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MEASURES WHOSE RANGE IS A BALL

NEIL W. RICKERT

It has been shown by R. P. Kaufman and the author that if μ is a measure of total variation 1 with values in R^n, then there is a measurable set E with

$$|\mu(E)| \geqq \frac{1}{2\pi^{1/2}} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)}.$$

The main purpose of this paper is to determine for which measures μ there is no set E with

$$|\mu(E)| > \frac{1}{2\pi^{1/2}} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)}.$$

It will be shown that they are the measures which satisfy the following two conditions:

(i) The measure of the whole space is zero.

(ii) The induced probability measure $\alpha \circ f(|\mu|)$ on the projective space P^{n-1} is orthogonally invariant, where $f = d\mu/d|\mu|$ maps the measure space to the sphere S^{n-1} and α is the natural map of S^{n-1} onto P^{n-1}.

A different, more geometric proof of the first inequality above has been given by Schwarz [5].

It is clear that condition (i) is equivalent to the centre of the range of μ being 0. It will be shown that for $n \geqq 2$, condition (ii) is equivalent to the range of μ being a ball of radius

$$\frac{1}{2\pi^{1/2}} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)}.$$

Of course for $n = 1$ condition (ii) is trivially satisfied by every measure with range in R^1 since P^0 consists of one point.

Let X be a space, Σ a σ-field of subsets of X, and μ a measure on Σ with values in R^n. By the range of μ we mean the set $\{\mu(E); E \in \Sigma\}$. For E in Σ, $|\mu|(E)$ will denote the total variation of μ on the set E. Note that $|\mu|$ is a positive measure on $\Sigma \cdot ||\mu||$ will denote $|\mu|(x)$.

If X' is another space, and Σ' a σ-field of subsets of X', and if f is a map of X into X', f will be called measurable if $f^{-1}(E) \in \Sigma$ whenever $E \in \Sigma'$. If f is measurable, and if μ is a measure on Σ, then $f(\mu)$ is that measure on Σ' defined by $f(\mu)(E) = \mu(f^{-1}(E))$. When X' is a topological space (such as a sphere or a projective space) we shall always understand that we are using the σ-field of Borel sets. By abuse of language we shall speak of measures on X, or measures on X' when we should be speaking of measures on Σ on Σ'.

If f denotes the Radon-Nikodym derivative $d\mu/d|\mu|, f$ is a measurable map of X into R^n. By redefining f on a set of $|\mu|$-measure zero if necessary, we may assume that f maps X into the sphere S^{n-1}. Thus $f(\mu)$ and $f(|\mu|)$ are Borel measures on S^{n-1}. Note that $f(\mu)$ and $f(|\mu|)$ are related by $f(\mu)(dx) = xf(|\mu|)(dx)$. It is easily seen from this that $|f(\mu)| = f(|\mu|)$ and $||f(\mu)|| = ||\mu||$.

If $x \in S^{n-1}$ we denote by H_x the hemisphere determined by x. That is $H_x = \{y \in S^{n-1}; \langle y, x\rangle \geqq 0\}$. For μ an R^n valued measure on X we denote by ρ_μ the function defined on S^{n-1} by $\rho_\mu(x) = \langle \mu(f^{-1}(H_x)), x\rangle$ where $f = d\mu/d|\mu|$.

Fix an arbitrary point x_0 in S^{n-1}. We denote by G the group $SO(n)$ and K the subgroup of G consisting of those elements which fix x_0. If we choose an orthonormal basis $e_1, e_2, \cdots, e_n$ for R^n such that $e_1 = x_0$, then G consists of orthogonal metrices of determinant 1, while K consists of matrices

$$\begin{pmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & & \\ 0 & & \\ \vdots & & A \\ 0 & & \end{pmatrix}$$

where A is an $(n-1) \times (n-1)$ orthogonal matrix of determinant 1.

The projective space P^{n-1} is the space obtained from S^{n-1} by identifying antipodal points. We shall denote by α the natural projection of S^{n-1} onto P^{n-1}.

