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ON INEQUALITIES GENERALIZING THE PYTHAGOREAN FUNCTIONAL EQUATION AND JENSEN'S FUNCTIONAL EQUATION

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Several authors have solved the Pythagorean functional equation

$$(1) \qquad |f(x+iy)|^2 = |f(x)|^2 + |f(iy)|^2 ,$$

where f is an entire function and x and y are real variables.

A simple computation shows that, if f is a solution of (1), then f is also a solution of

$$(2) \quad |f(z_1+z_2)|^2 + |f(z_1-z_2)|^2 = |f(z_1+\bar{z}_2)|^2 + |f(z_1-\bar{z}_2)|^2 ,$$

where z_1 and z_2 are complex variables. (If an entire function vanishes at the origin and is a solution of (2), then it is a solution of (1), and conversely.) If an entire function f is a solution of Jensen's functional equation

$$(3) \qquad f(z_1+z_2) + f(z_1-z_2) = 2f(z_1) ,$$

where z_1 and z_2 are complex variables, then it is also a solution of

$$(4) \quad |f(z_1+z_2) + f(z_1-z_2)| = |f(z_1+\bar{z}_2) + f(z_1-\bar{z}_2)| .$$

In this paper we shall prove that a solution of (4) is always a solution of (2). Then we shall solve certain functional inequalities derived from (2) and use the solutions to solve (1), (2), (3), and (4).

See [2], [3], [1] concerning (1). We shall use the following two lemmas to prove that (4) implies (2).

LEMMA 1. *If f is an entire function of z, then $\Delta |f(z)|^2 = 4|f'(z)|^2$ where Δ stands for the Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$ ($z = x+iy$, $i=\sqrt{-1}$, x, y real). (See [4].)*

Proof. Since this lemma is familiar, we omit the proof.

LEMMA 2. *Suppose that f, g are entire functions of z. If $|f'(z)| = |g'(z)|$ holds in $|z| < +\infty$ and $f(0) = g(0) = 0$ holds, then $|f(z)| = |g(z)|$ holds in $|z| < +\infty$.*

Proof. Since f, g are entire functions of z and $|f'(z)| = |g'(z)|$ holds in $|z| < +\infty$, we have $f'(z) = e^{i\theta}g'(z)$ in $|z| < +\infty$ where θ is a real constant. So, by the assumption $f(0) = g(0) = 0$ we have $f(z) =$

$e^{i\theta}g(z)$ in $|z| < +\infty$. So we have $|f(z)| = |g(z)|$ in $|z| < +\infty$.

We shall prove that (4) implies (2) if f is an entire function of z. By (4) we have

$$(5) \qquad |f(z_1 + z_2) + f(z_1 - z_2)|^2 = |f(z_1 + \bar{z}_2) + f(z_1 - \bar{z}_2)|^2 .$$

Taking the Laplacian $\partial^2/\partial s^2 + \partial^2/\partial t^2$ of both sides of (5) with respect to $z_1(z_1 = s + it, i = \sqrt{-1}, s, t$ real), by Lemma 1 we have

$$4|f'(z_1 + z_2) + f'(z_1 - z_2)|^2 = 4|f'(z_1 + \bar{z}_2) + f'(z_1 - \bar{z}_2)|^2 .$$

Hence

$$(6) \qquad |f'(z_1 + z_2) + f'(z_1 - z_2)| = |\overline{f'(z_1 + \bar{z}_2)} + \overline{f'(z_1 - \bar{z}_2)}| .$$

When z_1 is arbitrarily fixed, $f(z_1 + z_2) - f(z_1 - z_2)$ and $\overline{f(z_1 + \bar{z}_2)} - \overline{f(z_1 - \bar{z}_2)}$ are entire functions of z_2 with $(f(z_1 + z_2) - f(z_1 - z_2))_{z_2=0} = (\overline{f(z_1 + \bar{z}_2)} - \overline{f(z_1 - \bar{z}_2)})_{z_2=0} = 0$ and by (6)

$$\left|\frac{\partial}{\partial z_2}(f(z_1 + z_2) - f(z_1 - z_2))\right| = \left|\frac{\partial}{\partial z_2}(\overline{f(z_1 + \bar{z}_2)} - \overline{f(z_1 - \bar{z}_2)})\right| .$$

Hence, by Lemma 2

$$|f(z_1 + z_2) - f(z_1 - z_2)| = |\overline{f(z_1 + \bar{z}_2)} - \overline{f(z_1 - \bar{z}_2)}| .$$

So, we have

$$(7) \qquad |f(z_1 + z_2) - f(z_1 - z_2)|^2 = |f(z_1 + \bar{z}_2) - f(z_1 - \bar{z}_2)|^2 .$$

Adding (5), (7) and using the parallelogram identity $|a + b|^2 + |a - b|^2 = 2|a|^2 + 2|b|^2$ (a, b complex), we have

$$2|f(z_1 + z_2)|^2 + 2|f(z_1 - z_2)|^2 = 2|f(z_1 + \bar{z}_2)|^2 + 2|f(z_1 - \bar{z}_2)|^2 .$$

Hence (4) implies (2) if f is an entire function.

