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MATRIX SUMMABILITY OVER CERTAIN CLASSES OF SEQUENCES ORDERED WITH RESPECT TO RATE OF CONVERGENCE

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Let C_0 denote the set of all complex null sequences, and let S_0 denote the set of all sequences in C_0 which have at most a finite number of zero terms. If $a = \{a_p\} \in S_0$ and $b = \{b_p\} \in S_0$, we say that a converges faster than $b, a < b$, provided $\lim a_p/b_p = 0$. We say that a and b converge at the same rate, $a \sim b$, provided $0 < \liminf |a_p/b_p|$ and $\limsup |a_p/b_p| < \infty$. If $a \in S_0$, let $[a] = \{x \in S_0 : x \sim a\}$. Let $E_0 = \{[x] : x \in S_0\}$. If $[a], [b] \in E_0$, then we say that $[a]$ is less than $[b]$, $[a] <' [b]$, provided $a < b$. We note that E_0 is partially ordered with respect to $\leqq'$. In this paper we study matrix summability over subsets of S_0 and over elements of E_0. Open intervals in S_0 will be denoted by $(a, b), (a, -)$, and $(-, b)$, where $(a, -) = \{x \in S_0 : a < x\}$ and $(-, b) = \{x \in S_0 : x < b\}$. Some of our results characterize, for matrices, maximal summability intervals in S_0. Such intervals are of the form $(-, b)$, never of the form $(-, b] = \{x \in S_0$: either $x < b$ or $x \sim b\}$.

Notational conveniences used are as follows. If $A = (a_{pq})$ is a matrix and b is a sequence such that for each positive integer p, the series $\sum_{q=1}^{\infty} a_{pq} b_q$ converges, then $A(b)$ will denote the sequence $\{\sum_{q=1}^{\infty} a_{pq} b_q\}_{p=1}^{\infty}$. We will use A_b to denote the matrix $(a_{pq} b_q)$. If each of a and b is a sequence, then ab will be used to denote the sequence $\{a_p b_p\}$.

Playing a basic role throughout the paper are the two classical Silverman-Toeplitz (abbreviated $S - T$) conditions which are necessary and sufficient for a matrix A to be convergence preserving over (abbreviated c.p.o.) C_0. These conditions are

(1) $\{a_{pq}\}_{p=1}^{\infty}$ converges, $q = 1, 2, 3, \cdots$,

and

(2) there exists K such that $\sum_{q=1}^{\infty} |a_{pq}| < K, p = 1, 2, 3, \cdots$.

We note that the $S - T$ conditions are necessary and sufficient for a matrix A to be c.p.o. S_0.

REMARK 1. A matrix sums every sequence in some interval $(-, b)$ if and only if it has convergent columns.

REMARK 2. If the matrix A is c.p.o. $[b]$ and c is a sequence such that $\lim c_p/b_p = 0$, then $A(c)$ is convergent.

REMARK 2′. If A is c.p.o. $[b]$, then A is c.p.o. $(-, b]$.

REMARK 3. If A is c.p.o. $(a, -)$, then A is c.p.o. C_0.

LEMMA. *Suppose K and L are countable subsets of S_0 such that if $x \in K$ and $y \in L$, then $x < y$. Then there exists $z \in S_0$ such that if $x \in K$ and $y \in L$, then $x < z < y$.*

Proof. Our proof will be for the case that both K and L are infinite sets. Let $K = \{a^{(1)}, a^{(2)}, a^{(3)}, \cdots\}$ and $L = \{b^{(1)}, b^{(2)}, b^{(3)}, \cdots\}$. Let $\{n_p\}_{p=1}^{\infty}$ be an increasing sequence of positive integers such that if $i > n_p$, then

$$\left| \frac{b_i^{(j)}}{a_i^{(t)}} \right| > 2^p , \qquad j, t = 1, 2, \cdots, p .$$

