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A FAMILY OF FUNCTORS DEFINED ON GENERALIZED PRIMARY GROUPS

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Let G denote an abelian group; G is called a generalized p-primary group if $qG = G$ for all primes $q \neq p$. Let α be an ordinal, and let $\delta: G \to E_\alpha$ satisfy the following four conditions: (1) E_α is p^α Ext-injective, (2) $p^\alpha E_\alpha = 0$, (3) $\delta(G)$ is p^α-pure in E_α, (4) $\ker \delta = p^\alpha G$. Define $p^{\alpha *}G$ to be that subgroup of E_α such that $p^\alpha(E_\alpha/\delta(G)) = p^{\alpha *}G/\delta(G)$. If α is a limit ordinal, let $L_\alpha(G) = \varprojlim_{\beta<\alpha} G/p^\beta G$. Let

$$U(G) = \mathrm{Ext}(Z(p^\infty), G) \quad \textbf{and} \quad U_\alpha(G) = U(G)/p^\alpha U(G) .$$

Then we have the following p^α-pure containments: $G/p^\alpha G \cong \delta(G) \subseteqq U_\alpha(G) \subseteqq p^{\alpha *}(G) \subseteqq L_\alpha U_\alpha(G)$, whenever α is a countable limit of lesser hereditary ordinals. We have $p^{\alpha *}G = U_\alpha(G)$ for all groups G if and only if p^α Ext is hereditary. From this we obtain a new proof of the fact that p^α Ext is hereditary when α is a countable limit of lesser hereditary ordinals. We also obtain an example of a cotorsion group G such that $G/p^\alpha G$ is not equal to $L_\alpha(G)$, thus refuting a conjecture of Harrison. A group G is called generally complete if $L_\alpha(G)/\delta(G)$ is reduced for all limit ordinals α. A generalized p-primary group G is generally complete if and only if it is cotorsion.

A result of Kulikov [7] will be studied and generalized, and an application to the study of cotorsion groups will be given.

Troughout this paper the word "group" will mean "abelian group". The notation of [2] will be followed. The letter p will indicate a prime.

The elements of the group $\mathrm{Ext}(A, B)$ are equivalence classes of extensions $E: 0 \to B \to E \to A \to 0$. However, no distinction will be made between equivalence classes and an element of the equivalence class. Thus, it will be said that E is an element of $\mathrm{Ext}(A, B)$. Also, B will be considered as a subgroup of E. The arrow $\rightarrowtail$ will denote a monomorphism, and the arrow $\twoheadrightarrow$ will denote an epimorphism. The element $\mathrm{Ext}(f, g)E$, for $E \in \mathrm{Ext}(A, B)$, $f: B \to B'$, and $g: A' \to A$, will be denoted by gEf. All other notation will be that used in Chapter III of [8].

Recall that a subgroup H of a group G is said to be p^α-pure in G if the extension $H \rightarrowtail G \twoheadrightarrow G/H$ is an element of $p^\alpha\mathrm{Ext}(G/H, H)$; G/H is said to be a p^α-pure quotient of the group G. A group G is said to be p^α-projective if $p^\alpha \mathrm{Ext}(G, A) = 0$ for all groups A; G is called p^α-injective if $p^\alpha \mathrm{Ext}(A, G) = 0$ for all groups G.

The functor $p^\alpha \operatorname{Ext}(\cdot, \cdot)$ is said to be hereditary (or shorter, α is called a hereditary ordinal) if every p^α-pure subgroup of a p^α-projective is p^α-projective, or, equivalently, if every p^α-pure quotient of a p^α-injective is p^α-injective. In § 3 a new proof will be given to show that $p^\alpha \operatorname{Ext}$ is hereditary if α, is a countable limit of lesser hereditary ordinals.

We shall use the notation $\lambda(G)$ to denote the length of G; i.e., the least ordinal α satisfying $p^{\alpha+1}G = p^\alpha G$.

1. The functor p^α. In [9] it is shown that for all ordinals α there exists an exact sequence

$$Z \rightarrowtail G_\alpha \twoheadrightarrow H\alpha ,$$

such that for all group G the following hold.

(1) $p^\alpha G \rightarrowtail G \xrightarrow{\delta} \operatorname{Ext}(H_\alpha, G) \overset{\varepsilon}{\twoheadrightarrow} \operatorname{Ext}(G_\alpha, G)$
is exact, and $\operatorname{Im}(\delta)$ is p^α-pure in $\operatorname{Ext}(H_\alpha, G)$. Here we have identified G with $\operatorname{Hom}(Z, G)$ in the usual way;

(2) H_α is a p^α-projective p-group, so $p^\alpha \operatorname{Ext}(H_\alpha, G) = 0$, and $\operatorname{Ext}(H_\alpha, G)$ is p^α-injective;

(3) The sequences for α and $\alpha + n$ are connected by

$$\begin{tikzcd} Z \arrow[r, tail, "p^n"] \arrow[d, "p^n"'] & G_{\alpha+n} \arrow[r, two heads] \arrow[d, equal] & H_{\alpha+n} \arrow[d] \\ Z \arrow[r, tail] & G_\alpha \arrow[r, two heads] & H_\alpha \end{tikzcd} ;$$

(4) If α is a limit ordinal, then

$$H_\alpha = \bigoplus \sum_{\beta<\alpha} H_\beta ;$$

(5) $p^\alpha H_{\alpha+1}$ is cyclic of order p and $H_\alpha = H_{\alpha+1}/p^\alpha H_{\alpha+1}$;

(6) $p^\alpha H_\alpha = 0$.

