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ALGEBRAS OF GLOBAL DIMENSION ONE WITH A FINITE IDEAL LATTICE

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Let A denote a finite-dimensional (associative) algebra over an algebraically closed field K. It is well known that A has global dimension zero if and only if A is the direct sum of a finite number of full matrix algebras over K. In this paper a specific representation is given for those algebras A which have global dimension one (or less) and have only a finite number of (two-sided) ideals. It is shown that every such algebra is isomorphic to a (contracted) semigroup algebra $K[S]$ over a subsemigroup S of the semigroup of all $n \times n$ matrix units $\{e_{ij}\} \cup \{0\}$ which (i) contains $e_{11}, \cdots, e_{nn}$ and (ii) contains e_{ij} or e_{ji} whenever there are h and k such that e_{hi}, e_{ik} and e_{hj}, e_{jk} are in S. Conversely, if S satisfies (i) and (ii) then $K[S]$ has global dimension one or less and has a finite ideal lattice.

We use the definitions and notation of Cartan-Eilenberg ([2], VI, 2) and Jans ([11], 4). If A is a finite-dimensional algebra then A is Noetherian and therefore l. gl. dim. $A =$ r. gl. dim. A. In this case one writes gl. dim. A for this number. It is perhaps worthwhile to point out that if A is over an algebraically closed field, then gl. dim. A is precisely dim. A, the so-called Hochschild dimension of A (see [2], p. 176) and [8]). In [10] Hochschild proved that dim. $A \leqq 1$ if and only if A is segregated in every extension, i.e., every exact sequence of (finite-dimensional) algebras $B \rightarrow A \rightarrow 0$ splits. In [12] Jans gives a structure theorem for this class of algebras. By the above comments, for algebraically closed fields Jans' theorem is in fact a structure theorem for algebras of global dimension one or less. Unfortunately, however, we are unable at this time to relate the results of this paper to those of Jans.

Harada [9] has also given a characterization of semiprimary rings of global dimension $\leqq 1$ which is in spirit somewhat related to the methods of this paper. But again we are unable to deduce our results from Harada's.

On the other hand, Barry Mitchell has pointed out to the author that part of the main theorem of this paper is an immediate corollary of his work on the global dimension of abelian categories, see [15], pp. 229 ff. Specifically, one infers immediately from Mitchell's results that if S is a subsemigroup of $\{e_{ij}\} \cup \{0\}$ which contains all e_{ii}, then gl. dim. $K[S] \leqq 1$ if and only if whenever e_{hi}, e_{ik} and e_{hj}, e_{jk} are in S then either e_{ij} or e_{ji} is also in S. In this paper, however, we prefer to retain our original proofs since they require no special knowledge of category theory.

For convenience we define a *semigroup S of matrix units* (*of degree n*) to be a subsemigroup of the semigroup of all $n \times n$ matrix units $\{e_{ij}\} \cup \{0\}$ which contains all e_{ii}. If K is a field, $K[S]$ will denote the algebra of all $n \times n$ matrices over K which is spanned by S. If $n = 1$ and $S = \{e_{11}\}$, then $K[S] \cong K$ is the semigroup algebra of S over K; in all other cases S contains e_{11} and e_{22} and therefore contains $0 = e_{11} e_{22}$. In this case $K[S]$ is the so-called *contracted semigroup algebra of S over K*, *i.e.*, the semigroup algebra of S over K modulo the ideal generated by the zero of S(cf. [6]).

In general $K[S]$ has global dimension greater than one. The smallest $K[S]$ which is not of global dimension $\leqq 1$ is the algebra of all 4×4 matrices

$$\begin{pmatrix} x_{11} & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ x_{31} & 0 & x_{33} & 0 \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}, x_{ij} \text{ in } K .$$

Ignoring the zeros above the main diagonal, we note that the zero in the (3, 2)-position is "surrounded by nonzero positions". To describe this situation more precisely we introduce the *graph*

$$G(S) = \{(i, j): e_{ij} \in S\}$$

of a matrix units semigroup S. Clearly there is a one-one correspondence between transitive, reflexive (directed) graphs on n vertices and matrix units semigroups of degree n. We say that S (or $G(S)$) *surrounds no zeros* if whenever (h, i), (i, k) and (h, j), (j, k) are elements of $G(S)$, then either (i, j) or (j, i) is also in S. This is equivalent to the existence of unique paths of maximal length joining any two vertices. Mitchell ([15], p. 236) calls such a graph a *decision free* graph.

