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A GENERALIZED FATOU THEOREM FOR BANACH ALGEBRAS

DONALD CURTIS TAYLOR

Let B denote a commutative, semisimple Banach algebra with unit and let I be a fixed closed ideal in B. In the maximal ideal space M_B of B, fix a compact set X and put $E = X \cap h(I)$, where $h(I)$ is the hull of I. The main result of this note is the following

THEOREM 1.1. **Let I have an approximate unit that is uniformly bounded by the constant C and let g be a nonnegative continuous function on X of sup-norm <1 that vanishes on E. If h is an element of I and $\delta > 0$, then there exists an element f in I such that**

(1) $||f|| \leqq C$

(2) $\operatorname{Re} \hat{f}(x) \geqq g(x) + |\operatorname{Im} \hat{f}(x)| \qquad (x \in X)$

(3) $||fh - h|| < \delta$.

Here $\hat{f}$ denotes the Gelfand transform of an element f in B, $|| \; ||$ denotes the norm in B, and $|| \; ||_\infty$ denotes the sup-norm in the space of complex, continuous functions on X.

1. A theorem of Fatou.

Let A denote the sup-norm algebra of continuous functions on the closed unit disc which are analytic on the open disc, and let $J = J(E)$ be the ideal of functions in A which vanish on E, a closed set of Lebesgue measure 0 on the unit circle. In J there exists a sequence of functions, each of sup-norm $\leqq 1$, which converges uniformly to the constant function 1 on compact subsets of the complement of E; that is, J has an approximate unit. This fact may be deduced from a classical theorem of Fatou which guarantees the existence of a function in A which vanishes precisely on E and has positive real part elsewhere. The results in this paper stem from the observation that a reverse implication holds. By using only the existence of an approximate unit in J and the fact that E is a hull (This latter fact follows from the F. and M. Riesz theorem, for example.), we obtain the following *generalized Fatou theorem* for A: if g is a nonnegative continuous function on the closed unit disc of sup-norm <1 that vanishes on E, then there exists an f in J such that $||f||_\infty \leqq 1$ and $\operatorname{Re} f \geqq g + |\operatorname{Im} f|$.

Because of the central role played by approximate units, an analogous result can be established with surprising ease in the setting of a general Banach algebra. Specifically, let B denote a commutative, semisimple Banach algebra with unit and let I be a fixed closed ideal in B. In the maximal ideal space M_B of B, fix a compact set X and put $E = X \cap h(I)$, where $h(I)$ is the hull of I. Our main result is

THEOREM 1.1. *Let I have an approximate unit that is uniformly bounded by the constant C and let g be a nonnegative continuous function on X of sup-norm <1 that vanishes on E. If h is an element of I and $\delta > 0$, then there exists an element f in I such that*

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(3) $\|fh - h\| < \delta$.

Here $\hat{f}$ denotes the Gelfand transform of an element f in B, $\| \ \|$ denotes the norm in B, and $\| \ \|_\infty$ denotes the sup-norm in the space of complex, continuous functions on X. (For definitions and concepts we refer the reader to [2, p. 80] and [3].)

P. C. Curtis and A. Figá-Talamanca have already proved, by an application of their factorization theorem (see [1, p. 171]), a version of Theorem 1.1. They proved that if A is a commutative semisimple Banach algebra with approximate unit, then the Gelfand transforms of elements of A vanish arbitrarily slowly at infinity [1, Th. 4.1 and Th. 4.3, p. 180]. But their argument apparently does not give information about conditions (1), (2), and (3) of Theorem 1.1. For our applications this is crucial because conditions (1), (2), and (3) enable us to give a useful characterization of an approximate unit in I (see Corollary 2.4). Furthermore, condition (2) gives us information about $\operatorname{Re}\hat{f}$. This is important because we need estimates on the supremum norm of $\exp(\hat{f})$. These estimates and condition (1) provide the information necessary to prove the interpolation Theorem 2.5.

2. Proof of Theorem 1.1.

We will first give a precise definition of an approximate unit in I and then we will state and prove two preliminary results.

DEFINITION 2.1. The closed ideal I is said to have an approximate unit, if there exists a real number $C \geqq 1$ and a collection $\{e_\lambda : \lambda \in \Lambda\}$ of elements of I, where the index set Λ is a directed set, such that the following two conditions are satisfied: $\|e_\lambda\| \leqq C$, for each λ, and $\lim e_\lambda f = f$, for $f \in I$.

