PACIFIC JOURNAL OF MATHEMATICS
Vol. 26, No. 2, 1968

ABSOLUTE $(C, 1)\cdot(N, p_n)$ SUMMABILITY OF A FOURIER SERIES AND ITS CONJUGATE SERIES

H. P. DIKSHIT

In the present paper we have established theorems concerning absolute $(C, 1)\cdot(N, p_n)$ summability of a Fourier series and its conjugate series. Incidentally these theorems include as special cases previous theorems on the $|C|$-summability of Fourier series and its conjugate series due to Bosanquet and Bosanquet and Hyslop.

1. Definitions and notations. Let Σa_n be a given infinite series with the sequence of partial sums $\{\mathscr{S}_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write $P_n = p_0 + p_1 + \cdots + p_n$; $P_{-1} = p_{-1} = 0$.

The sequence-to-sequence transformation:

$$t_n = \sum_{\nu=0}^{n} p_\nu \mathscr{S}_{n-\nu}/P_n = \sum_{\nu=0}^{n} P_\nu a_{n-\nu}/P_n \qquad (P_n \neq 0) , \tag{1.1}$$

defines the sequence $\{t_n\}$ of Nörlund means [9] of the sequence $\{\mathscr{S}_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series Σa_n is said to be summable (N, p_n) to the sum $\mathscr{S}$ if $\lim_{n\to\infty} t_n$ exists and is equal to $\mathscr{S}$, and is said to be absolutely summable (N, p_n) or summable $|N, p_n|$ [8], if the sequence $\{t_n\}$ is of bounded variation, that is, the infinite series $\sum_n |t_n - t_{n-1}| < \infty$.[1] In the special case in which

$$p_n = \binom{n + \alpha - 1}{\alpha - 1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} \qquad (\alpha > -1) , \tag{1.2}$$

the Nörlund mean reduces to the familiar (C, α) mean.

The summability $|N, p_n|$, where $\{p_n\}$ is defined by (1.2), is the same as summability $|C, \alpha|$.

The conditions for the regularity of the (N, p_n) method of summation are

$$\lim_{n\to\infty} p_n/P_n = 0 \quad \text{and} \quad \sum_{k=0}^{n} |p_k| = O(|P_n|) , \qquad n \to \infty . \tag{1.3}$$

We define the $(C, 1)\cdot(N, p_n)$ mean of $\{\mathscr{S}_n\}$ as the $(C, 1)$ mean of $\{t_n\}$, the sequence of Nörlund means of $\{\mathscr{S}_n\}$. We write t_n^1 and u_n^1 for the $(C, 1)$ means of $\{t_n\}$ and $\{u_n\} = \{n(t_n - t_{n-1})\}$, respectively. Thus the $(C, 1)\cdot(N, p_n)$ mean of $\{\mathscr{S}_n\}$ is

[1] Symbolically, $\{t_n\} \in BV$. Similarly by '$f(x) \in BV(a, b)$' we mean that $f(x)$ is a function of bounded variation in the interval (a, b) and by $\{\gamma_n\} \in B$ that $\{\gamma_n\}$ is a bounded sequence.

$$t_n^1 = \sum_{\nu=0}^{n} t_\nu/(n+1) . \tag{1.4}$$

The series Σa_n is said to be summable $(C, 1)\cdot(N, p_n)$ to the sum t if $\lim_{n\to\infty} t_n^1$ exists and is equal to t and is said to be absolutely summable $(C, 1)\cdot(N, p_n)$ or summable $|(C, 1)\cdot(N, p_n)|$, if $\sum_n |t_n^1 - t_{n-1}^1| < \infty$.

Let $f(t)$ be a periodic function, with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. We assume without any loss of generality that the constant term in the Fourier series of $f(t)$ is zero, so that $\int_{-\pi}^{\pi} f(t)dt = 0$, and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) . \tag{1.5}$$

Then the conjugate series of (1.5) is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t) . \tag{1.6}$$

We write throughout:

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\} ; \qquad \psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\} ;$$

$$\varphi_1(t) = t^{-1}\Phi_1(t) ; \qquad \Phi_1(t) = \int_0^t \varphi(u)du ;$$

$$R_n = (np_n)/P_n ; \qquad c_n = \sum_{\nu=n}^{\infty} \{(\nu+1)P_\nu\}^{-1} ;$$

$$S_n = \sum_{\nu=0}^{n} P_\nu(\nu+1)^{-1}/P_n ; \qquad \Delta_n f_n = f_n - f_{n+1} ;$$

$\tau = [\pi/t]$, i.e., the greatest integer contained in π/t.