So, if (1) or (3) holds, then (2) holds. But the converse is not true as the example $f(z) = \cos z$ shows.

Now, we consider the following two functional inequalities where z_1, z_2 are complex variables:

$$(8) \qquad |f(z_1 + z_2)|^2 + |f(z_1 - z_2)|^2 \leqq |f(z_1 + \bar{z}_2)|^2 + |f(z_1 - \bar{z}_2)|^2$$

with $|z_1| < +\infty$ and $\mathrm{Im}\,(z_2^2) \geqq 0$,

$$(9) \qquad |f(z_1 + z_2)|^2 + |f(z_1 - z_2)|^2 \geqq |f(z_1 + \bar{z}_2)|^2 + |f(z_1 - \bar{z}_2)|^2$$

with $|z_1| < +\infty$ and $\mathrm{Im}\,(z_2^2) \geqq 0$.

In this paper we shall solve the two functional inequalities (8), (9) which are extensions of (2), and then by the results obtained we shall solve the functional equations (1), (2), (3), (4). Our results are extensions of the result of E. Hille (see [2]).

2. Solutions of (8) and (9).

We shall use the following lemma:

LEMMA 3. *Suppose that H is an entire function. If $A(t) = |H(te^{i\varphi})|^2$ where t, φ are real and φ is fixed, then we have*

$$A''(0) = 2\operatorname{Re}(e^{2i\varphi}H''(0)\overline{H(0)}) + 2|H'(0)|^2 .$$

Proof. Since it is easy, we omit it.

THEOREM 1. *An entire function f is a solution of the functional inequality* (8) *if and only if*

$$f(z) = a \sin h((\beta + i\gamma)z) + b \cos h((\beta + i\gamma)z)$$

or $f(z) = az + b$ where a, b are arbitrary complex constants and β, γ are nonnegative real constants.

Proof. Putting $z_2 = te^{i\varphi}$ in (8) where t, φ are real and φ is fixed with $0 < \varphi < \pi/2$, we have

$$|f(z_1 + te^{i\varphi})|^2 + |f(z_1 - te^{i\varphi})|^2 \leqq |f(z_1 + te^{-i\varphi})|^2 + |f(z_1 - te^{-i\varphi})|^2 .$$

Then, for each fixed z (complex),

$$p(t) = |f(z + te^{-i\varphi})|^2 + |f(z - te^{-i\varphi})|^2 - |f(z + te^{i\varphi})|^2 - |f(z - te^{i\varphi})|^2$$

is a twice differentiable function of the real variable t which has a minimum at $t = 0$ ($p(0) = 0$). Hence, $p''(0) \geqq 0$. It follows from Lemma 3 that $8 \sin 2\varphi \operatorname{Im}(f''(z)\overline{f(z)}) \geqq 0$ for each complex number z, and since $\sin 2\varphi > 0$, this implies that

$$\operatorname{Im}(f''(z)\overline{f(z)}) \geqq 0 . \tag{10}$$

We may assume that $f(z) \not\equiv 0$. Then it follows from (10) that $\operatorname{Im}(f''(z)/f(z)) \geqq 0$ in the domain where $f(z) \neq 0$. Since $f''(z)/f(z)$ is a meromorphic function and the set of zeros of $f(z)$ is countable, it follows from Picard's little theorem that $f''(z)/f(z) = A$ for all z where A is a complex constant such that $\operatorname{Im}(A) \geqq 0$. The solutions of this differential equation are precisely those functions listed in Theorem 1.

Conversely, we shall prove that these two functions satisfy (8).

First, let us prove that $f(z) = a \sin h((\beta + i\gamma)z) + b \cos h((\beta + i\gamma)z)$ satisfies (8).

By the parallelogram identity we have

$$
\begin{aligned}
&|f(z_1+te^{-i\varphi})|^2+|f(z_1-te^{-i\varphi})|^2\\
(11)\qquad &\qquad =\frac{1}{2}(|f(z_1+te^{-i\varphi})+f(z_1-te^{-i\varphi})|^2\\
&\qquad\qquad +|f(z_1+te^{-i\varphi})-f(z_1-te^{-i\varphi})|^2)\,,
\end{aligned}
$$

$$
\begin{aligned}
&|f(z_1+te^{i\varphi})|^2+|f(z_1-te^{i\varphi})|^2\\
(12)\qquad &\qquad =\frac{1}{2}(|f(z_1+te^{i\varphi})+f(z_1-te^{i\varphi})|^2\\
&\qquad\qquad +|f(z_1+te^{i\varphi})-f(z_1-te^{i\varphi})|^2)\,,
\end{aligned}
$$

where $0 \leqq \varphi \leqq \pi/2$.

