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A STABILITY THEOREM FOR A THIRD ORDER NONLINEAR DIFFERENTIAL EQUATION

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A stability theorem and a corollary are proved for a nonlinear nonautonomous third order differential equation. A remark shows that the results do not hold for the linear case.

THEOREM. **Let** $p'(t)$ **and** $q(t)$ **be continuous and** $q(t) \geqq 0$, $p(t) < 0$ **with** $p'(t) \geqq 0$. **For any A and B suppose**

$$A + Bt - \int_{t_1}^{t} q(s)ds < 0$$

for large t **where** $Q(t) = \int_{t_0}^{t} q(s)ds$, **then any nonoscillatory solution** $x(t)$ **of the equation**

$$\dddot{x} = p(t)\dot{x} + q(t)x^{2n+1} = 0, n = 1, 2, 3, \cdots ,$$

has the following properties;

$$\operatorname{sgn} x = \operatorname{sgn} \ddot{x}, \neq \operatorname{sgn} \dot{x}, \lim_{t\to\infty} \ddot{x}(t) = \lim_{t\to\infty} \dot{x}(t) = 0, \lim_{t\to\infty} |x(t)| = L \geqq 0 ,$$

and $x(t)\dot{x}(t), \ddot{x}(t)$ **are monotone functions.**

COROLLARY. **If** $q(t) > \in > 0$ **for large** t, **then** $\lim_{t\to\infty} x(t) = 0$.

In this paper, a nonoscillatory solution $x(t)$ of a differential equation is one that is continuable for large t and for which there exists a t_0 such that if $t > t_0$ then $x(t) \neq 0$. Under above conditions on $p(t)$ and $q(t)$ there always exist continuable nonoscillatory solutions of the equation

$$\dddot{x} + p(t)\dot{x} + q(t)x^{2n+1} = 0 . \tag{1}$$

This follows from an exercise in [1] by letting

$$x(t) = y_1(t), \dot{x}(t) = -y_2(t), \ddot{x}(t) = y_3(t) ,$$

so that

$$\begin{aligned} \dot{y}_1 &= -y_2 \\ \dot{y}_2 &= -y_3 \\ \dot{y}_3 &= -[q(t)y_1^{2n+1} - p(t)y_2] . \end{aligned}$$

Equation (1) can then be written as the system $\bar{y}' = -f(t, \bar{y})$ where $f(t, \bar{o}) = \bar{o}$, $f(t, \bar{y})$ continuous for $t \geqq 0, y_1, y_2, y_3, \geqq 0$ and $f_k(t, \bar{y}) \geqq 0$, $k = 1, 2, 3,$ for $y_k > 0$. In fact $\| \bar{y}(0) \|$ may be prescribed.

THEOREM 1.[1] *If p and q satisfy the following conditions for large t,*

(i) $q(t) \geqq$ *and q continuous,*

(ii) $p(t) < 0$ *with* $p'(t) \geqq 0$ *and continuous,*

(iii) *for any A and B,* $A + Bt - \int_{t_0}^{t} Q(s)ds < 0$ *for large t where* $Q(t) = \int_{t_0}^{t} q(s)ds$,

then for any nonoscillatory solution $x(t)$ of (1) *the following properties hold for large t:*

(a) $\operatorname{sgn} x = \operatorname{sgn} \ddot{x} \neq \operatorname{sgn} \dot{x}$, *where* $\operatorname{sgn} x = \begin{Bmatrix} 1 \text{ if } x \ 0 \\ -1 \text{ if } x \ 0 \end{Bmatrix}$.

(b) $\lim_{t\to\infty} \ddot{x}(t) = \lim_{t\to\infty} \dot{x}(t) = 0, \lim_{t\to\infty} /x(t)/ = L \geqq 0$.

(c) $x(t), \dot{x}(t), \ddot{x}(t)$ *are monotone functions.*

Proof. Suppose $x(t)$ is a solution that does not oscillate. Let a be a large positive number such that $x(t) \neq 0$ for $t \geqq a$.

Since $-x(t)$ is also a solution of (1), without loss of generality, assume that $x(t) > 0$ for $t \geqq a$. (1) may be written in the form

$$\frac{\dddot{x}(t)}{x^{2n+1}(t)} + \frac{p(t)\dot{x}(t)}{x^{2n+1}(t)} = -q(t) \text{ for } t \leqq a \,. \tag{2}$$

An integration from a to t, an integration by parts, and another integration from a to t yield

$$\begin{aligned} &\frac{\dot{x}(t)}{x^{2n+1}(t)} + \frac{2n+1}{2}\int_a^t \frac{(\dot{x}(s))^2}{x^{2n+2}(s)}ds \\ &\quad + (2n+1)(n+1)\int_a^t \frac{(t-s)(\dot{x}(s))^3}{x^{2n+3}(s)}ds \\ &\quad - \frac{1}{2n}\int_a^t \frac{p(s)}{x^{2n}(s)}ds + \frac{1}{2n}\int_a^t \frac{(t-s)p'(s)}{x^{2n}(s)}ds \\ &= M + Kt - \int_a^t Q(s)ds \,. \end{aligned} \tag{3}$$

Assertion 1. For any $t_a > a$, $\dot{x}(t)$ cannot be nonnegative for all $t > t_a$. Suppose that $\dot{x}(t) \geqq 0$ for all $t > t_a$. Let t_p be so large that the conditions of the theorem hold for all $t \geqq t_p$ and $t_p \geqq t_a$. For $t \geqq t_p$ the following holds

$$\begin{aligned} &\frac{\dot{x}(t)}{x^{2n+1}(t)} + (2n+1)(n+1)\int_{a_p}^t \frac{(t-s)(\dot{x}(s))^3}{x^{2n+3}(s)}ds - \frac{1}{2n}\int_{t_p}^t \frac{p(s)}{x^{2n(s)}}ds \\ &\quad + \frac{1}{2n}\int_{t_p}^t \frac{(t-s)p'(s)}{x^{2n}(s)}ds \leqq \bar{M} + Kt - \int_a^t Q(s)ds \,, \end{aligned} \tag{4}$$

[1] This theorem appears in the author's Ph. D. dissertation written at the University of Missouri under the direction of W. R. Utz.

where all constants are combined and named $\bar{M}$. For sufficiently large t the right side, $\bar{M} + \bar{K}t - \int_0^t Q(s)ds$, is negative and the left side positive, this is clearly impossible.