K, denotes a positive constant not necessarily the same at each occurrence.

2. Introduction.

Astrachan has proved the following theorem for the $(N, p_n)\cdot(C, 1)$ summability of a Fourier series.

THEOREM A[2]. *The* $(N, p_n)\cdot(C, 1)$ *method is* K_α*-effective* $(0 < \alpha \leqq 1)$, *provided the generating sequence satisfies the conditions*

$$\{R_n\} \in B , \tag{2.1}$$

$$\left\{\sum_{k=1}^{n} k\,|\Delta p_{k-1}|/P_n\right\} \in B \tag{2.2}$$

[2] Astrachan [1], § 11, Theorem II. In Astrachan's notations $(N, p_n)\cdot(C, 1)$ is denoted by $(N, p_n)\cdot C_1$. In [4] the present author has indicated and supplied a deficiency in the proof of this theorem.

and

$$\left\{\sum_{k=1}^{n} k^{-1} \mid P_k \mid / P_n\right\} \in B \, . \tag{2.3}$$

Theorem A implies *inter alia* that the Fourier series of $f(t)$ is summable $(N, p_n)\cdot(C, 1)$, at every point $t = x$, at which $\lim_{t\to 0} \varphi_1(t) = f(x)$.

Since bounded variation is the property associated with absolute summability in the same sense in which continuity is associated with ordinary summability it may be expected that the bounded variation of $\varphi_1(t)$ over $(0, \pi)$ along with the bounded variation of sequences in (2.1)–(2.3) may be sufficient to ensure the $|(C, 1)\cdot(N, p_n)|$ summability of the Fourier series of $f(t)$, at $t = x$. That, this is indeed, true, in one of the most important cases: when $\{p_n\}$ is a positive, monotonic nonincreasing sequence, is established in our Theorem 1. Since in this case if $\{R_n\} \in BV$ then the sequence in (2.2) is automatically of bounded variation and $\{S_n\} \in BV$ is equivalent to $P_n c_n = O(1)$ ([14]; [12]) which in its turn is equivalent to $\{S_n\} \in B$ ([12]; [13]), therefore the hypotheses: $\{R_n\} \in BV$ and $P_n c_n = O(1)$, or equivalently, $\{R_n\} \in BV$ and $\{S_n\} \in B$, are sufficient to ensure the $|(C, 1)\cdot(N, p_n)|$ summability of the Fourier series at a point.

Incidentally, the following form of a result of Bosanquet follows as a corollary from our Theorem 1, when we observe that the (C, δ) mean is a special case of the (N, p_n) mean and appeal to Kogbetliantz [6].

THEOREM B [2]. *If $\varphi_1(t) \in BV(0, \pi)$, then the Fourier series of $f(t)$, at $t = x$, is summable $|C, 1 + \delta|$, for every $\delta > 0$.*

Concerning the $|N, p_n|$ summability of the conjugate series Pati has recently proved the following theorem.

THEOREM C [12]. *If $\psi(t) \in BV(0, \pi)$, $\int_0^\pi t^{-1} |\psi(t)| \, dt \leqq K$ and $\{p_n\}$ is a positive sequence such that $\{R_n\} \in BV$ and $\{S_n\} \in BV$ then the conjugate series of the Fourier series of $f(t)$, at $t = x$, is summable $|N, p_n|$.*

The object of our Theorem 2 and Theorem 3 is to study the $|(C, 1)\cdot(N, p_n)|$ summability of the conjugate series under each of the two conditions on $\psi(t)$ used in Theorem C. We observe here that our Theorem 2 and Theorem 3 contain as special cases the following two theorems of Bosanquet and Hyslop on the $|C|$ summability of the conjugate series, respectively.

THEOREM D[3]. *If* $\int_0^{\pi} t^{-1} |\psi(t)| dt \leqq K$, *then the conjugate series of the Fourier series of* $f(t)$, *at* $t = x$, *is summable* $|C, 1 + \delta|$, *for every* $\delta > 0$.

THEOREM E[4]. *If* $\psi(t) \in BV(0, \pi)$, *then the sequence* $\{nB_n(x)\}$ *is summable* $|C, 1 + \delta|$, *for every* $\delta > 0$.

3. We establish the following theorems.

THEOREM 1. *If* $\phi_1(t) \in BV(0, \pi)$ *and* $\{p_n\}$ *is a positive, monotonic nonincreasing sequence, such that* (i) $\{R_n\} \in BV$ *and* (ii) $\{P_n c_n\} \in B$, *then the Fourier series of* $f(t)$, *at* $t = x$, *is summable* $|(C, 1)\cdot(N, p_n)|$.