Let $p^{\alpha *}G$ denote $\varepsilon^{-1}(p^\alpha \operatorname{Ext}(G_\alpha, G))$; then $G/p^\alpha G = \operatorname{Im}\delta$ is a p^α-pure subgroup of $p^{\alpha *}G$.

THEOREM 1.1. *Let E be p^α-injective such that $p^\alpha E = 0$, that there exists a homomorphism $\gamma: G \to E$ with kernel $p^\alpha G$, and that $\operatorname{Im}\gamma$ a p^α-pure subgroup of E. Let G^* denote the subgroup of E satisfying $G^*/\gamma(G) = p^\alpha(E/\gamma(G))$. Then there exists an isomorphism $g: p^{\alpha *}G \to G^*$, such that $g\delta = \gamma$.*

Proof. For convenience in the remainder of this paper we will denote $\operatorname{Ext}(H_\alpha, G)$ and $\operatorname{Ext}(G_\alpha, G)$ by $E_\alpha(G)$ and $F_\alpha(G)$, respectively, or simply by E_α and F_α if no confusion can result. For this proof let $E/\gamma(G) = F$. Replace $\mathrm{Im}\,\gamma$ and $\mathrm{Im}\,\delta$ by $G/p^\alpha G$. Then the following sequences are exact:

$$G/p^\alpha G \rightarrowtail E_\alpha \twoheadrightarrow F_\alpha \ ,$$
$$G/p^\alpha G \rightarrowtail E \twoheadrightarrow F \ .$$

Before continuing with the proof we prove the following:

LEMMA 1.2. *If f, g are homomorphisms from E_α to E (or E to E_α) such that $f \mid G/p^\alpha G = g \mid G/p^\alpha G$, then $f \mid p^{\alpha^*}G = g \mid p^{\alpha^*}G$ ($f \mid G^* = g \mid G^*$).*

Proof. Assume $f, g\colon E_\alpha \to E$, the proof for $f, g\colon E \to E_\alpha$ being the same. Let $h = f - g$; then $h(G/p^\alpha G) = 0$. Therefore, h can be lifted to a homomorphism h^* of F_α into E. Since $p^\alpha E = 0$, we have $h^* \mid p^\alpha F = 0$. Thus, $h \mid p^{\alpha^*}G = 0$; so $f \mid p^{\alpha^*}G = g \mid p^{\alpha^*}G$.

We now continue the proof of Theorem 1.1. Since E is p^α-injective, there exists a homomorphism $g'\colon E_\alpha \to E$ such that the following diagram commutes.

$$\begin{tikzcd} G/p^\alpha G \arrow[r, tail] \arrow[d, equal] & E_\alpha \arrow[r, two heads] \arrow[d, "g'"'] & F_\alpha \arrow[d, "\bar{g}"] \\ G/p^\alpha G \arrow[r, tail] & E \arrow[r, two heads] & F \end{tikzcd} ;$$

$\bar{g}$ arises in the usual way. Let $g = g' \mid p^{\alpha^*}G$. Since $\bar{g}(p^\alpha F_\alpha) \subseteqq p^\alpha F$, it follows that $g(p^{\alpha^*}G) \subseteqq G^*$. Similarly, there exists a homomorphism $f'\colon E \to E_\alpha$ such that

$$\begin{tikzcd} G/p^\alpha G \arrow[r, tail] \arrow[d, equal] & E \arrow[r, two heads] \arrow[d, "f'"'] & F \arrow[d, "\bar{f}"] \\ G/p^\alpha G \arrow[r, tail] & E_\alpha \arrow[r, two heads] & F_\alpha \end{tikzcd}$$

commutes. Let $f = f' \mid G^*$; then clearly $f(G^*) \subseteqq p^{\alpha^*}G$. Consider $f' \circ g'\colon E_\alpha \to E_\alpha$. By Lemma 1.2

$$f \circ g = f' \circ g' \mid p^{\alpha^*}G = 1_{E_\alpha} \mid p^{\alpha^*}G = 1p^{\alpha^*}G \ .$$

Similarly, $g \circ f = g' \circ f' \mid_{G^*} = 1_{G^*}$. Thus, g is an isomorphism of $p^{\alpha^*}G \to G^*$, and clearly $g\delta = \gamma$.