Define

$$\begin{aligned} c_i &= b_i^{(1)},\ i = 1, 2, \cdots, n_2 , \\ c_i &= (1/p) \min [|\, b_i^{(1)} \,|, |\, b_i^{(2)} \,|, \cdots, |\, b_i^{(p)} \,|] , \\ &\qquad n_p < i \leqq n_{p+1},\ p = 2, 3, 4, \cdots . \end{aligned}$$

Let r be a positive integer. If $p > r$ and q is a positive integer such that $n_p < q \leqq n_{p+1}$, then we have $|\, b_q^{(r)}/c_q \,| \geqq p$, and, since $c_q = |\, b_q^{(t)} \,|/p$ for some $t \in \{1, 2, \cdots, p\}$, we have $|\, c_q/a_q^{(r)} \,| > 2^p/p$. Thus $a^{(r)} < c < b^{(r)}$. This completes the proof.

THEOREM 1. *If A is* c.p.o. $[b]$, *then there exists $b' \in S_0$ such that $b < b'$ and A is* c.p.o. $[b']$.

Proof. Since A is c.p.o. $[b]$, then by Remarks 1 and 2′, A has convergent columns. Let $a_q = \lim_{p\to\infty} a_{pq}$. By Remark 2, A sums every null sequence x such that $\lim x_p/b_p = 0$. Thus A_b sums every null sequence. Therefore from (2) of the $S - T$ conditions there exists M such that if n is a positive integer, then $\sum_{p=1}^{\infty} |\, a_{np}b_p \,| < M$. Clearly $\sum_{q=1}^{\infty} |\, a_q b_q \,| \leqq M$. Let $C = (c_{pq})$ be the matrix defined by $c_{pq} = a_{pq}b_q - a_q b_q$. Let $D = (d_{pq})$ be the matrix defined by $d_{pq} = a_q b_q$. Then $A_b = C + D$. We wish to show that the sequence

$$(*) \qquad \left\{ \sum_{p=1}^{\infty} |\, a_{np}b_p - a_p b_p \,| \right\}_{n=1}^{\infty}$$

converges to zero. We note that (*) is bounded. Suppose (*) has a subsequence which converges to $\mu > 0$. Note that each column of C converges to zero. Let n_1 be a positive integer such that

$$\left| \sum_{p=1}^{\infty} |\, c_{n_1 p} \,| - \mu \right| < \mu/8 .$$

Let k_1 be a positive integer such that $\sum_{p=1}^{k_1} | c_{n_1 p} | > 7\mu/8$. Let $N_1 > n_1$ be an integer such that if $q > N_1$, then $\sum_{p=1}^{k_1} | c_{qp} | < \mu/8$. Let $n_2 > N_1$ be an integer such that

$$\left| \sum_{p=1}^{\infty} | c_{n_2 p} | - \mu \right| < \mu/8 .$$

Let $k_2 > k_1$ be an integer such that $\sum_{p=1}^{k_2} | c_{n_2 p} | > 7\mu/8$. Let $N_2 > n_2$ be an integer such that if $q > N_2$, then $\sum_{p=1}^{k_2} | c_{qp} | < \mu/8$. Continue the process to obtain increasing sequences $\{n_p\}_{p=1}^{\infty}$ and $\{k_p\}_{p=1}^{\infty}$ of positive integers. Define $t_{pq} = | c_{pq} |/c_{pq}$ if $c_{pq} \neq 0$, $t_{pq} = 1$ if $c_{pq} = 0$. Define

$$\begin{aligned} s_p &= 1, p = 1, 2, \cdots, k_1 , \\ s_p &= (-1)^{q+1} t_{n_q p}, k_{q-1} < p \leqq k_q, q = 2, 3, 4, \cdots . \end{aligned}$$