There are two possibilities for $\dot{x}(t)$.

Case 1. $\dot{x}(t) < 0$ for $t > \bar{t}$, for some $\bar{t}$.

Case 2. For each $t \in (a, \infty)$ there is a $\bar{t} > t$ such that $\dot{x}(\bar{t}) \geqq 0$.

Assertion 2. Case 2 is impossible.

Let t_1 be a large t such that $\dot{x}(t_1) \geqq 0$. There exists a number $t_2 > t_1$ such that $\dot{x}(t_2) < 0$. Let r be the greatest zero of $\dot{x}(t)$ less than t_2. There exists a number $t_3 > t_2$ such that $\dot{x}(t_3) \geqq 0$. Let s be the smallest zero of $\dot{x}(t)$ greater than t_2. Multiply the original differential Equation (1) by $\dot{x}(t)$ to obtain

$$\dddot{x}(t)\dot{x}(t) + p(t)[\dot{x}(t)]^2 + q(t)x^{2n+1}(t) = 0 ,$$

integrating from r to s and using integration by parts on the first integral gives

$$-\int_r^s [\ddot{x}(t)]^2 dt + \int_r^s p(t)[\dot{x}(t)]^2 dt + \int_r^s q(t)x^{2n+1}(t)\dot{x}(t)dt = 0 .$$

The left side is negative, this is clearly impossible and Assertion 2 is proved. Therefore, there exists a $\bar{t}$ such that $\dot{x}(t) < 0$ for $t > \bar{t}$.

Consider Equation (1) written in the form

$$\dddot{x}(t) = -p(t)\dot{x}(t) - q(t)x^{2n+1}(t) ,$$

the right side is negative for large t. Therefore, $\dddot{x}(t) < 0$ for $t > \bar{t}$. This implies that $\ddot{x}(t)$ is a decreasing function and $\dot{x}(t)$ is concave downward for $t > \bar{t}$. Since $\ddot{x}(t)$ is eventually of one sign, there are three possibilities for $\dot{x}(t)$.

Case 1. $\lim_{t\to\infty} \dot{x}(t) = -\infty$

Case 2. $\lim_{t\to\infty} \dot{x}(t) = c < 0$

Case 3. $\lim_{t\to\infty} \dot{x}(t) = 0$.

Case 1 is impossible since it implies that $x(t)$ is negative for large t. Case 2 also implies that $x(t)$ is negative for large t. Therefore, the only remaining possibility is

$$\lim_{t\to\infty} \dot{x}(t) = 0 .$$

Since $\ddot{x}(t)$ is decreasing and must remain positive for large t, $\dot{x}(t)$ is eventually monotone increasing. Since $\ddot{x}(t)$ is monotone decreasing and positive, $\lim_{t\to\infty} (\ddot{x})t$ exists. Suppose that $\lim_{t\to\infty}\ddot{x}(t) = c > 0$. Then $x(t)$ eventually has slope larger than $c/2$, this is impossible since $\dot{x}(t) < 0$ for large t. Therefore, $\lim_{t\to\infty} \ddot{x}(t) = 0$. Thus $x(t)$ is positive, decreasing and concave upward for large t.

COROLLARY. *If* $q(t) > \varepsilon > 0$ *for large* t, *then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose $\lim_{t\to\infty} x(t) = L, L \neq 0$. Since $-x(t)$ is a solution whenever $x(t)$ is a solution, it can be assumed without loss of generality that $L > 0$. Consider Equation (1) in the form

$$\dddot{x}(t) = -\, p(t)\dot{x}(t) - q(t)x^{2n1}(t) \;.$$

Since $\lim_{t\to\infty} \dot{x}(t) = 0$ and $\lim_{t\to\infty} p(t) = p$, where $p \leqq 0$, given any α such that

$$0 < \frac{\alpha}{2} < L^{2n+1} \;, \quad \text{for large } t$$

$L^{2n+1} - \alpha/2 < x^{2n+1}(t) < L^{2n+1} + \alpha/2$ and $p(t)\dot{x}(t) > 0$. Therefore, $\dddot{x}(t) = -\, p(t)\dot{x}(t) - q(t)x^{2n+1}(t) < -\, \varepsilon(L^{2n+1} - \alpha/2) < 0$ and $\ddot{x}(t)$ must then tend to $-\infty$ as t tends to $+\infty$, this is impossible. This $L = 0$.

REMARK. The following example illustrates the theorem.

$$\dddot{x} - \frac{1}{2}\dot{x} + \frac{e^{2t}}{2}x^3 = 0 \;.$$

$x = e^{-t}$ is a solution with the required properties .

REMARK. The theorem does not hold for $n = 0$, i. e., in the linear case.

Proof. Consider $\dddot{x} - 2\dot{x} + x = 0$, $x = e^t$ is a solution.

REFERENCE

1. Hartman, *Ordinary Differential Equations*, John Wiley, 1964.

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