It follows that, if E is a p^α-injective having the following properties:

(1) There exists a homomorphism $\gamma\colon G \to E$ with $\ker \gamma = p^\alpha G$ and $\mathrm{Im}\,\gamma$ p^α-pure in E;

(2) $p^\alpha E = 0$,

then $p^{\alpha^*}G$ can be taken as the subgroup of E with the property that

$p^{\alpha*}G/\gamma(G) = p^{\alpha}(E/\gamma(G))$.

Let $U(G) = \operatorname{Ext}(Z(p^{\infty}), G)$ and $U_{\alpha}(G) = U(G)/p^{\alpha}U(G)$. In [11] it is shown that for all ordinals α, $U_{\alpha}(G)$ is contained in $p^{\alpha*}G$ and $\delta(G) \subseteqq U_{\alpha}(G)$. In [11] Nunke has shown that α is a hereditary ordinal if and only if $U_{\alpha}(G) = p^{\alpha*}(G)$ for all groups G.

The remaining part of this section will be spent in proving the following theorem.

THEOREM 1.3. *Let α be an ordinal such that for all $\gamma < \alpha$ there exists a hereditary β with $\gamma < \beta < \alpha$. Then $p^{\alpha*}G \subseteqq \varprojlim_{\beta<\alpha} U_{\beta}(G)$.*

The proof of this theorem follows from a series of lemmas. We first observe that $\{U_{\beta}(G), \pi^{\beta}_{\gamma}\}$ is an inverse system, where for $\beta > \gamma$ $\pi^{\beta}_{\gamma}\colon U_{\beta}(G) \to U_{\gamma}(G)$ is the natural projection with kernel $p^{\beta}U_{\gamma}(G)$.

LEMMA 1.4. *Let β and γ be ordinals with $\gamma < \beta$. Then there exists a homomorphism $\pi^{\beta}_{\gamma}\colon p^{\beta*}G \to p^{\gamma*}G$ agreeing with the natural projection of $G/p^{\beta}G$ onto $G/p^{\gamma}G$ when restricted to $G/p^{\beta}G$. Moreover if $\alpha < \beta < \gamma$, then $\pi^{\beta}_{\gamma}\pi^{\alpha}_{\beta} = \pi^{\alpha}_{\gamma}$.*

Proof. The extensions

$$G/p^{\beta}G \rightarrowtail E_{\beta} \twoheadrightarrow F_{\beta}$$

and

$$G/p^{\gamma}G \rightarrowtail E_{\gamma} \longrightarrow F_{\gamma}$$

are p^{β}-pure and p^{γ}-pure, respectively. Since $\beta > \gamma$, the top extension is also p^{γ}-pure. As E_{γ} is p^{γ}-injective, there exists a map μ^{β}_{γ} of E_{β} into E_{γ} such that the following diagram commutes:

$$\begin{tikzcd}
p^{\gamma}G/p^{\beta}G \arrow[d] & & \\
G/p^{\beta}G \arrow[r] \arrow[d, "\pi"'] & E_{\beta} \arrow[r] \arrow[d, "\mu^{\beta}_{\gamma}"] & F_{\beta} \arrow[d, "\lambda^{\beta}_{\gamma}"] \\
G/p^{\gamma}G \arrow[r] & E_{\gamma} \arrow[r] & F_{\gamma}
\end{tikzcd}$$

where π is the canonical projection. The homomorphism λ^{β}_{γ} arises in the usual way. Define π^{β}_{γ} by $\pi^{\beta}_{\gamma} = \mu^{\beta}_{\gamma} \mid p^{\beta*}G$.

As in the proof of Theorem 1.1, $\operatorname{Im}\pi^{\beta}_{\gamma}$ is contained in $p^{\gamma*}G$, and, as in Lemma 1.2, the homomorphism is unique. If $\alpha < \beta < \gamma$, then let $\mu^{\alpha}_{\gamma} = \mu^{\beta}_{\gamma}\mu^{\alpha}_{\beta}$.

LEMMA 1.5. *Let β and γ be ordinals with $\beta < \gamma$. Let π denote the canonical projection of $G/p^\beta G$ onto $G/p^\gamma G$. If π_γ^β is a homomorphism of $U_\beta(G)$ into $p^{\gamma*}(G)$ agreeing with π on $G/p^\beta G$, then π_γ^β is the canonical projection of $U_\beta(G)$ onto $U_\gamma(G)$.*

Proof. Let μ denote the natural projection of $U_\gamma(G)$ onto $U_\beta(G)$. Consider the homomorphism $\pi_\gamma^\beta - \mu$. On the group $G/p^\beta G$ the homomorphism $\pi_\gamma^\beta - \mu = 0$. Thus, there exists a homomorphism λ: $U_\beta(G)/\delta(G)$ into $p^{\gamma*}(G)$ such that the following diagram commutes:

$$\begin{tikzcd} U_\beta(G) \arrow[r] \arrow[d, "\pi_\gamma^\beta - \mu"'] & U_\beta(G)/\delta(G) \arrow[dl, "\lambda"] \\ p^{\gamma*}G & \end{tikzcd}$$

Since $p^\gamma(p^{\gamma*}G) = 0$ and $U_\beta(G)/\delta(G)$ is divisible, λ must be the zero homomorphism. Thus $\pi_\gamma^\beta - \mu = 0$.