We now state our main result:

THEOREM. *Let A be a finite-dimensional algebra with identity over an algebraically closed field K. Then $A \cong K[S]$ for some semigroup S of matrix units which surrounds no zeros if and only if* gl. dim. $A \leqq 1$ *and A has only a finite number of ideals.*

The remainder of the paper will be devoted to a proof of this theorem.

LEMMA 1. *If $A = K[S]$ where S is a semigroup of matrix units* *which surrounds no zeros, then* gl. dim $A \leqq 1$.

Proof. Let $e_i = e_{ii}$, and let N denote the radical of A. Since $1 = \Sigma e_i$ and the e_i are primitive, each simple A-module has the form Ae_i/Ne_i, [7]. It therefore suffices by ([11], p. 56) to show that Ne_i is projective for each i.

As in the introduction to [4], we may assume that the vertex incidence matrix $C = (c_{ij})$ ($c_{ij} = 0$ if $e_{ij} \notin S$ and $c_{ij} = 1$ if $e_{ij} \notin S$) has the block triangular form

$$(1) \qquad \begin{pmatrix} C_{11} & & & & \\ C_{21} & C_{22} & & & 0 \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & & \cdot \\ C_{m1} & C_{m2} & \cdot & \cdot\ \cdot & C_{mm} \end{pmatrix}$$

where the diagonal blocks are square matrices each entry of which is 1 and each of the blocks below the diagonal is either a zero block or else has all entries equal to 1.

Let J be the set of i such that $e_{ip} \in Ne_p$. Clearly, Ne_p is the vector space direct sum of the Ke_{ip}, $i \in J$. Let $G = G(S)$. Set $n_1 = \min J$ and define

$$J_1 = \{i\colon (i, n_1) \in G\} .$$

Since $n_1 \in J$, $(n_1, p) \in G$; hence for $i \in J_1$ we have $e_{ip} = e_{in_1} e_{n_1 p} \in Ne_p$. Thus $J_1 \subseteqq J$.

Having defined integers $n_1 < n_2 < \cdots < n_k$ in J and subsets $J_1, \cdots, J_k$ of J such that for each $t \leqq k$ $n_t = \min J - (J_1 \cup \cdots \cup J_{t-1})$ and $J_t = \{i\colon (i, n_t) \in G\}$, if $J \neq J_1 \cup \cdots \cup J_k$, let us define

$$n_{k+1} = \min J - (J_1 \cup \cdots \cup J_k)$$

and $J_{k+1} = \{i\colon (i, n_{k+1}) \in G\}$. As in the case above for n_1 we have $J_{k+1} \subseteqq J$. By the way we chose n_t for $t \leqq k$, clearly $n_{k+1} > n_k$. Since J is finite this process must end, and so, there is an integer m such that $J = J_1 \cup \cdots \cup J_m$.

We now show that the J_i are pair-wise disjoint. Assume that for $s > t$, $J_s \cap J_t$ contains a nonzero element i. Then (i, n_s) and $(i, n_t) \in G$. Since $n_s, n_t \in J$ we also have (n_s, p) and (n_t, p) in G. But, $(n_s, n_t) \notin G$ since $s > t$ implies that $n_s \notin J_t$. Now since S surrounds no zeros we must have (n_t, n_s) in G. This implies that the vertex incidence matrix C has a diagonal block (not necessarily one of the C_{ii}) of the form:

$$\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

where the three ones correspond to the edges (n_t, n_t), (n_t, n_s), (n_s, n_s) and the zero represents the fact that $(n_s, n_t) \notin G$. But this contradicts the fact that C has square diagonal blocks with all entries 1. Hence we have $J_s \cap J_t = \varnothing$.

Let now $M_i = \Sigma\{Ke_{jp}: j \in J_i\}$. We claim that M_i is a left ideal of A. To show this it suffices to show that if $e_{kj} \in S$ where $j \in J_i$, then $k \in J_i$: Note that $j \in J_i$ if and only if $(j, n_i) \in G$; hence if $(k, j) \in G$ and $j \in J_i$, by transitivity $(k, n_i) \in G$ and so $k \in J_i$. Since the J_i are pairwise disjoint we have $Ne_p = M_1 \oplus \cdots \oplus M_m$.

Observe next that $Ae_{n_i} = \Sigma\{Ke_{jn_i} : j \in J_i\}$. Now one easily verifies that the mapping $\varphi: M_i \to Ae_{n_i}$ defined by $\varphi(\Sigma\alpha_j e_{jp}) = \Sigma\alpha_j e_{jn_i}$ is an A-isomorphism. It follows that Ne_p is isomorphic to the direct sum of the projective A-modules Ae_{n_i}, and is therefore itself projective. This completes the proof of the lemma.

