \in X^*$ such that $F(g) \neq 0$. Let $g_j = f_{i_j} - (F(f_{i_j})/F(g))g$ for each $j \in \omega$ and

$$g_0 = f_0 - \frac{\alpha}{F(g)} g .$$

Then $F(g_j) = 0$ for each $j \in \omega$, but $F(g_0) \neq 0$. For every $x \in X$,

$$g_j(x) \to f_0(x) - \frac{\alpha}{F(g)} g(x) = g_0(x) ,$$

so that $g_j \xrightarrow{w^*} g_0$. Let S be the norm-closed subspace of X^* spanned by $\{g_j : j \in \omega\}$. Then $g_0 \in K_X(S)$, but $g_0 \notin S$, since $F(g_0) \neq 0$ whereas $F(f) = 0$ for all $f \in S$. Thus $S \notin \mathscr{T}_0(S)$, and hence X does not have property P_0, so that $Q^{(0)}(X) = \infty$ by Theorem 1.

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