LEMMA 1.6. *If $\gamma < \beta$ and β is a hereditary ordinal, then the homomorphism π_γ^β: $p^{\beta*}G \to p^{\gamma*}G$ defined in Lemma 1.4 is the natural projection of $U_\beta(G)$ onto $U_\gamma(G)$.*

Proof. If β is a hereditary ordinal, then $p^{\beta*}G = U_\beta(G)$. Lemma 1.5 completes the proof.

Let α be a limit ordinal. Then the group H_α is $\Sigma_{\beta<\alpha} H_\beta$. This shows that the group $E_\alpha = \Pi_{\beta<\alpha} E_\beta$, since

$$E_\alpha = \operatorname{Ext}(H_\alpha, G) = \operatorname{Ext}(\Sigma H_\beta, G) = \Pi \operatorname{Ext}(H_\beta, G) = \Pi\, E_\beta\ .$$

The homomorphism δ: $G \to E_\beta$ can be defined in terms of δ_β: $G \to E_\beta$ by $\delta(x)_\beta = \delta_\beta(x)$. Then the homomorphism μ_β^α used in the proof of Lemma 1.4 can be taken as the natural coordinate projection. So the intersection over all $\beta < \alpha$ of $\operatorname{Ker} \pi_\beta^\alpha$ is zero.

THEOREM 1.7. *If α is a limit ordinal, then the set $\{p^{\beta*}G, \pi_\sigma^\beta\}_{\beta<\alpha}$ is an inverse system, and there is an isomorphic copy of $p^{\alpha*}G$ in $\varprojlim_{\beta<\alpha} p^{\beta*}G$.*

Proof. Lemma 1.4 shows that $\{p^{\beta*}G, \pi_\gamma^\beta\}$ is an inverse system. The homomorphisms π_β^α: $p^{\alpha*}G \to p^{\beta*}G$ gives a family of maps of the group $p^{\alpha*}G$ into this inverse system satisfying $\pi_\gamma^\beta \pi_\beta^\alpha = \pi_\gamma^\alpha$. Thus, there is a homomorphism μ: $p^{\alpha*}G \to \varprojlim_{\beta<\alpha} p^{\beta*}G$. The $\ker \mu = \bigcap_{\beta<\alpha} \ker \pi_\beta^\alpha = 0$. Thus, μ is a monomorphism.

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. We will show that for all $\gamma < \alpha$ the image of π_γ^α is contained in $U_\gamma(G)$. Let $\gamma < \alpha$; then there exists a hereditary ordinal β such that $\gamma < \beta < \alpha$. Since $p^{\beta *}G = U_\beta(G)$, it follows that the image of π_β^α is contained in $U_\beta(G)$. Lemma 1.4 and 1.5 show that π_γ^α maps $p^{\alpha *}G$ into $U_\gamma(G)$. Since $\{U_\beta(G), \pi_\gamma^\beta\}$ is an inverse family and $\pi_\gamma^\alpha \pi_\beta^\alpha$, it follows that there exists a homomorphism

$$\mu \colon p^{\alpha *}G \longrightarrow \varprojlim_{\beta<\alpha} U_\beta(G) .$$

As in the proof of Theorem 1.7, ker $\mu = 0$. Thus μ is a monomorphism.

COROLLARY 1.8. *The group $G/p^\alpha G$ is a p^α-pure subgroup of the group $\varprojlim_{\beta<\alpha} U_\beta(G)$.*

Proof. Since $\Pi_{\beta<\alpha} U_\beta(G) \subseteqq E_\alpha$, it follows that $\varprojlim_{\beta<\alpha} U_\beta(G) \subseteqq E_\alpha$. The group $G/p^\alpha G$ is a p^α-pure subgroup of E_α, and

$$G/p^\alpha G \subseteqq p^{\alpha *}G \subseteqq \varprojlim_{\beta<\alpha} U_\beta(G) .$$

2. **The functor L_α.** Let G be a group and α a limit ordinal. Then the family $\{p^\beta G\}_{\beta<\alpha}$ forms a neighborhood system at zero for the group G. This topology will be called the natural topology. If the length of $G = \lambda(G) = \alpha$, then the topology is a Hausdorff topology. If $\alpha \neq \lambda(G)$, then $\{p^\beta G\}_{\beta<\alpha}$ leads to a topology on $G/p^\alpha G$, given by $\{p^\beta G/p^\alpha G\}_{\beta<\alpha}$. This topology is a Hausdorff topology on $G/p^\alpha G$. The family, $\{p^\beta G\}_{\beta<\alpha}$, leads to a uniformity on G, respectively $G/p^\alpha G$. Therefore, we can consider the completion of G, $(G/p^\alpha G)$ with respect to this uniformity. Let $L_\alpha(G)$ denote the completion of G if $\lambda(G) = \alpha$, or completion of $G/p^\alpha G$ if $\lambda(G) > \alpha$.