One easily verifies that every ideal I of a matrix units semigroup algebra $K[S]$ is generated by $I \cap S$ and so $K[S]$ has only a finite number of ideals. This fact with Lemma 1 proves one half of our theorem.

Recall that a ring R is called *hereditary* if every left ideal is projective. It is well known that $l.\,\mathrm{gl.\,dim.}\,R \leqq 1$ if and only if R is hereditary (see [2], p. 112). If $X \subseteqq R$, let $l(X)$ denote the left annihilator of X in R. After Kaplansky [13], we call a ring *Baer* if it has an identity and the left annihilator of every subset is generated by an idempotent.

LEMMA 2. *A hereditary, Artinian ring (with identity) is a Baer ring.*

Proof. If $a \in R$, then, Ra is a left ideal of R and therefore projective. Hence $R \xrightarrow{\varphi} Ra \to 0$ splits where $\varphi(r) = ra$. This says that $\ker \varphi = l(a)$ is a direct summand of ${}_RR$. If ${}_RR = L \oplus l(a)$, $1 = f + e$ where $f \in L$, $e \in L$, $e \in l(a)$, then $l(a) = Re$ and $e^2 = e$.

By an argument due to Maeda [14] (which we include for the convenience of the reader) we can extend this to two elements: Let $a, b \in R$. If $l(a) = Re$ and $l(b) = Rf$ where $e^2 = e, f^2 = f$, then $l(a) = l(1 - e)$ and $l(b) = l(1 - f)$. As shown above there is an idempotent g such that $l(e(1 - f)) = Rg$. It is now straightforward to show that $(ge)^2 = ge$ and that $l(a,b) = l(1 - e, 1 - f) = Rge$.

Now since R is Artinian one show easily that $l(X) = Ae$ for any subset X of R.

DEFINITION. Let S be a semigroup of matrix units. By a *twisted matrix units semigroup algebra of S over a field K* we shall mean an algebra $K_\varphi[S]$ which has a basis $\{a_{ij}: (i, j) \in G(S)\}$ which multiplies as follows: $a_{ij}a_{jk} = \varphi(i, j, k)a_{ik}$ where $\varphi(i, j, k)$ is a *nonzero* element of K; all other products are zero.

In case $\varphi = 1$ (when defined), we clearly have that $K_\varphi[S] \cong K[S]$ where $K[S]$ is the (contracted) semigroup algebra of S over K.

The following lemma follows immediately from results in [5], however, for the sake of completeness we give a proof here.

LEMMA 3. *Let A be a finite dimensional algebra over an algebraically closed field K. If A is Baer and has a finite ideal lattice then A is a twisted matrix units semigroup algebra over K.*

Proof. First we note that if A is Baer and if e is an idempotent in A, then eAe is Baer (see [14]). Hence, if e is primitive then eAe has only one idempotent and therefore the left annihilator of every nonzero subset is zero. This implies that the radical of eAe is zero and that eAe is a divisor ring. Since K is algebraically closed we have then that $eAe = Ke$.

Now let $1 = \Sigma e_i$ where $\{e_i\}$ is a family of pairwise orthogonal idempotents. Let us first show that if $e_ixe_jye_k = 0$, then $e_ixe_j = 0$ or $e_iye_k = 0$: Suppose $e_jye_k \neq 0$ and let $Af, f^2 = f$, be the left annihilator of e_jye_k. Now $fe_jye_k = 0$ and hence $e_jfe_jye_k = 0$; since $e_jfe_j \in Ke_j$ we must have $e_jfe_j = 0$. On the other hand, $e_ixe_j \in Af$ and so $e_ixe_j = e_ixe_jf$. Multiplying on the right by e_j we obtain $e_ixe_j = 0$.

Now it is clear that to complete the proof of this lemma it suffices to show that e_iAe_j has dimension $\leqq 1$ over K for all i, j. Since eAe has finite ideal lattice for all $e^2 = e$, it suffices to assume that $1 = e_1 + e_2$. First suppose that $e_1Ae_2 \neq 0$ and $e_2Ae_1 \neq 0$. Then, as shown above, $e_2Ae_1Ae_2 \neq 0$ and therefore $e_2Ae_1Ae_2 = e_2Ae_2$. Hence, there exist $e_{21} \in e_2Ae_1$ and $e_{12} \in e_1Ae_2$ such that $e_{21}e_{12} = e_2$. Now, if $x_{21} \in e_2Ae_1$ we have $x_{21} = e_2x_{21} = e_{21}e_{12}x_{21} = e_{21}(\alpha e_1) = \alpha e_{21}$. Thus, $[e_2Ae_1: K] = 1$. Similarly, $[e_1Ae_2: K] = 1$. In the remaining case assume that $e_1Ae_2 = 0$. It follows that $A = Ke_1 + Ke_2 + e_2Ae_1$. One easily shows that any K-subspace of e_2Ae_1 is an ideal of A. Since K is infinite and A has only finitely many ideals we can only conclude that $[e_2Ae_1: K] \leqq 1$.