Suppose q is a positive even integer. Then

$$\begin{aligned} &\left| \sum_{p=1}^{\infty} c_{n_q p} s_p - (-\mu) \right| \\ &\quad = \left| \sum_{p=1}^{k_{q-1}} c_{n_q p} s_p + \sum_{p=k_{q-1}+1}^{k_q} c_{n_q p} s_p + \sum_{p=k_q+1}^{\infty} c_{n_q p} s_p + \mu \right| \\ &\quad \leqq \sum_{p=1}^{k_{q-1}} | c_{n_q p} | + \sum_{p=k_q+1}^{\infty} | c_{n_q p} | + \left| \sum_{p=k_{q-1}+1}^{k_q} c_{n_q p} s_p + \mu \right| \\ &\quad < \mu/8 + \mu/4 + \left| - \sum_{p=k_{q-1}+1}^{k_q} | c_{n_q p} | + \mu \right| \\ &\quad < \mu/8 + \mu/4 + \mu/4 . \end{aligned}$$

Similarly, if q is a positive odd integer, then

$$\left| \sum_{p=1}^{\infty} c_{n_q p} s_p - \mu \right| < 5\mu/8 .$$

Thus $C(s)$ is divergent. But $A_b(s)$ is convergent since $A_b(s) = A(bs)$ and $bs \in [b]$. Clearly $D(s)$ is convergent. Hence $C(s)$ is convergent since $C(s) = A_b(s) - D(s)$. Therefore we have a contradiction. Thus (*) converges to zero since the assumption to the contrary leads to a contradiction.

Let j_1 be a positive integer such that if $q > j_1$, then $\sum_{p=1}^{\infty} | c_{pq} | < 1/4$. Let K be a number such that $\sum_{p=1}^{\infty} | c_{np} | < K, n = 1, 2, 3, \cdots$. Let i_1 be a positive integer such that $\sum_{p=i_1+1}^{\infty} | c_{np} | < 1/4, n = 1, 2, \cdots, j_1$. Let $j_2 > j_1$ be an integer such that if $q > j_2$, then $\sum_{p=1}^{\infty} | c_{qp} | < 1/4^2$. Let $i_2 > i_1$ be an integer such that $\sum_{p=i_2+1}^{\infty} | c_{np} | < 1/4^2, n = 1, 2, \cdots, j_2$. Continue the process to obtain increasing sequences $\{j_p\}_{p=1}^{\infty}$ and $\{i_p\}_{p=1}^{\infty}$ of positive integers. Define

$$e_n = 1,\ n = 1, 2, \cdots, i_1\ ,$$
$$e_n = 2^t,\ i_t < n \leqq i_{t+1},\ t = 1, 2, 3, \cdots\ .$$

Consider the matrix C_e. If q is a positive integer, then

$$\begin{aligned}\sum_{p=1}^{\infty} | c_{qp}e_p | &= \sum_{p=1}^{i_1} | c_{qp}e_p | + \sum_{t=1}^{\infty} \Big(\sum_{p=i_t+1}^{i_{t+1}} | c_{qp}e_p | \Big) \\ &< K + \sum_{t=1}^{\infty} \Big(2^t \cdot \sum_{p=i_t+1}^{i_{t+1}} | c_{qp} | \Big) \\ &\leqq K + \sum_{t=1}^{\infty} 2^t/4^t \\ &= K + 1\ .\end{aligned}$$

Let $\{r_p\}$ be an increasing sequence of positive integers such that

$$\sum_{p=r_n+1}^{\infty} | a_p b_p | < 1/4^n\ .$$

Define

$$f_p = 1,\ p = 1, 2, \cdots, r_1\ ,$$
$$f_p = 2^q,\ r_q < p \leqq r_{q+1},\ q = 1, 2, 3, \cdots\ .$$

Then

$$\begin{aligned}\sum_{p=1}^{\infty} | a_p b_p f_p | &= \sum_{p=1}^{r_1} | a_p b_p f_p | + \sum_{q=1}^{\infty} \Big(\sum_{p=r_q+1}^{r_{q+1}} | a_p b_p f_p | \Big) \\ &\leqq M + \sum_{q=1}^{\infty} \Big(2^q \cdot \sum_{p=r_q+1}^{r_{q+1}} | a_p b_p | \Big) \\ &< M + \sum_{q=1}^{\infty} 2^q/4^q\ .\end{aligned}$$