We denote by m the unique probability measure on S^{n-1} which is invariant under orthogonal transformations. Up to a scalar factor m is the usual surface measure on S^{n-1}. Note that $\alpha(m)$ is the invariant probability measure on P^{n-1}.

LEMMA 1. *ρ_μ is a continuous function on S^{n-1} and*

$$\int_{S^{n-1}} \rho_\mu(x) m(dx) = \frac{1}{2\pi^{1/2}} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)}$$

if μ is an valued measure of total variation 1 on X.

Proof. The second assertion was essentially proved in the proof of Theorem 3 of [3]. For the first assertion observe that

$$\begin{aligned} \rho_\mu(x) - \rho_\mu(y) &= \langle x, \mu(f^{-1}(H_x))\rangle - \langle y, \mu(f^{-1}(H_y))\rangle \\ &= \int_X \max(0, \langle x, f(t)\rangle) \,|\mu|\,(dt) - \int_X \max(0, \langle y, f(t)\rangle)\,|\mu|\,(dt) \\ &\leqq \int_X \langle x - y, f(t)\rangle \,|\mu|\,(dt) \leqq |x - y| \,. \end{aligned}$$

The continuity of ρ_μ follows immediately.

LEMMA 2. *Let λ be a probability measure on S^{n-1} and suppose that $\left\langle x, \int_{H_x} y\lambda(dy)\right\rangle$ is independent of x. Suppose furthermore that λ is K-invariant (i.e. $\lambda(kE) = \lambda(E)$ for $k \in K$). Then $\alpha(\lambda)$ is the invariant probability measure on P^{n-1}.*

The proof of Lemma 2 will require some properties of spherical functions, so we postpone the proof until the end of this paper.

LEMMA 3. *Let μ be an R^n valued measure of total variation 1 on Σ, a σ-field of subsets of X. Then a necessary and sufficient condition that ρ_μ be a constant function on S^{n-1} is that $\mu(X) = 0$ and $\alpha(f(|\mu|)$ is the invariant probability measure on P^{n-1}.*

Proof. Suppose first that $\mu(X) = 0$ and $\alpha(f(|\mu|)$ is the invariant probability measure on P^{n-1}. Let x be an element of S^{n-1}. Define the function h on P^{n-1} by $h(\alpha(y)) = \langle x, y\rangle$ if $y \in H_x$. Note that if y and z are in H_x, and if $\alpha(y) = \alpha(z)$ it follows that $\langle x, y\rangle = 0 = \langle x, z\rangle$ whence it follows that h is well defined. Since $\alpha(f(|\mu|)) = \alpha(m)$ it follows that

$$\begin{aligned}
\frac{1}{\pi^{1/2}}\frac{\Gamma(n/2)}{\Gamma((n+1)/2)} &= \int_{P^{n-1}} h(p)\alpha(m)(dp) \\
&= \int_{P^{n-1}} h(p)\alpha(f(|\mu|)(dp) \\
&= \int_{H_x} \langle x, y\rangle f(|\mu|)(dy) - \int_{H_{-x}} \langle x, y\rangle f(|\mu|)(dy) \\
&= \left\langle x, \int_{H_x} yf(|\mu|)(dy)\right\rangle + \left\langle -x, \int_{H_{-x}} yf(|\mu|)(dy)\right\rangle \\
&= \rho_\mu(x) + \rho_\mu(-x)\ .
\end{aligned}$$

Also $\rho_\mu(x) - \rho_\mu(-x) = \langle\mu(X), x\rangle = 0$. Thus

$$\rho_\mu(x) = \frac{1}{2\pi^{1/2}}\frac{\Gamma(n/2)}{\Gamma((n+1)/2)}$$

and ρ_μ is a constant function.

Conversely suppose that ρ_μ is constant. For $x \in S^{n-1}$, since

$$\begin{aligned}
\langle x, \mu(f^{-1}(H_x \cap H_{-x})\rangle &= 0\ , \\
\langle x, \mu(X)\rangle = \langle x, \mu(f^{-1}(H_x))\rangle &+ \langle x, \mu(f^{-1}(H_{-x}))\rangle \\
&= \rho_\mu(x) - \rho_\mu(-x) = 0\ .
\end{aligned}$$

Thus $\mu(X) = 0$.