In [12], Zelinsky showed that $L_\alpha(G) = \varprojlim_{\beta<\alpha} G/p^\beta G$. We remark that notation $L_\alpha(G)$ is consistent with the notation used by Harrison in [4]. Let $\pi_\beta \colon L_\alpha(G) \to G/p^\beta G$ be the natural projection of $\varprojlim G/p^\beta G$ onto $G/p^\beta G$. A base for the topology on $L_\alpha G$ is given by $\{\ker \pi_\beta\}_{\beta<\alpha}$. We shall call this topology the induced topology. We shall now study the functor L_α on the following class of groups introduced by Kulikov in [6] and [7].

DEFINITION 2.1. A group G is a generalized p-primary group (g.p. group), if G is divisible by all primes other than p.

The following theorem is due to Kulikov [7].

THEOREM 2.2. *Let G be a* g.p. *group. Let α be an ordinal less than or equal to the length of G, satisfying the following condition:*

(*) *There exists a countable increasing sequence of ordinals whose limit is* α.

Then if δ *is the natural map of* G *into* $\varprojlim_{\beta<\alpha} G/p^\beta G$, *with kernel equal to* $p^\alpha G$:

(1) $\delta(G) + p^\beta L_\alpha(G) = L_\alpha(G)$, *for all* $\beta < \alpha$;

(2) $L_\alpha(G)/\delta(G)$ *is divisible*;

(3) $\delta(G) \cap p^\beta L_\alpha(G) = p^\beta \delta(G)$ *for all* $\beta < \alpha$;

(4) $G/p^\beta G = L_\alpha(G)/p^\beta L_\alpha(G)$, *for all* $\beta < \alpha$.

Notice that condition (1) states that $\delta(G)$ is dense in $L_\alpha(G)$ in the natural topology; and condition (4) shows that $L_\alpha(G)$ is complete in the natural topology, since

$$L_\alpha(L_\alpha(G)) = \varprojlim_{\beta<\alpha} L_\alpha(G)/p^\beta L_\alpha(G) = \varprojlim G/p^\beta G = L_\alpha(G) .$$

We will show that conditions (1), (2), and (4) are equivalent and that when they happen, the natural topology and the induced topology on $L_\alpha(G)$ are the same. However, we first shall prove the following.

THEOREM 2.3. *If* G *is a* g.p. *group and* α *is a limit ordinal, then* $G/p^\alpha G$ *is* p^α*-pure in* $L_\alpha(G)$.

Proof. Since $G/p^\beta G$ is contained in E_β, it follows that

$$L_\alpha(G) \subseteqq \Pi_{\beta<\alpha} G/p^\beta G \subseteqq \Pi E_\beta = E_\alpha .$$

The embedding $\delta: G \to L_\alpha(G)$ is the map, $\delta: G \to E_\alpha$, with its range cut down to $L_\alpha(G)$. Since $G/p^\alpha G$ is a p^α-pure in E_α, the theorem follows.

Notice that this theorem generalized condition (3) of Kulikov's theorem.

THEOREM 2.4. *If* G *is a* g.p. *group and* α *is a limit ordinal less than or equal to the length of* G, *then the following are equivalent*:

(1) $\delta(G)$ *is dense in* $L_\alpha(G)$ *in the natural topology*; *i.e.*, $\delta(G) + p^\beta L_\alpha(G) = L_\alpha(G)$ *for all* $\beta < \alpha$.

(2) $L_\alpha(G)/\delta(G)$ *is divisible.*

(3) $p^\beta L_\alpha(G) = \ker \pi_\beta$ *for* $\beta < \alpha$, *where* π_β *is the natural projection,* $L_\alpha(G)$, *onto* $G/p^\beta G$; *i.e., the natural topology and the induced topology are the same.*

Proof. First we shall show that (1) implies (3). Note that $\pi_\beta L_\alpha(G) \subseteqq G/p^\beta G$; it follows that $p^\beta L_\alpha G \subseteqq \ker \pi_\beta$. If $x \in \ker \pi_\beta$, then $x = y + z$, with $y \in \delta(G)$ and $z \in p^\beta L_\alpha G$. Then $z \in \ker \pi_\beta$. Thus, $y \in \delta(G) \cap \ker \pi_\beta = p^\beta G$. It follows that $x \in p^\beta G + p^\beta L_\alpha G = p^\beta L_\alpha G$. Thus, $\ker \pi_\beta = p^\beta L_\alpha(G)$.