On the other hand

$$
\begin{aligned}
&|f(z_1+te^{-i\varphi})+f(z_1-te^{-i\varphi})|^2-|f(z_1+te^{i\varphi})+f(z_1-te^{i\varphi})|^2\\
(13)\qquad &\quad =4\,|a\sin h((\beta+i\gamma)z_1)+b\cos h((\beta+i\gamma)z_1)|^2\\
&\qquad \times(|\cos h((\beta+i\gamma)te^{-i\varphi})|^2-|\cos h((\beta+i\gamma)te^{i\varphi})|^2)\,.
\end{aligned}
$$

Since $|\cos h(c+id)|^2=\cos^2 hc-\sin^2 d$ where c, d are arbitrary real numbers, we have

$$
\begin{aligned}
&|\cos h((\beta+i\gamma)te^{-i\varphi})|^2-|\cos h((\beta+i\gamma)te^{i\varphi})|^2\\
&\quad =\cos^2 h(\beta t\cos\varphi+\gamma t\sin\varphi)-\sin^2(-\beta t\sin\varphi+\gamma t\cos\varphi)\\
&\qquad -\cos^2 h(\beta t\cos\varphi-\gamma t\sin\varphi)+\sin^2(\beta t\sin\varphi+\gamma t\cos\varphi)\\
(14)\qquad &\quad =4\sin h(\beta t\cos\varphi)\cos h(\beta t\cos\varphi)\sin h(\gamma t\sin\varphi)\cos h(\gamma t\sin\varphi)\\
&\qquad +4\sin(\beta t\sin\varphi)\cos(\beta t\sin\varphi)\sin(\gamma t\cos\varphi)\cos(\gamma t\cos\varphi)\\
&\quad =\sin h(2\beta t\cos\varphi)\sin h(2\gamma t\sin\varphi)+\sin(2\beta t\sin\varphi)\sin(2\gamma t\cos\varphi)\,.
\end{aligned}
$$

Here we may assume that $t \geqq 0$. Since $\sin hx \geqq x$ in $x \geqq 0$, by $\beta \geqq 0, \gamma \geqq 0, t \geqq 0, \cos\varphi \geqq 0, \sin\varphi \geqq 0$ we have

$$(15)\qquad \sin h(2\beta t\cos\varphi) \geqq 2\beta t\cos\varphi \geqq 0\,,$$

$$(16)\qquad \sin h(2\gamma t\sin\varphi) \geqq 2\gamma t\sin\varphi \geqq 0\,.$$

By (15), (16)

$$(17)\qquad \sin h(2\beta t\cos\varphi)\sin h(2\gamma t\sin\varphi) \geqq 4\beta\gamma t^2\cos\varphi\sin\varphi\,.$$

Since $x \geqq |\sin x|$ in $x \geqq 0$, by $\beta \geqq 0$, $\gamma \geqq 0$, $t \geqq 0$, $\cos\varphi \geqq 0$, $\sin\varphi \geqq 0$ we have

$$(18)\qquad 2\beta t\sin\varphi \geqq |\sin(2\beta t\sin\varphi)|\,,$$

$$(19)\qquad 2\gamma t\cos\varphi \geqq |\sin(2\gamma t\cos\varphi)|\,.$$

By (18), (19) we have

$$(20)\qquad 4\beta\gamma t^2\sin\varphi\cos\varphi \geqq |\sin(2\beta t\sin\varphi)\sin(2\gamma t\cos\varphi)|\,.$$

Further we have

$$
\begin{aligned}
(21)\qquad & |\sin(2\beta t\sin\varphi)\sin(2\gamma t\cos\varphi)| \\
& \qquad \geqq -\sin(2\beta t\sin\varphi)\sin(2\gamma t\cos\varphi)\,.
\end{aligned}
$$

By (17), (20), (21)

$$
\begin{aligned}
(22)\qquad & \sin h(2\beta t\cos\varphi)\sin h(2\gamma t\sin\varphi) \\
& \qquad + \sin(2\beta t\sin\varphi)\sin(2\gamma t\cos\varphi) \geqq 0\,.
\end{aligned}
$$

By (13), (14), (22) we have

$$
\begin{aligned}
(23)\qquad & |f(z_1 + te^{-i\varphi}) + f(z_1 - te^{-i\varphi})|^2 \\
& \qquad \geqq |f(z_1 + te^{i\varphi}) + f(z_1 - te^{i\varphi})|^2\,.
\end{aligned}
$$