Now let λ be the probability measure on S^{n-1} defined by

$$\lambda(E) = \int_K f(|\mu|)(kE)dk ,$$

so that if φ is a continuous function on S^{n-1},

$$\int_{S^{n-1}} \varphi(y)\lambda(dy) = \int_K \int_{S^{n-1}} \varphi(k^{-1}y)f(|\mu|)(dy)dk .$$

Note that λ is K-invariant, and that

$$\begin{aligned}\left\langle x, \int_{H_x} y\lambda(dy)\right\rangle &= \int_{H_x} \langle x, y\rangle\lambda(dy) = \int_{S^{n-1}} \max(0, \langle x, y\rangle)\lambda(dy) \\ &= \int_K \int_{S^{n-1}} \max(0, \langle x, k^{-1}y\rangle)f(|\mu|)(dy) \\ &= \int_K \int_{S^{n-1}} \max(0, \langle kx, y\rangle)f(|\mu|)(dy) = \int_K \rho_\mu(kx)dk .\end{aligned}$$

Since ρ_μ is constant it follows that λ satisfies the hypotheses of Lemma 2. We may thus conclude that $\alpha(\lambda)$ is the invariant measure $\alpha(m)$ on P^{n-1}. We then conclude that if h is a continuous function on P^{n-1} whose value at a point y depends only on the distance from y to $\alpha(x_0)$

$$\int h(y)\alpha(f(|\mu|))(dy) = \int h(y)\alpha(\lambda)dy = \int h(y)\alpha(m)dy .$$

By x_0 was an arbitrary point of S^{n-1}, so the same assertion is true as long as the value of h at y depends only on the distance from y to some point x in P^{n-1} (where x depends on h but not y). But linear combinations of such continuous functions are dense in all continuous functions on P^{n-1} (see [4]) so we conclude that $\alpha(f(|\mu|)) = \alpha(m)$, as required.

THEOREM 1. *Let μ be an R^n-valued measure $(n \geqq 2)$ of total variation 1 on Σ, a σ-field of subsets of X. Then the following conditions are equivalent:*

(1) *If $f = d\mu/d|\mu|$ and α is the natural projection of S^{n-1} onto P^{n-1}, then $\mu(X) = 0$ and $\alpha(f(|\mu|))$ is the invariant measure on P^{n-1}.*

(2) *The range of μ is the ball with centre 0 and radius*

$$\frac{1}{2\pi^{1/2}} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)} .$$

(3) *The convex hull of the range of μ is a ball with centre 0.*

(4) *For each $E \in \Sigma$, $|\mu(E)| \leqq (1/2\pi^{1/2})(\Gamma(n/2)/\Gamma((n+1)/2)$.*

Proof. (4) $\Rightarrow$ (1). It follows from (4) that

$$\rho_\mu(x) \leqq \frac{1}{2\pi^{1/2}} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)}$$

for $x \in S^{n-1}$. But ρ_μ is continuous, and

$$\int_{S^{n-1}} \rho_\mu(x) m(dx) = \frac{1}{2\pi^{1/2}} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)}$$

as was shown in Lemma 1. It follows that ρ_μ is constant, and (1) now follows from Lemma 3.

(1) $\Rightarrow$ (4). Assume that (1) is true. Suppose that there is a set $E \in \Sigma$ with $|\mu(E)| > (1/2\pi^{1/2})(\Gamma(n/2)/\Gamma((n+1)/2))$. Then there is an $x \in S^{n-1}$ such that $\langle x, \mu(E)\rangle > (1/2\pi^{1/2})(\Gamma(n/2)/\Gamma((n+1)/2))$. But clearly $\langle x, \mu(f^{-1}(H_x))\rangle \geqq \langle x, \mu(E)\rangle$ so $\rho_\mu(x) > (1/2\pi^{1/2})(\Gamma(n/2)/\Gamma((n+1)/2))$ contradicting what was proved in Lemma 3. It follows that (4) must be true.