LEMMA 2.2. *If S is a compact subset of $M_B \backslash h(I)$ and $\{e_\lambda\}$ is an approximate unit in I, then $\{\hat{e}_\lambda\}$ converges uniformly to 1 on S.*

Proof. For each x in S choose an f_x belonging to I such that $\hat{f}_x(x) = 1$ and then define

$$V_x = \{y \in M_B : |\hat{f}_x(y)| > 1/2\} . \tag{2.1}$$

The family of sets $\{V_x : x \in S\}$ is an open cover of S and therefore, by the compactness of S, there exists a finite subcover

$$\{V_{x_1}, V_{x_2}, \cdots, V_{x_n}\} \tag{2.2}$$

of S. Let $\{f_{x_1}, f_{x_2}, \cdots, f_{x_n}\}$ denote those elements in I from which (2.2) is defined. It then follows that the inequality

$$|1 - \hat{e}_\lambda(y)| < 2 \max\{\|f_{x_j} - f_{x_j} e_\lambda\|\} \qquad (j = 1, 2, 3, \cdots, n)$$

holds for each y in S. Since $\lim e_\lambda f_{x_j} = f_{x_j}$, for $j = 1, 2, \cdots, n$, we conclude that $\{\hat{e}_\lambda\}$ converges uniformly to 1 on S. Hence our proof is complete.

LEMMA 2.3. *Let I have an approximate unit that is uniformly bounded by the constant C. If K is a compact G_δ subset of M_B that contains $h(I)$, then there is a closed ideal I_0 in B such that*

(1) $I_0 \subset I$,

(2) *$h(I_0)$ is a compact G_δ subset of M_B,*

(3) *$h(I_0) \subset K$, and*

(4) *I_0 has a countable approximate unit that is uniformly bounded by C.*

Proof. Since K is a compact G_δ subset of M_B, there exists a descending sequence $\{V_n\}$ of open subsets of M_B such that $K = \bigcap V_n$. Let $\{e_\lambda\}$ be an approximate unit of I that is uniformly bounded by C. Then, by virtue of Lemma 2.2, there exists an e_{λ_1} in I, $\|e_{\lambda_1}\| \leqq C$, such that $|1 - \hat{e}_{\lambda_1}| < 1/2$ on $M_B \backslash V_1$. Suppose that $e_{\lambda_1}, e_{\lambda_2}, \cdots, e_{\lambda_n}$ have been defined in I. Since $\lim e_\lambda e_{\lambda_j} = e_{\lambda_j}$ for $j = 1, 2, \cdots, n$, it follows that there exists (by Lemma 2.2) an $e_{\lambda_{n+1}}$ in I, $\|e_{\lambda_{n+1}}\| \leqq C$, such that $|1 - \hat{e}_{\lambda_{n+1}}| < 1/2$ on $M_B \backslash V_{n+1}$ and such that the inequality

$$\|e_{\lambda_j} e_{\lambda_{n+1}} - e_{\lambda_j}\| < 1/2^{n+1}$$

holds for $j = 1, 2, \cdots, n$. Thus, by induction, we have defined a sequence of elements $\{e_{\lambda_n}\}$ in I, $\|e_{\lambda_n}\| \leqq C$, such that $e_{\lambda_n} e_{\lambda_j} \to e_{\lambda_j}$ as $n \to \infty (j = 1, 2, 3, \cdots)$ and such that $\{\hat{e}_{\lambda_n}\}$ converges pointwise to 1 on $M_B \backslash K$. It is easy to show that the subset I_0 of I defined by

$$I_0 = \{f \in I : \lim e_{\lambda_n} f = f\}$$

is a closed ideal in B that satisfies (1), (3), and (4). To see that I_0 satisfies (2), we observe that $h(I_0) = \bigcap_{n=1}^\infty \{x \in M_B : |\hat{e}_{\lambda_n}(x)| < 1/n\}$. Hence our proof is complete.

Proof of Theorem 1.1. By virtue of Lemma 2.3 we may clearly assume that E is a compact G_δ subset of M_B and g vanishes precisely on E.