Lemma 3 together with Lemma 2 tells us that a finite-dimensional, hereditary algebra with finite ideal lattice over an algebraically closed field K is isomorphic to a twisted matrix units semigroup algebra $K_\varphi[S]$. Thus, to complete the proof of our main result we need only show that such an S surrounds no zeros and that $K_\varphi[S] \cong K[S]$.

LEMMA 4. *Let S be a semigroup of matrix units such that*

$A = K_\varphi[S]$ is Baer. Then, S surrounds no zeros.

Proof. Let a_{pq}, $(p, q) \in G = G(S)$ be a basis for A satisfying the conditions in the above definition. Clearly we may choose a_{pp} to be idempotent. We write $e_p = a_{pp}$.

Now suppose that S does surround a zero. Then there exist $(h, i), (i, k), (h, j), (j, k) \in G$, with (i, j) and (j, i) not in G. Let $e = e_h + e_i + e_j + e_k$. Since A is Baer, eAe is also (see [13]). But eAe is not Baer. To see this observe that eAe has a basis consisting exactly of the elements of the array:

$$\begin{array}{llll} a_{kk} & & & \\ a_{ik} & a_{ii} & & \\ a_{jk} & & a_{jj} & \\ a_{hk} & a_{hi} & a_{hj} & a_{hh} . \end{array}$$

Note that $(p, q) \in G$ if and only if $e_p A e_q \neq 0$. Now suppose that $e_k A e_i \neq 0$, then a_{ki} is a basis element of A. But then

$$0 \neq a_{jk} a_{ki} \in e_j A e_i .$$

Hence $(j, i) \in G$, contrary to our assumption. Similar arguments show that the basis elements a_{pq} in the above array are indeed the only ones which survive in eAe. It follows easily that eAe is isomorphic to the algebra of all 4×4 matrices

$$x = \begin{pmatrix} x_{11} & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ x_{31} & 0 & x_{33} & 0 \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}$$

where $x_{ij} \in K$. But the left annihilator of the element x where $x_{21} = x_{31} = 1$ and all other entries are zero is not generated by an idempotent. This establishes the lemma.

LEMMA 5. *Let S be a semigroup of matrix units which surrounds no zeros. Then, $K_\varphi[S] \cong K[S]$.*

Proof. Let $A = K_\varphi[S]$. As in the proof of Lemma 1, we assume that the vertex incidence matrix C of the graph G of S has the normalized form (1). If C_{ii} is an $n_i \times n_i$ block, then $A/\mathrm{rad}\, A$ is isomorphic to the direct sum of algebras $K_{\varphi_i}[T_i]$ where T_i is the semigroup of all matrix units of degree n_i. One shows easily that

$$K_{\varphi_i}[T_i] \cong K[T_i]$$

and hence that

$$A/\mathrm{rad}\, A \cong K[S]/\mathrm{rad}\, K[S] .$$

We conclude from this that if the reduced (basic) ring of A is isomorphic to that of $K[S]$, then A is isomorphic to $K[S]$ (see [1]).

Now from the block triangular form of C, it is clear that the reduced ring of A is of the form $K_\varphi(S')$ where S' is a semigroup of matrix units whose associated graph is strictly triangular (not just block triangular). Thus, without loss of generality we may assume that $S = S'$ and, hence that $(i, j) \in G(S)$ implies that $j \leqq i$.

We now show that if $\{a_{ij}: (i, j) \in G = G(S)\}$ is a basis for A (which satisfies the conditions in the above definition) then each a_{ij} may be replaced by a nonzero K-multiple a'_{ij} of a_{ij} so that the basis

$$\{a'_{ij}: (i, j) \in G\}$$

together with zero is a semigroup (necessarily isomorphic to S). First we choose a'_{ii} so that a'_{ii} is idempotent. Clearly this can be done since $a_{ii}^2 = \alpha(i, i, i)\, a_{ii}$ and $\alpha(i, i, i) \neq 0$. Now replace a_{ij} by $a'_{ii}a_{ij}a'_{jj}$, so that without loss of generality we may assume that $a_{ii}^2 = a_{ii}$ and $a_{ii}a_{ij} = a_{ij}a_{jj} = a_{ij}$.