(1) $\Rightarrow$ (2). From the implication (1) $\Rightarrow$ (4) we know that the range of μ is contained in the ball with centre 0 and radius

$$\frac{1}{2\pi^{1/2}} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)} .$$

Also from (1) it follows that μ is an atom free measure, since $\alpha(f(|\mu|))$ is atom free. Hence the range of μ is convex (see [1]). It suffices to show therefore that every point on the surface of the ball is in the range of μ. But from (1) and Lemma 3 it follows that for $x \in S^{n-1}$, $\rho_\mu(x) = (1/2\pi^{1/2})(\Gamma(n/2)/\Gamma((n+1)/2))$. Whence we conclude, using (4), that

$$\mu(f^{-1}(H_x)) = \frac{1}{2\pi^{1/2}} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)} x .$$

Thus (2) is true.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). Assume that the convex hull of the range of μ is the ball with centre 0, radius r. Since the range of μ is closed (see [1]) it includes every extreme point rx, for $x \in S^{n-1}$. But if $\mu(E) = rx$,

$$\langle x, \mu(f^{-1}(H_x))\rangle \geqq \langle x, \mu(E)\rangle = r .$$

On the other hand $|\mu(f^{-1}(H_x))| \leqq r$, so it follows that $\rho_\mu(x) = r$ for $x \in S^{n-1}$. Applying Lemma 3, we see that (1) is true.

REMARK. From the implication (3) $\Rightarrow$ (2) of the above theorem, it follows that if the convex hull of the range of a measure is a ball with centre 0, then the measure is nonatomic, and the range is actually convex. In fact this can be seen directly, and does not depend on the centre of the ball being 0. For if there were an atom, the convex hull of the range of μ would have a straight edge—that is there would be a line segment on the boundary of the convex hull of the range. This is not the case when the convex hull of the range is a ball, whence we conclude that μ is atom free, and has a convex range.

We now turn our attention to characterizing measures with range a ball whose centre need not be 0.

LEMMA 4. *If μ is an R^n valued measure on X, then the range of μ contains* 0 *and is symmetric about* $\mu(X)/2$.

Proof. The measure of the empty set is 0, and the map

$$\mu(E) \longrightarrow \mu(X - E)$$

is a symmetry of the range of μ about $\mu(X)/2$.

LEMMA 5. *Assume μ is an R^n valued measure on X, and that F is a measurable set. Define the measure λ on X by*

$$\lambda(E) = \mu(E - F) - \mu(E \cap F) .$$

Set $f = d\mu/d\,|\,\mu\,|$ and $g = d\lambda/d\,|\,\lambda\,|$. Then $\alpha(f(|\,\mu\,|)) = \alpha(g(|\,\lambda\,|))$.

Proof. Clearly $|\,\mu\,| = |\,\lambda\,|$ and $g(t) = f(t)$ or $g(t) = -f(t)$ depending on whether $t \in X - F$ or $t \in F$. Since $\alpha(x) = \alpha(-x)$ it follows that $\alpha(f(t)) = \alpha(g(t))$ for $t \in X$. Since $|\,\mu\,| = |\,\lambda\,|$ the result follows.

THEOREM 2. *Let μ be an R^n valued measure on X, and define $f = d\mu/d\,|\,\mu\,|$. Let α be the natural projection of S^{n-1} onto P^{n-1}. A necessary and sufficient condition that the range of μ be a ball is that the measure $\alpha(f(|\,\mu\,|))$ on P^{n-1} be invariant under orthogonal transformations. In this case the centre of the ball is $\mu(X)/2$ and its radius is $(1/2\pi^{1/2})(\Gamma(n/2)/\Gamma((n+1)/2))\,||\,\mu\,||$.*

Proof. Without loss of generality we may assume that $||\,\mu\,|| = 1$. We may also assume that the range of μ is convex (see the remark following Theorem 1). Thus there is a measurable set F with $\mu(F) = \mu(X)/2$. Define λ by

$$\lambda(E) = \mu(E - F) - \mu(E \cap F) .$$

Notice that if E is any measurable set,

$$\lambda((E - F) \cup (F - E)) = \mu(E) - \mu(X)/2$$

and

$$\lambda(E) = \mu((E - F) \cup (F - E)) - \mu(X)/2 .$$

It follows that the range of λ is just the range of μ translated by $-\mu(X)/2$. Note also that $\lambda(X) = 0$. The result now follows easily from Theorem 1 and Lemmas 4 and 5.