Let $0 < \lambda < 1/4C$ and $\varepsilon = \min\{1/32, (1 - \|g\|_\infty)/2\}$. Let K_p denote the compact subset of X defined by

$$K_p = \{x \in X : g(x) \geqq (1-\lambda)^{p-1}/8\} \qquad (p = 1, 2, 3, \cdots) .$$

We shall by induction construct a sequence of elements $e_1, e_2, e_3, \cdots$ in I and a sequence of compact subsets $D_1, D_2, D_3, \cdots$ of $X \backslash E$ such that for each positive integer p the following hold:

(a) $\|e_p\| \leqq C$ and $\|e_p h - h\| < \delta$.

(b) $K_{p+1} \cup D_p \subset D_{p+1}$.

(c) $\operatorname{Re} \hat{e}_p > 1 - \varepsilon^p$ and $|\operatorname{Im} \hat{e}_p| < \varepsilon^p$ on D_p.

(d) $\sum_{k=1}^{p} \lambda(1-\lambda)^{k-1} |\hat{e}_k| \leqq (1/32)\cdot(1-\lambda)^p$ on $X \backslash D_{p+1}$.

Let $D_1 = K_1$. Then there exists, by virtue of Lemma 2.2, an element e_1 in I that satisfies conditions (a) and (c). Suppose that $e_1, e_2, e_3, \cdots, e_n$ have been defined in I and that $D_1, D_2, \cdots, D_p$ have been defined in $X \backslash E$. Since $\sum_{k=1}^{p} \lambda(1-\lambda)^{k-1}\hat{e}_k$ is a continuous function on X that vanishes on E, there exists a compact subset D_{p+1} of $X \backslash E$ that satisfies conditions (b) and (d). Thus, as before, we have by virture of Lemma 2.2 an element e_{p+1} in I that satisfies conditions (a) and (c). Hence our construction is complete.

Let f denote the element in I defined by

$$f = \sum_{k=1}^{\infty} \lambda(1-\lambda)^{k-1} e_k .$$

It is clear that $\|f\| \leqq C$ and that $\|fh - h\| < \delta$. Therefore, we complete our proof by showing that

$$\operatorname{Re} \hat{f}(x) \geqq g(x) + |\operatorname{Im} \hat{f}(x)| \qquad (x \in X) . \tag{2.3}$$

Let x belong to X. If $x \in E$, then it is clear that (2.3) holds. If x is an element of D_1, then (2.3) follows directly from (b) and (c). Otherwise, since g vanishes precisely on E, there exists a positive integer p such that x belongs to $D_{p+1} \backslash D_p$. Consequently, by conditions (a), (c), and (d), we have

$$\begin{aligned}
\operatorname{Re} \hat{f}(x) &\geqq \sum_{k=p+1}^{\infty} \lambda(1-\lambda)^{k-1} \operatorname{Re} \hat{e}_k(x) - \lambda(1-\lambda)^{p-1} \|e_p\| \\
&\qquad - \sum_{k=1}^{p-1} \lambda(1-\lambda)^{k-1} |\hat{e}_k(x)| \\
&\geqq \sum_{k=p+1}^{\infty} \lambda(1-\lambda)^{k-1}(1-\varepsilon^k) - (1-\lambda)^{p-1}/4 - (1-\lambda)^{p-1}/32 \qquad (2.4) \\
&\geqq \sum_{k=p+1}^{\infty} \lambda(1-\lambda)^{k-1} - \sum_{k=p+1}^{\infty} \varepsilon\lambda(1-\lambda)^{k-1}\varepsilon^{k-1} - (9/32)\cdot(1-\lambda)^{p-1} \\
&\geqq (1-\lambda)^p - \varepsilon(1-\lambda)^{p-1} - (9/32)\cdot(1-\lambda)^{p-1} \\
&\geqq (14/32)\cdot(1-\lambda)^{p-1}
\end{aligned}$$

and

$$
(2.5) \qquad \begin{aligned} |\operatorname{Im} \hat{f}(x)| &\leqq \sum_{k=p+1}^{\infty} \lambda(1-\lambda)^{k-1}\varepsilon^k + (1-\lambda)^{p-1}/4 + (1-\lambda)^{p-1}/32 \\ &\leqq (5/16)\cdot(1-\lambda)^{p-1} . \end{aligned}
$$

Since x does not belong to K_p, we see by combining (2.4) and (2.5) that (2.3) holds.