Next we have

$$
\begin{aligned}
& |f(z_1 + te^{-i\varphi}) - f(z_1 - te^{-i\varphi})|^2 - |f(z_1 + te^{i\varphi}) - f(z_1 - te^{i\varphi})|^2 \\
(24)\qquad & \quad = 4\,|a\cos h((\beta + i\gamma)z_1) + b\sin h((\beta + i\gamma)z_1)|^2 \\
& \quad \times (|\sin h((\beta + i\gamma)te^{-i\varphi})|^2 - |\sin h((\beta + i\gamma)te^{i\varphi})|^2)\,.
\end{aligned}
$$

Since $|\sin h(c + id)|^2 = \cos^2 hc + \sin^2 d - 1$, by the same way as in (14)

$$
\begin{aligned}
(25)\qquad & |\sin h((\beta + i\gamma)te^{-i\varphi})|^2 - |\sin h((\beta + i\gamma)te^{i\varphi})|^2 \\
& = \sin h(2\beta t\cos\varphi)\sin h(2\gamma t\sin\varphi) - \sin(2\beta t\sin\varphi)\sin(2\gamma t\cos\varphi)\,.
\end{aligned}
$$

By replacing (21) by

$$
|\sin(2\beta t\sin\varphi)\sin(2\gamma t\cos\varphi)| \geqq \sin(2\beta t\sin\varphi)\sin(2\gamma t\cos\varphi)
$$

in the above calculation from (15) to (22) we have

$$
\begin{aligned}
(26)\qquad & \sin h(2\beta t\cos\varphi)\sin h(2\gamma t\sin\varphi) \\
& \qquad - \sin(2\beta t\sin\varphi)\sin(2\gamma t\cos\varphi) \geqq 0\,.
\end{aligned}
$$

By (24), (25), (26) we have

$$
\begin{aligned}
(27)\qquad & |f(z_1 + te^{-i\varphi}) - f(z_1 - te^{-i\varphi})|^2 \\
& \qquad \geqq |f(z_1 + te^{i\varphi}) - f(z_1 - te^{i\varphi})|^2\,.
\end{aligned}
$$

By (11), (12), (23), (27), we can conclude that

$$
f(z) = a\sin h((\beta + i\gamma)z) + b\cos h((\beta + i\gamma)z)
$$

satisfies (8).

Next, when $f(z) = az + b$, we have

$$
|f(z_1 + z_2)|^2 + |f(z_1 - z_2)|^2 = |f(z_1 + \bar{z}_2)|^2 + |f(z_1 - \bar{z}_2)|^2
$$

in $|z_1| < +\infty$ and in $|z_2| < +\infty$. Thus the theorem is proved.

THEOREM 2. *An entire function f is a solution of* (9) *if and only if* $f(z) = a \sin h((\beta - i\gamma)z) + b \cos h((\beta - i\gamma)z)$ *or* $f(z) = az + b$ *where a, b are arbitrary complex constants and β, γ are nonnegative real constants.*

Proof. Putting $f^*(z) = \overline{f(\bar{z})}$, $f^*(z)$ is an entire function of z. By (9) $f^*(z)$ satisfies the following functional inequality:

$$(9') \qquad |f^*(z_1 + z_2)|^2 + |f^*(z_1 - z_2)|^2 \leqq |f^*(z_1 + \bar{z}_2)|^2 + |f^*(z_1 - \bar{z}_2)|^2 ,$$

where z_1, z_2 are complex variables with $|z_1| < +\infty$ and $\operatorname{Im}(z_2^2) \geqq 0$. Hence, by (9′) and by Theorem 1 the theorem is proved.

By Theorem 1 we can solve (1), (2), (3), (4). An entire function f is a solution of (1) if and only if $f(z) = a \sin \alpha z$ or $f(z) = a \sin h\alpha z$ or $f(z) = az$ where a is an arbitrary complex constant and α is an arbitrary real constant.

An entire function f is a solution of (2) or (4) if and only if $f(z) = a \sin \alpha z + b \cos \alpha z$ or $f(z) = a \sin h\alpha z + b \cos h\alpha z$ or $f(z) = az + b$ where a, b are arbitrary complex constants and α is an arbitrary real constant.

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REFERENCES

1. H. Haruki, *On a certain family of meromorphic functions*, Proceedings of the Japan Academy **40** (1964), 92.
2. E. Hille, *A Pythagorean functional equation*, Ann. Math. **24** (1923), 175-180.
3. R. M. Robinson, *A curious trigonometric identity*. Amer. Math. Monthly **64** (1957), 83-85.
4. Pólya und Szegö, *Aufgaben und Lehrsätze aus der Analysis I* (1954), 94.

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