We turn our attention now to the proof of Lemma 2. We shall need to investigate certain properties of spherical functions. A more general discussion of spherical functions on spheres can be found in [4] and [2]. S^{n-1} is a symmetric space (seen [2] for the definition of symmetric spaces) and can be written as G/K where G and K are the groups introduced earlier in the paper. Likewise P^{n-1} is a symmetric space. For technical reasons part of our discussion will apply only to the case $n \geqq 3$, although the arguments could be suitably modified to apply to the case $n = 2$. In any case Lemma 2 was already proved for the case $n = 2$ in [3].

Since the G-invariant differential operators on S^{n-1} are all polynomials in the Laplace-Beltrami operator Δ (see [2] p. 397), a function f on S^{n-1} is a spherical function if and only if

(i) f is K-invariant (that is $f(kx) = f(x)$ for $k \in K$).

(ii) $f(x_0) = 1$.

(iii) f is an eigenfunction of the operator Δ.

To determine the spherical functions we coordinatize S^{n-1} as follows. A point x in S^{n-1} is given coordinates $(r, z_1, z_2, \cdots, z_{n-2})$. Here r is the distance from x_0 to x measured along the surface of S^{n-1} (that is r is the angle between the vectors x_0 and x, so that $\langle x_0, x\rangle = \cos r$). If we project x onto the $n - 1$ dimensional plane orthogonal to the vector x_0, and then produce the corresponding vector until it intersects the sphere in the plane, we obtain a point in S^{n-2}. Then $(z_1, \cdots, z_{n-2})$ are coordinates of this point in some local coordinates for S^{n-2}. In this way we obtain coordinates for S^{n-1} except at $r = 0$ or $r = \pi$.

Assume that on S^{n-2} the Riemannian metric is $\sum a_{ij}dz_i dz_j$. Then clearly the Riemannian metric on S^{n-1} is $dr^2 + (\sin r)^2 \sum a_{ij}dz_i dz_j$. Denoting by (b_{ij}) the $(n - 2) \times (n - 2)$ matrix inverse to (a_{ij}) the Laplace-Beltrami operator is given by

$$\Delta f = \frac{1}{(\sin r)^{n-2}} \frac{\partial}{\partial r}\left((\sin r)^{n-2}\frac{\partial f}{\partial r}\right) + \frac{1}{(\sin r)^2 A} \sum_k \frac{\partial}{\partial z_k} \sum_j b_{jk} A \frac{\partial f}{\partial z_j}$$

where $A^2 = \det(a_{ij})$.

But if f is K-invariant it is a function only of r and we then have

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + (n-2)\frac{\cos r}{\sin r}\frac{\partial f}{\partial r}\ .$$

Now if f is a spherical function it satisfies $\Delta f + \lambda f = 0$ for some complex number λ. Making a change of variables, we set $f(r) = \varphi(\cos r)$ for some function φ. Then if f is a spherical function φ will satisfy

$$(1-t)^2\frac{d^2\varphi}{dt^2} - (n-1)t\frac{d\varphi}{dt} + \lambda\varphi = 0\ .$$

On the other hand suppose that φ is twice continuously differentiable on $[-1, 1]$ and satisfies the above differential equation. The function f defined by $f(x) = \varphi(\cos r) = \varphi(\langle x_0, x\rangle)$ will then be twice continuously differentiable on S^{n-1} and except at x_0 and $-x_0$ it will satisfy

$$\Delta f + \lambda f = 0\ .$$

By continuity this equation will also be satisfied at the exceptional points. From the proof on p. 400 of [2] it can be seen that $f(x_0) \neq 0$ unless f vanishes identically, so after a suitable normalization f is a spherical function. We therefore seek solutions of the equation

$$(1-t^2)\frac{d^2\varphi}{dt^2} - (n-1)t\frac{d\varphi}{dt} + \lambda\varphi = 0\ .$$