Let now m denote the degree of S and let $n < m$. Assume inductively that we have replaced all a_{ij} for $i < n$ by nonzero K-multiples a'_{ij} so that the set T of all a'_{ij}, $(i, j) \in G$ and $i < n$, together with zero is a semigroup. Let the "n-th row" of S consist of

$$a_{ni_p}, \cdots, a_{ni_1}, a_{nn} .$$

Assume that we have replaced the last $s + 1$ elements of this row by $a'_{ni_s}, \cdots, a'_{ni_1}, a'_{nn} = a_{nn}$, so that $T_s = T \cup \{a'_{ni_s}, \cdots, a'_{ni_1}, a'_{nn}\}$ satisfies the following condition:

(*) *If* $j, k \in \{i_s, \cdots, i_1, n\}$, *then* $a'_{nj}a'_{jk} = a'_{nk}$.

Choose $a'_{ni_{s+1}} = a'_{nt}a'_{ti_{s+1}}$, if there exists $t \in \{i_s, \cdots, i_1\}$ such that $(t, i_{s+1}) \in G$. If there is no such t, let $a'_{ni_{s+1}} = a_{ni_{s+1}}$. We now claim that T_{s+1} satisfies (*) with $s + 1$ replacing s: Since $(j, k) \in G$ implies that $k \leqq j$, it suffices to show that for any j such that $n > j > i_{s+1}$ and $(j, i_{s+1}) \in G$ we have $a'_{nj}a'_{ji_{s+1}} = a'_{nt}a'_{ti_{s+1}}$: (The case where $a'_{ni_{s+1}} = a_{ni_{s+1}}$ is clear). This means that (n, j), (j, i_{s+1}) and (n, t), (t, i_{s+1}) are in G. Since S surrounds no zeros we must therefore have (j, t) or (t, j) in G. Assume that $(t, j) \in G$ then $a'_{ti_{s+1}} = a'_{tj}a'_{ji_{s+1}}$ and hence by our inductive hypotheses $a'_{nt}a'_{ti_{s+1}} = a'_{nt}a'_{tj}a'_{ji_{s+1}} = a'_{nj}a'_{ji_{s+1}}$. A similar argument takes care of the case $(j, t) \in G$. This shows that T_{s+1} satisfies (*). By induction T_p satisfies (*) with $s = p$, i.e., T_p together with zero in a semigroup. Now it is clear that by induction on n, we can choose $\{a'_{ij}\}$ as desired. This completes the proof of the lemma and therefore of the theorem.

REMARKS. Examples show that the situation gets much more complicated if one weakens any of the hypotheses of our main theorem. On the other hand, the only place that the algebraic closure of K is needed is in Lemma 3. That it is essential there is shown by the real algebra of all 2×2 matrices of the form

$$\begin{pmatrix} z_1 & 0 \\ z_2 & t \end{pmatrix}$$

where z_i are complex and t is real.

A partial generalization in one direction may be obtained as follows: Let A be finite-dimensional over an algebraically closed field K. Instead of assuming that A is hereditary assume the weaker condition that gl. dim. $A/N^2 < \infty$. Stephen Chase [3] has shown that this is equivalent to the existence of a complete set of mutually orthogonal idempotents $e_1, \cdots, e_n$ such that $e_iNe_j = 0$ if $i \geqq j$. Now, if we assume further that A has a finite ideal lattice, a slight extension of the argument in the proof of Lemma 3 (above) shows that $[e_iAe_j : K] \leqq 1$ for all i, j. Thus, A has a basis $t_{ij} = e_it_{ij}e_j$ such that

$$t_{ij}t_{jk} = \varphi(i, j, k)t_{ik}$$

where $\varphi(i, j, k)$ is some element (possibly zero) of K. However, even if we assume that $\varphi(i, j, k) \neq 0$ (when defined) it is in general impossible to replace $\varphi(i, j, k)$ by 1 and get an isomorphic algebra (see the example in [5]). It is, of course, quite possible that one might be able to find reasonable necessary and sufficient conditions on the graph $G = \{(i, j) : e_iAe_j \neq 0\}$ and the function φ in order that A have *global dimension* $\leqq n$.

We wish also to point out that Lemma 1 together with Lemma 2 yields a new and somewhat less complicated (modulo basic results on the global dimension of Artinian rings) proof of our theorem in [4] that for a divisor ring K, $K[S]$ is a Baer ring if S is a semigroup of matrix units which surrounds no zeros.

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