We will now show (3) implies (1). A neighborhood system for $L_\alpha(G)$ in the product topology is given by $\{\ker \pi_\alpha \mid \beta < \alpha\}$. If condition (3) holds, then $\{p^\beta L_\alpha G \mid \beta < \alpha\}$ is a neighborhood system for $L_\alpha G$. The group $\delta(G)$ is dense in $L_\alpha(G)$ in the product topology. If condition (3) holds, then $\delta(G)$ is dense in $L_\alpha(G)$ in the natural topology.

In order to show (1) is equivalent to (2), we first observe that, since G is generalized primary, all groups in question are divisible by all primes other than p. Thus, it only has to be shown that $\delta(G)$ is dense in $L_\alpha(G)$ if and only if $L_\alpha(G)/\delta(G)$ is a p-divisible. The proof of this fact follows from a series of lemmas.

LEMMA 2.5. *If $\beta < \alpha$ and π_β is the map defined in* (3) *of Theorem* 2.4, *then $L_\alpha G = \delta(G) + \ker \pi_\beta$.*

Proof. If $x \in L_\alpha G$, then there exists $y \in G$ such that $y + p^\beta G = \pi_\beta(x)$. Then $\delta(y) - x \in \ker \pi_\beta$.

LEMMA 2.6. *Let G, $L_\alpha G$, π_β be as above. If $x \in \ker \pi_\beta$ and the image of x in $L_\alpha(G)/\delta(G)$ is in $p^\beta(L_\alpha G/\delta(G))$, then $x \in p^\beta L_\alpha(G)$.*

Proof. The proof is by induction on β. If $\beta = 1$, then $\pi_1(x) = 0$, and x maps into $p(L_\alpha G/\delta(G))$. Thus, there exists a $y \in L_\alpha(G)$ such that $x + \delta(G) = py + \delta(G)$, and so $x - py \in \delta(G)$. Since $\pi_1(x - py) = 0$, $x - py \in \ker \pi_1 \cap \delta(G) = p\delta(G)$. Thus, there exists a $z \in G$ such that $x - py = p\delta(z)$, or $x = p(y + \delta(z)) \in pL_\alpha G$.

If $\beta > \gamma$, then let π_γ^β be the natural projection of $G/p^\beta G \to G/p^\gamma G$. If $\beta = \gamma + 1$, then $0 = \pi_\gamma^\beta\ \pi_\beta(x) = \pi_\gamma(x)$. So $x \in \ker \pi_\gamma$, and x maps into $p^\gamma(L_\alpha G/\delta(G))$. Hence, $x \in p^\gamma L_\alpha(G)$. We must show $x \in p^{\gamma+1}(G)$. Since $x \in p^\beta[L_\alpha(G)/\delta(G)]$, there exists a $y' \in L_\alpha(G)$ such that

$$y' + \delta(G) \in p^\gamma(L_\alpha(G)/\delta(G)) \quad \text{and} \quad x + \delta(G) = py' + \delta(G)\ ;$$

thus, $x - py' \in \delta(G)$. Since $x \in p^\gamma L_\alpha(G)$, we see that

$$x - py' \in pL_\alpha G \cap \delta(G) = p\delta(G)\ ;$$

so $x = p(y' + z)$ for some $z \in \delta(G)$. Let $y = y' + z$. Then $x = py$ and $y + \delta(G) = y' + \delta(G) \in p^\gamma(L_\alpha(G)/\delta(G))$. By Lemma 2.5, $L_\alpha(G) = \delta(G) + \ker \pi_\gamma$. So there exists $y'' \in \ker \pi_\gamma$, $g \in \delta(G)$, such that $y = y'' + g$. Then $y'' + \delta(G) = y + \delta(G) \in p^\gamma(L_\alpha(G)/\delta(G))$. Thus, $y'' \in p^\gamma L_\alpha(G)$ by the induction hypothesis. It follows that $py'' \in p^\beta L_\alpha G \subseteq \ker \pi_\beta$. Thus, $pg = x - py'' \in \ker \pi_\beta$, so $pg \in \delta(G) \ker \pi_\beta = p^\beta\delta(G)$, and we see that $x \in p^\beta L_\alpha(G)$.

Let β be a limit ordinal. Then

$$\pi_\gamma(x) - \pi_\gamma^\beta\pi_\beta(x) = 0\ , \quad \text{and} \quad x + \delta(G) \in p^\beta(L_\alpha(G)/\delta(G)) \subseteq p^\gamma(L_\alpha(G)/\delta(G))\ .$$

So by the induction hypothesis we see that $x \in p^{\gamma}L_{\alpha}(G)$ for all $\gamma < \beta$, and thus $x \in \bigcap_{\beta<\gamma} p^{\gamma}L_{\alpha}(G) = p^{\beta}L_{\alpha}(G)$.