If φ is a polynomial of degree k then necessarily $\lambda = k(n+k-2)$ as is readily verified by checking the term of degree k. If we can show that for every nonnegative integer k there is a polynomial φ_k^{n-1} of degree k which satisfies the equation with $\lambda = k(n+k-2)$ and such that $\varphi_k^{n-1}(1) = 1$ (this last condition is equivalent to the corresponding spherical function being 1 at x_0, and can be achieved by a suitable normalization), then these φ_k^{n-1} will give rise to spherical functions on S^{n-1}. Furthermore using essentially the argument on p. 404 of [2] it can be deduced that all spherical functions arise in this way.

Notice that φ_k^1 satisfies $\varphi_k^1(\cos r) = \cos kr$. Thus

$$\varphi_k^1(t) = \sum_{j \leqq k/2} \binom{k}{2j} t^{k-2j}(t^2-1)^j\ .$$

Notice also that φ_k^2 is just the Legendre polynomial

$$\varphi_k^2(t) = (\text{constant})\left(\frac{d}{dt}\right)^k (t^2-1)^k\ .$$

By differentiating the differential equation, observe that

$$\varphi_{k-1}^{n+2}(t) = (\text{constant})\frac{d}{dt}\varphi_k^n(t)\ .$$

Thus we have proved

THEOREM 3. *The spherical functions on* S^{n-1} *are the functions* $f_k^{n-1}(x) = \varphi_k^{n-1}(\langle x_0, x\rangle)$ *where* φ_k^{n-1} *satisfies*

$$(1 - t^2)\frac{d^2\varphi}{dt^2} - (n-1)t\frac{d\varphi}{dt} + k(n+k-2)\varphi = 0 .$$

φ_k^{n-1} *is a polynomial of degree* k *and* $\varphi_k^{n-1}(1) = 1$. *Furthermore* φ_k^{n-1} *is an odd polynomial if* k *is odd, and is an even polynomial if* k *is even. If* k *is even (respectively odd) and if* j *is an even (respectively odd) integer,* $0 \leqq j \leqq k$, *then the coefficient of* t^j *in the polynomial* $\varphi_k^{n-1}(t)$ *does not vanish. In particular* $\varphi_k^{n-1}(0) \neq 0$ *if* k *is even.*

If μ is a K-invariant measure on S^{n-1} we recall that its Fourier-Stieltjes coefficients are defined by $\hat{\mu}(f_k^{n-1}) = \int f_k^{n-1}(x)\mu(dx)$ (see [2]). Likewise the Fourier coefficients of K-invariant functions on S^{n-1} can be defined. We wish to investigate the Fourier coefficients of the function ψ defined by $\psi(x) = \max(0, \langle x_0, x\rangle)$.

LEMMA 6. *For* $n \geqq 3$, $\hat{\psi}(f_k^{n-1}) \neq 0$ *if* k *is even.*

Proof. Writing f for f_k^{n-1} and φ for φ_k^{n-1} we have

$$\begin{aligned}\hat{\psi}(f) &= \int_{S^{n-1}} \psi(x)f(x)m(dx)\\ &= \int_{S^{n-1}} \max(0, \cos r)\varphi(\cos r)m(dx)\\ &= K\int_0^\pi \max(0, \cos r)\varphi(\cos r)(\sin r)^{n-2}dr\end{aligned}$$

where K is chosen so that $K\int_0^\pi (\sin r)^{n-2}dr = 1$. Thus

$$\begin{aligned}\hat{\psi}(f) &= K\int_{-1}^1 \max(0, t)\varphi(t)(1-t^2)^{(n-3)/2}dt\\ &= K\int_0^1 t\varphi(t)(1-t^2)^{(n-3)/2}dt .\end{aligned}$$