Let $g_p = \min\,[e_p, f_p],\ p = 1, 2, 3, \cdots$. Then $g_p \to \infty$ as $p \to \infty$. Thus $b < bg$. If n is a positive integer, then

$$\begin{aligned}\sum_{p=1}^{\infty} |a_{np} b_p g_p | &\leqq \sum_{p=1}^{\infty} | c_{np} g_p | + \sum_{p=1}^{\infty} | a_p b_p g_p | \\ &\leqq \sum_{p=1}^{\infty} | c_{np} e_p | + \sum_{p=1}^{\infty} | a_p b_p f_p | \\ &< K + 1 + M + 1\ .\end{aligned}$$

Therefore the matrix A_{bg} sums every null sequence. Thus if $b < b' < bg$, then A is c.p.o. $[b']$. The existence of a sequence b' such that $b < b' < bg$ follows from the lemma. This completes the proof of the theorem.

REMARK 4. We note that the matrix A, defined by $a_{pq} = 1$ if $p \neq q$, $a_{pq} = 2^{p-1}$ if $p = q$, has a maximal interval $(-, b)$ over which it is convergence preserving. For example $b = \{1/2^{p-1}\}$.

On the other hand, the matrix A, defined by $a_{pq} = 0$ if $q > p$, $a_{pq} = 1$ if $p \geqq q$, has no such maximal interval. This is easily shown by supposing that $(-, b)$ is a maximal summability interval for A. Then A_b is c.p.o. C_0 and hence satisfies the $S - T$ conditions. Thus $\sum_{p=1}^{\infty} | b_p |$ converges. It is easy to find $c \in S_0$ such that $b < c$ and $\sum_{p=1}^{\infty} | c_p |$ converges. Thus A_c satisfies the $S - T$ conditions and hence is c.p.o. C_0. Therefore A is c.p.o. $(-, c)$.

It is easy to show that if there exist numbers r and R such that $0 < r < | a_{pq} | < R$, $p, q = 1, 2, 3, \cdots$, then $A = (a_{pq})$ has no maximal summability interval. The proof will be omitted.

REMARK 5. Let Δ be a chain in S_0 unbounded above. If $a \in \Delta$, let $a' = \{a_1, 1/2, a_2, 1/4, a_3, 1/8, \cdots\}$. Let $\Delta' = \{a' : a \in \Delta\}$. Then Δ' is a chain in S_0 which is unbounded above. Let $A = (a_{pq})$ be defined by $a_{pq} = 1/2^n$ if $q = 2n - 1$, $a_{pq} = 1$ if q is an even integer. Clearly if $a' \in \Delta'$, then A is c.p.o. $[a']$. But A is not c.p.o. C_0.

THEOREM 2. *If A is* c.p.o. *each of the sets* $[b^{(1)}], [b^{(2)}], [b^{(3)}], \cdots$, *then there exists $d \in S_0$ such that $b^{(p)} < d$, $p = 1, 2, 3, \cdots$, and A is* c.p.o. $[d]$.

Proof. By Theorem 1 we can find $t^{(n)}$ in S_0 such that $t^{(n)} > b^{(n)}$ and A is c.p.o. $[t^{(n)}]$, $n = 1, 2, 3, \cdots$. If n is a positive integer, let $\alpha^{(n)} \in [t^{(n)}]$ such that $0 < \alpha_p^{(n)} < 1$, $p = 1, 2, 3, \cdots$. If n is a positive integer, let M_n be a number which exceeds $\sum_{q=1}^{\infty} | a_{pq}\alpha_q^{(n)} |$, $p = 1, 2, 3, \cdots$. If n is a positive integer, let

$$\beta_p^{(n)} = \frac{\alpha_p^{(n)}}{2^n[M_n + 1]}, \qquad p = 1, 2, 3, \cdots .$$