We can now show the equivalence of conditions (1) and (2) of Theorem 2.4. Since $L_{\alpha}(G) = \delta(G) + \ker \pi_{\beta}$, we see that every element of $p^{\beta}(L_{\alpha}(G)/\delta(G))$ is the image of an element of $\ker \pi_{\beta}$. Lemma 2.6 then assures us that every element of $p^{\beta}(L_{\alpha}(G)/\delta(G))$ is the image of an element in $p^{\beta}L_{\alpha}(G)$ under the homomorphism

$$p^{\beta}L_{\alpha}(G) \longrightarrow (\delta(G) + p^{\beta}L_{\alpha}(G))/\delta(G) .$$

Since $(\delta(G) + p^{\beta}L_{\alpha}(G))/\delta(G) \subseteqq p^{\beta}(L_{\alpha}(G)/\delta(G))$, it then follows that

$$(\delta(G) + p^{\beta}L_{\alpha}(G))/\delta(G) = p^{\beta}(L_{\alpha}(G)/\delta(G)) .$$

If $L_{\alpha}(G)/\delta(G)$ is p-divisible, then $p^{\beta}(L_{\alpha}(G)/\delta(G)) = L_{\alpha}(G)/\delta(G)$; and so $L_{\alpha}(G) = \delta(G) + p^{\beta}L_{\alpha}(G)$. Conversely, if $L_{\alpha}(G) = \delta(G) + p^{\beta}L_{\alpha}(G)$, then $p^{\beta}(L_{\alpha}(G)/\delta(G)) = L_{\alpha}(G)/\delta(G)$. This completes the proof.

3. Some applications.

The following definition is due to Harrison [4].

DEFINITION 3.1. A g.p. group is called fully complete if $L_{\alpha}G = G/p^{\alpha}G$ for all limit ordinals α less than or equal to the length of G.

Harrison [4] conjectured that a g.p. group is cotorsion if and only if G is fully complete. Using Theorems 1.3 and 2.4, we can find an example of a g.p. cotorsion group G which is not fully complete.

Let Ω be the first uncountable ordinal. Nunke [11] has shown that $p^{\Omega}\,\mathrm{Ext}$ is not hereditary. Therefore, by Proposition 4.1, [11] and Theorem 13 we have that $U_{\Omega}(G) \subsetneqq p^{\Omega^*}G \subseteqq L_{\Omega}U_{\Omega}(G)$, for some group G. The group $U_{\Omega}(G)$ is a g.p. cotorsion group and is not fully complete.

Let $Z \rightarrowtail G_{\Omega} \twoheadrightarrow H_{\Omega}$ define p^{Ω}. Let M_{Ω} be the torsion subgroup of G_{Ω}. Nunke [11] has shown that M_{Ω} is not p^{Ω} Ext-projective. In showing that α is hereditary if and only if $U_{\alpha}(G) = p^{\alpha^*}(G)$ for all groups G, Nunke actually showed that $U_{\alpha}(G) = p^{\alpha^*}(G)$ if and only if $p^{\alpha}\ \mathrm{Ext}\,(M_{\alpha}, G) = 0$, for G fixed.

LEMMA 3.2. $p^{\Omega}\ \mathrm{Ext}\,(M_{\Omega}, \mathrm{Tor}\,(M_{\Omega}, M_{\Omega})) \neq 0$.

Proof. In [11] it is shown that

$$\begin{tikzcd}
 & M_{\Omega} \arrow[r, equal] \arrow[d, tail] & M_{\Omega} \arrow[d, tail] \\
Z \arrow[r, tail] \arrow[d, equal] & G_{\Omega} \arrow[r, two heads] \arrow[d, two heads] & H_{\Omega} \arrow[d, two heads] \\
Z \arrow[r, tail] & Q_p \arrow[r, two heads] & Z(p^{\infty})
\end{tikzcd}$$

is exact and the last column is p^{Ω}-pure. Here $Q_p = \{a/b \in Q \mid b = p^n$ for some $n\}$. From this we obtain

$$\begin{array}{ccccc}
(\operatorname{Tor} M_\Omega, M_\Omega) & & & & \\
\downarrow & & & & \\
(\operatorname{Tor} H_\Omega, M_\Omega) & \longrightarrow & M_\Omega \otimes Z & \xrightarrow{\beta} & M_\Omega \otimes G_\Omega \\
\gamma\downarrow & & \| & & \\
M_\Omega = \operatorname{Tor}(Z(p^\infty), M_\Omega) & \longrightarrow & M_\Omega \otimes Z\,. & &
\end{array}$$