COROLLARY 2.4. *If I has an approximate unit that is uniformly bounded by the constant C, then I has an approximate unit $\{e_\lambda: \lambda \in \Lambda\}$, $\|e_\lambda\| \leqq C$, such that*

$$
(2.6) \qquad \operatorname{Re} \hat{e}_\lambda \geqq |\operatorname{Im} \hat{e}_\lambda|
$$

on M_B for each λ.

Proof. Let $\{h_\alpha: \alpha \in A\}$ be an approximate unit in I and let Λ be the directed set defined by

$$
\Lambda = \{(\alpha, n): \alpha \in A, n \text{ a positive integer}\}
$$

under the usual partial ordering; that is, $(\alpha_1, n_1) \leqq (\alpha_2, n_2)$, if and only if, $\alpha_1 \leqq \alpha_2$ and $n_1 \leqq n_2$. For each $\lambda = (\alpha, n) \in \Lambda$ we may choose, by virtue of Theorem 1.1, an element e_λ in I, $\|e_\lambda\| \leqq C$, such that
(a) $\|e_\lambda h_\alpha - h_\alpha\| < 1/n$
(b) $\operatorname{Re} \hat{e}_\lambda(x) \geqq |\operatorname{Im} \hat{e}_\lambda(x)| \qquad (x \in M_B)$.
Now for arbitrary $f \in I$ and $\varepsilon > 0$, there is an $\alpha_0 \in A$ such that

$$
\|h_\alpha f - f\| < \varepsilon/4C \qquad \text{for } \alpha \geqq \alpha_0 .
$$

Choose $\lambda_0 = (\alpha_0, n)$, where $\|f\|/n < \varepsilon/2$. Then for $\lambda = (\alpha, n) \geqq \lambda_0$ we have

$$
\begin{aligned} \|e_\lambda f - f\| &= \|e_\lambda f - e_\lambda h_\alpha f + e_\lambda h_\alpha f - h_\alpha f + h_\alpha f - f\| . \\ &\leqq 2C\|h_\alpha f - f\| + \|f\|\cdot\|e_\lambda h_\alpha - h_\alpha\| < \varepsilon . \end{aligned}
$$

In other words, $\lim e_\lambda f = f$ for every $f \in I$. Hence our proof is complete.

THEOREM 2.5. *Suppose I has an approximate unit that is uniformly bounded by the constant C and suppose g is an element in B. If Δ is a positive continuous function on X such that $\Delta \geqq |\hat{g}|$ on E and $\varepsilon > 0$, then there exists an element f in B such that*
(1) $\hat{f} = \hat{g}$ on E, $|\hat{f}| \leqq \Delta$ on X, and
(2) $\|f\| \leqq \|p\|_\infty^C \|g\| + \varepsilon$.
Here p is the positive continuous function on X defined by $p(x) = \max\{1, |\hat{g}(x)|/\Delta(x)\}$.

Proof. Since $\log(p)$ is a nonnegative continuous function on X that vanishes on E, we have by virtue of Theorem 1.1 an element h in I such that $-\operatorname{Re}\hat{h} \leqq -\log(p)$ on X and such that the norm $\|h\| \leqq C\cdot\log\|p\|_\infty + \eta$, where $\eta = \log(1 + \varepsilon/\|p\|_\infty^C\|g\|)$. Set $f = g\cdot\exp(-h)$. It is clear that $\hat{f}\,|\,E = \hat{g}\,|\,E$ and since $|\hat{g}(x)|/p(x) \leqq \Delta(x)$, $x \in X$, the inequalities

$$|\hat{f}(x)| = |\hat{g}(x)|\exp(-\operatorname{Re}\hat{h}(x)) \leqq |\hat{g}(x)|/p(x) \leqq \Delta(x)$$

hold, $x \in X$. We complete the proof by showing (2) holds. To see this, we observe that the norm

$$\|f\| \leqq \|g\|\exp\|h\| \leqq \|g\|\exp(\log\|p\|_\infty^C + \eta) \leqq \|p\|_\infty^C\|g\| + \varepsilon\,.$$

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