If p is a positive integer, let $c_p = \sum_{n=1}^{\infty} \beta_p^{(n)}$. We wish to show that $c \in S_0$. Let $\mu > 0$, and let k be a positive integer such that $2^{-k} < \mu/2$. Let R be a positive integer such that if $q > R$, then $\beta_q^{(p)} < \mu/2^{k+1}$, $p = 1, 2, \cdots, k$. Then if $n > R$, we have

$$c_n = \sum_{p=1}^{\infty} \beta_n^{(p)} = \sum_{p=1}^{k} \beta_n^{(p)} + \sum_{p=k+1}^{\infty} \beta_n^{(p)} < \mu/2 + 2^{-k} < \mu .$$

Thus $c \in S_0$. If q is a positive integer, then, using the double sum theorem, we have

$$\begin{aligned}
\sum_{p=1}^{\infty} | a_{qp} c_p | &= \sum_{p=1}^{\infty} | a_{qp} | \left(\sum_{n=1}^{\infty} \beta_p^{(n)} \right) \\
&= \sum_{p=1}^{\infty} \sum_{n=1}^{\infty} | a_{qp} | \beta_p^{(n)} \\
&= \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} | a_{qp} | \beta_p^{(n)} \\
&= \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} | a_{qp} | \cdot \frac{\alpha_p^{(n)}}{2^n[M_n + 1]} \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n[M_n + 1]} \sum_{p=1}^{\infty} | a_{qp} | \alpha_p^{(n)} \\
&< \sum_{n=1}^{\infty} 2^{-n} .
\end{aligned}$$

Thus A_c sums every null sequence. Therefore A sums every sequence $x \in S_0$ such that $x < c$. We note that if n is a positive integer, then $c_p/\beta_p^{(n)} > 1$, $p = 1, 2, 3, \cdots$. Thus if n is a positive integer, then

$$\lim_{p \to \infty} \left| \frac{c_p}{b_p^{(n)}} \right| = \lim_{p \to \infty} \frac{c_p}{\beta_p^{(n)}} \cdot \frac{\beta_p^{(n)}}{t_p^{(n)}} \cdot \frac{t_p^{(n)}}{| b_p^{(n)} |} = \infty .$$

Hence $b^{(n)} < c$, $n = 1, 2, 3, \cdots$. By the lemma, there exists $d \in S_0$ such that $b^{(n)} < d < c$, $n = 1, 2, 3, \cdots$. A is c.p.o. $[d]$ since $d < c$ and A sums every sequence $x \in S_0$ such that $x < c$. This completes the proof of the theorem.

COROLLARY. *Suppose M is a countable set of matrices and L is a countable subset of E_0 such that if $A \in M$ and $[b] \in L$, then A is* c.p.o. $[b]$. *Then there exists $[c] \in E_0$ such that if $A \in M$ and $[b] \in L$, then $[b] <' [c]$ and A is* c.p.o. $[c]$.

Proof. The proof will be for the case that both M and L are infinite sets. Let $M = \{A^{(1)}, A^{(2)}, A^{(3)}, \cdots\}$ and $L = \{[b^{(1)}], [b^{(2)}], [b^{(3)}], \cdots\}$. By Theorem 2, if p is a positive integer, there exists $c^{(p)} \in S_0$ such that $b^{(n)} < c^{(p)}$, $n = 1, 2, 3, \cdots$, and $A^{(p)}$ is c.p.o. $[c^{(p)}]$. Let $L' = \{[c^{(1)}], [c^{(2)}], [c^{(3)}], \cdots\}$. By the lemma, there exists $c \in S_0$ such that if $b^{(s)} \in L$ and $c^{(t)} \in L'$, then $b^{(s)} < c < c^{(t)}$. If j is a positive integer, then by Remark 2′, $A^{(j)}$ is c.p.o. $[c]$ since $A^{(j)}$ is c.p.o. $[c^{(j)}]$ and $[c] \subset (-, c^{(j)}]$. This completes the proof.

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