Here β is the zero map; for if $x \otimes n \in M_\Omega \otimes Z$, then $\beta(x \otimes n) = x \otimes n$. However, $n \in p^{\Omega} G_\Omega$. Thus $x \otimes n = 0$ in $M_\Omega \otimes G_\Omega$. Thus γ is onto. By Theorem 3.9 of [9], the sequence

$$E\colon \operatorname{Tor}(M_\Omega, M_\Omega) \longrightarrow \operatorname{Tor}(H_\Omega, M_\Omega) \longrightarrow \operatorname{Tor}(Z(p^\infty), M_\Omega) = M_\Omega$$

is p^{Ω}-pure. Since M_Ω is not p^{Ω}-projective, M_Ω is not a summand of $\operatorname{Tor}(H_\Omega, M_\Omega)$, Theorem [3.1] of [9]. Thus $E \neq 0$, and

$$p^{\Omega} \operatorname{Ext}(M_\Omega, \operatorname{Tor}(M_\Omega, M_\Omega)) \neq 0\,.$$

This shows that $p^{\Omega *}(\operatorname{Tor}(M_\Omega, M_\Omega)) \neq U_\Omega(\operatorname{Tor}(M_\Omega, M_\Omega))$. So, the group $U_\Omega(\operatorname{Tor}(M_\Omega, M_\Omega))$ serves as a counter example to Harrison's conjecture.

We are now in a position to examine condition (*) of Theorem 2.2. Let $G = U_\Omega(\operatorname{Tor}(M_\Omega, M_\Omega))$. Then $L_\Omega G/G \neq 0$. Also, as $L_\Omega G$ and G are cotorsion, $L_\Omega G/G$ is reduced. Theorem 2.4 now tells us that conditions (1), (2), and (4) of Theorem 2.2 do not hold. It follows that if α is not a countable limit of lesser ordinals, then G need not be dense in $L_\alpha G$ in the natural topology. Also, the induced topology on $L_\alpha G$ need not be the natural topology on $L_\alpha G$.

DEFINITION 3.3. A g.p. group G is called generally complete provided $L_\alpha(G)/\delta(G)$ is reduced for all limit ordinals α less than or equal to the length of G.

Notice that if the length of $G = \lambda(G)$ is less than Ω and if G is generally complete, then G is fully complete.

THEOREM 3.4. *A necessary and sufficient condition for a* g.p. *group to be cotorsion is that it be generally complete.*

Proof. Let G be g.p. cotorsion group. Then $G/p^{\beta}G$ is cotorsion for all β. By Theorem 5.3 of [9], $L_\alpha(G)$ is cotorsion. It follows that $L_\alpha(G)/\delta(G)$ is cotorsion and so reduced. Therefore, G is generally complete.

Let G be a g.p. generally complete group. Then $G/p^{\beta}G$ is generally complete for all β. We will show by transfinite induction on α that $G/p^\alpha G$ is cotorsion for all α. If $\alpha = 0$, there is nothing to prove. Let $\alpha = \beta + 1$ for some ordinal β. The sequence $p^\beta G/p^\alpha G \rightarrowtail G/p^\alpha G \twoheadrightarrow G/p^\beta G$ is exact with ends cotorsion groups. Therefore, $G/p^\alpha G$ is cotorsion. Let α be a limit ordinal. Then, since G is generally complete, $L(G)/\delta(G)$ is reduced. The group $L_\alpha(G)$ is cotorsion, since by the induction hypothesis it is an inverse limit of cotorsion groups by Theorem 5.3 of [9]. Therefore, $\delta(G) = G/p^\alpha G$ is cotorsion.

This last theorem answers Question 3 posed by Fuchs in [3].

In [11] Nunke showed that p^α Ext is hereditary, if α is a limit ordinal less than Ω. In proving this he relied heavily upon Ulm's theorem. We now give a proof of this theorem which does not use Ulm's theorem.

THEOREM 3.5. *If α is an ordinal which satisfies condition* (*) *of theorem* 2.4, *then p^α* Ext *is hereditary.*

Proof. Since α satisfies condition (*) of Theorem 2.4 $L_\alpha U_\alpha(G)/U_\alpha(G)$ is divisible. However, $L_\alpha U_\alpha(G)$ and $U_\alpha(G)$ are cotorsion groups; therefore, $L_\alpha U_\alpha(G)/U_\alpha(G)$ must be reduced. Thus, $L_\alpha U_\alpha(G) = U_\alpha(G)$, for all groups G.

Let β be a hereditary ordinal; then $\beta + n$ is also hereditary Proposition 4.2 of [11]. If $\alpha < \Omega$, Proposition 4.1 of [11] and Theorem 1.3 give the desired result. If $\alpha \geqq \Omega$, then $\alpha + \omega + n$ is hereditary if n is any integer, by Proposition 4.2 of [11]. This fact together with Theorem 1.3 give the desired result.

We remark that for all other ordinals β p^β Ext is not hereditary. A proof of this fact may be found in [11].

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