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# SOME CONTINUITY PROPERTIES OF THE SCHNIRELMANN DENSITY

R. L. DUNCAN

**Let $S$ denote the set of all infinite increasing sequences of positive integers. For all $A = \{a_n\}$ and $B = \{b_n\}$ in $S$, define the metric $\rho(A, B) = 0$ if $A = B$, i.e., if $a_n = b_n$ for all $n$ and $\rho(A, B) = 1/k$ otherwise, where $k$ is the smallest value of $n$ for which $a_n \neq b_n$. Similar metrics have been considered previously [1, 2].**

**Our purpose here is to discuss several continuity properties of the Schnirelmann density $d(A) = \inf A(n)/n$, where $A(n)$ is the number of elements of $A$ not exceeding $n$. In particular, we obtain a characterization of the set of all sequences having density zero as the set of points of continuity of $d(A)$.**

THEOREM 1. *$d(A)$ is upper semicontinuous on $S$.*

*Proof.* For each $\varepsilon > 0$ there is a $k$ such that $A(k)/k < d(A) + \varepsilon$. If $\rho(A_n, A) = 1/k_n \to 0$, then $k_n \to \infty$ and there is an $N$ such that $k_n > k$ for all $n > N$. Hence $d(A_n) \leqq A_n(k)/k = A(k)/k < d(A) + \varepsilon$ for all $n > N$ and the desired result follows.

Let $L_a = \{A \in S \mid d(A) = a\}$ $(0 \leqq a \leqq 1)$ denote the level sets of $d(A)$ and define $M_a = \{A \in S \mid d(A) \geqq a\}$. Also, let $\bar{L}_a$ denote the closure of $L_a$.

THEOREM 2. $\bar{L}_a = M_a$.

*Proof.* If $\lim \rho(A_n, A) = 0$ and $A_n \in L_a$ for all $n$, then $d(A) \geqq \lim\sup d(A_n) = a$ by Theorem 1 and $\bar{L}_a \subset M_a$.

Now let $B = \{b_k\} \in M_a$ and $d(B) = b \geqq a > 0$. Also, let $A = \{a_k\}$, where $a_k = 1 + [k - 1/a]$, and define $B_k = \{b_1, \cdots, b_k, a_{k+1}, a_{k+2}, \cdots\}$. Then $an \leqq A(n) < an + 1$ and it follows that $d(A) = a$. Also, $k = B(b_k) \geqq bb_k$ and $b_k \leqq k/b < [k/b] + 1 \leqq [k/a] + 1 = a_{k+1}$. Hence, $B_k$ is an increasing sequence, and $B_k(n)/n \geqq a$ for all $n$ and $k$. Thus $d(B_k) = \lim B_k(n)/n = a$ and $B_k \in L_a$ for all $k$. Since $\lim \rho(B_k, B) = 0$, we have $M_a \subset \bar{L}_a$ for $a > 0$.

Finally, if $a = 0$ we define $B_k = \{b_1, \cdots, b_k, b_{k+1}^2, b_{k+2}^2, \cdots\}$. Then it is obvious that $B_k \in L_0$ and $\lim \rho(B_k, B) = 0$. Hence $M_0 \subset \bar{L}_0$ and $\bar{L}_a = M_a$ for $0 \leqq a \leqq 1$.

COROLLARY. $\bar{L}_0 = S$.

It follows from the above corollary and Theorem 1 that $d(A)$ is continuous at $A$ if and only if $d(A) = 0$. It also follows from Theorem

2 that $L_a$ is closed if and only if $a = 1$ and it is easily shown that $L_a$ is never open. However, it is a consequence of the following theorem that $L_a$ is a $G_\delta$ set and that a description of the graph of $d(A)$ can be given [3].

THEOREM 3. *$d(A)$ is a function of Baire class one.*

*Proof.* Let

$$d_n(A) = \inf_{1 \leqq k \leqq n} \frac{A(k)}{k} .$$

Then $d_n(A) = d_n(B)$ if $\rho(A, B) < 1/n$ and it follows that $d_n(A)$ is continuous (uniformly) on $S$. It remains to be shown that $\lim d_n(A) = d(A)$ for all $A \in S$.

Now let $k_n$ be the smallest value of $k$ for which $d_n(A) = A(k)/k$. Then $\lim d_n(A)$ exists since $d_n(A) \geqq d_{n+1}(A) \geqq 0$ for all $n$. Also, $d_n(A) = A(k_n)/k_n \geqq d(A)$ for all $n$. Hence $\lim d_n(A) \geqq d(A)$. Since the sequence $\{k_n\}$ is nondecreasing and these numbers are integers, we have either $k_n = k$ for all $n \geqq N$ or $k_n \rightarrow \infty$. In the first case it is clear that $d_n(A) = d(A)$ for all $n \geqq N$ and $\lim d_n(A) = d(A)$ Suppose that $k_n \rightarrow \infty$. Then $A(k)/k \geqq A(k_n)/k_n = d_n(A)$ for all $k \leqq k_n$. Hence $A(k)/k \geqq \lim d_n(A)$ for all $k$ and $\lim d_n(A) \leqq d(A)$ since the sequence $\{d_n(A)\}$ is nonincreasing. Thus $\lim d_n(A) = d(A)$ in this case also.

## REFERENCES

1. R. L. Duncan, *A topology for sequences of integers I*, Amer. Math. Monthly **66** (1959), 34-39.

2. ———, *A topology for sequences of integers II*, Amer. Math. Monthly **67** (1960), 537-539.

3. E. S. Thomas, Jr., *Some characterizations of functions of Baire class* 1, Proc. Amer. Math. Soc. **17** (1966), 456-461.

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LOCK HAVEN STATE COLLEGE AND
PENNSYLVANIA STATE UNIVERSITY