Integrating twice by parts yields

$$\begin{aligned}&\int_0^1 t\varphi(t)(1-t^2)^{(n-3)/2}dt\\ &= \frac{1}{n-1}\varphi(0) + \frac{1}{n-1}\int_0^1 (1-t^2)^{(n-1)/2}\varphi'(t)dt\\ &= \frac{1}{n-1}\varphi(0) + \frac{1}{n-1}\int_0^1 t(1-t^2)^{(n-1)/2}\varphi''(t)dt + \int_0^1 t^2(1-t^2)^{(n-3)/2}\varphi'(t)dt .\end{aligned}$$

Substituting from the differential equation

$$(1 - t^2)\varphi''(t) - (n - 1)t\varphi'(t) + k(n + k - 2)\varphi(t) = 0$$

we see that

$$(1 - k(n + k - 2)/(n - 1))\int_0^1 t\varphi(t)(1 - t^2)^{(n-3)/2}dt = \frac{1}{n - 1}\varphi(0) .$$

But when k is even, $\varphi(0) = \varphi_k^{n-1}(0) \neq 0$ so the desired conclusion follows.

We now proceed with the proof of Lemma 2. Thus assume that λ is a K-invariant probability measure on S^{n-1} such that

$$\left\langle x, \int_{H_x} y\lambda(dy)\right\rangle$$

is independent of x. Let K' be the subgroup of G consisting of elements of G which map x_0 into $-x_0$. P^{n-1} is then the symmetric space G/K'. Since λ is K-invariant it is clear that $\alpha(\lambda)$ is K'-invariant. Thus to show that $\alpha(\lambda)$ and $\alpha(m)$ are equal it suffices to show that they have the same Fourier-Stieltjes coefficients. But if h is a spherical function on P^{n-1} it is clear that the map $x \to h(\alpha(x))$ defines a spherical function on S^{n-1}. It follows then that the spherical functions on P^{n-1} are given by $h(\alpha(x)) = f_k^{n-1}(x)$ for k even. To prove that $\alpha(\lambda) = \alpha(m)$ it thus suffices to show that $\hat{\lambda}(f_k^{n-1}) = \hat{m}(f_k^{n-1})$ for k even. But $\hat{m}(f_0^{n-1}) = \hat{\lambda}(f_0^{n-1}) = 1$ since both are probability measures, and $\hat{m}(f_k^{n-1}) = 0$ for k a positive integer. We must therefore show that $\hat{\lambda}(f_k^{n-1}) = 0$ for k an even positive integer. Denote by μ the right K-invariant measure on G which projects to the measure λ on S^{n-1}. (In the notation of [3] $\mu = \tilde{\lambda}$). Then

$$\begin{aligned}
\left\langle x, \int_{H_x} y\lambda(dy)\right\rangle &= \int_{H_x}\langle y, x\rangle\lambda(dy) \\
&= \int_{S^{n-1}} \max(0, \langle y, gx_0\rangle)\lambda(dy) \\
&\quad \text{(where } g \in G \text{ is such that } gx_0 = x) \\
&= \int_{S^{n-1}} \max(0, \langle g^{-1}y, x_0\rangle)\lambda(dy) = \int_{S^{n-1}}\psi(g^{-1}y)\lambda(dy) \\
&= \int_G \psi(g^{-1}g'x_0)\mu(dg') = \int_G \psi((g')^{-1}gx_0)\mu(dg') \\
&= \lambda^*\psi(gx_0) = \lambda^*\psi(x)
\end{aligned}$$

where $\lambda^*\psi$ is the convolution product of λ and ψ on the symmetric space S^{n-1}. Thus the hypotheses of Lemma 2 guarantee that $\lambda^*\psi$ is a constant function on S^{n-1} and so its Fourier coefficients vanish except at the spherical function f_0^{n-1}. Thus $0 = \hat{\psi}(f_k^{n-1})\hat{\lambda}(f_k^{n-1})$ for k

a positive even integer, so on account of Lemma 6, $\hat{\lambda}(f_k^{n-1}) = 0$ for k an even positive integer. This completes the proof of Lemma 2.

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