THEOREM 2. *If* $\int_0^{\pi} t^{-1} |\psi(t)| dt \leqq K$, *and* $\{p_n\}$ *satisfies the same conditions as in Theorem* 1, *then the conjugate series of the Fourier series of* $f(t)$, *at* $t = x$, *is summable* $|(C, 1)\cdot(N, p_n)|$.

THEOREM 3. *If* $\psi(t) \in BV(0, \pi)$ *and* $\{p_n\}$ *satisfies the same conditions as in Theorem* 1, *then the sequence* $\{nB_n(x)\}$ *is summable* $|(C, 1)\cdot(N, p_n)|$.

4. We require the following lemmas for the proof of our theorems.

LEMMA 1 [7]. *If* $\{q_n\}$ *is nonnegative and nonincreasing, then for* $0 \leqq a \leqq b \leqq \infty, 0 \leqq t \leqq \pi$, *and any* n

$$\left| \sum_{k=a}^{b} q_k \exp\{i(n-k)t\} \right| \leqq KQ_\tau ,$$

where $\tau = [\pi/t]$ *and* $Q_m = q_0 + q_1 + \cdots + q_m$.

LEMMA 2[5]. *If* $\{p_n\}$ *is a positive sequence and* (i) *and* (ii) *hold, then uniformly in* $0 < t \leqq \pi$,

$$\sum_n \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} (P_n p_k - p_n P_k) \frac{\sin (n-k)t}{n-k} \right| \leqq K .$$

5. **Proof of Theorem 1.** We have by a well known identity of Kogbetliantz [6],

$$t_n^1 - t_{n-1}^1 = n^{-1} u_n^1 .$$

[3] Bosanquet and Hyslop [**3**], Theorem 1, when $\alpha = 0$.

[4] Bosanquet and Hyslop [**3**], Theorem 5.

[5] Pati [**12**] and Varshney [**14**]. For a more general result see Dikshit [**5**].

Therefore, in order to prove the theorem, it is sufficient to show that

$$\sum_n n^{-1} | u_n^1 | \leqq K .$$

Now, as in Pati [10], for the series (1.5), we have

$$t_n - t_{n-1} = \frac{2}{\pi}\int_0^\pi \left\{ \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k) \cos (n-k)t \right\} \varphi(t) dt .$$

Hence

$$\begin{aligned} n^{-1} u_n^1 &= \frac{2}{\pi}\int_0^\pi \left\{ \frac{1}{n(n+1)} \sum_{\nu=1}^{n} \frac{\nu}{P_\nu P_{\nu-1}} \sum_{k=0}^{\nu-1} (P_\nu p_k - p_\nu P_k) \cos (\nu - k)t \right\} \varphi(t) dt \\ &= \frac{2}{\pi}\int_0^\pi g^1(n, t) \varphi(t) dt , \end{aligned}$$

where

$$(5.1) \qquad g^1(n, t) = \frac{1}{n(n+1)} \sum_{\nu=1}^{n} \frac{\nu}{P_\nu P_{\nu-1}} \sum_{k=0}^{\nu-1} (P_\nu p_k - p_\nu P_k) \cos (\nu - k)t .$$

Integrating by parts, we get

$$\begin{aligned} \int_0^\pi \varphi(t) g^1(n, t) dt &= [\Phi_1(t) g^1(n, t)]_0^\pi - \int_0^\pi \Phi_1(t) \frac{d}{dt} g^1(n, t) dt \\ &= [t\varphi_1(t) g^1(n, t)]_{t=\pi} - \int_0^\pi \left(t \frac{d}{dt} g^1(n, t) \right) \varphi_1(t) dt \\ &= [t\varphi_1(t) g^1(n, t)]_{t=\pi} - \left[\varphi_1(t) \int_0^t u \frac{d}{du} g^1(n, u) du \right]_0^\pi \\ &\quad + \int_0^\pi \left\{ \int_0^t u \frac{d}{du} g^1(n, u) du \right\} d\varphi_1(t) . \end{aligned}$$

But

$$\int_0^t u \frac{d}{du} g^1(n, u) du = t g^1(n, t) - \int_0^t g^1(n, u) du .$$

Thus

$$\int_0^\pi \varphi(t) g^1(n, t) dt = \int_0^\pi \left\{ t g^1(n, t) - \int_0^t g^1(n, u) du \right\} d\varphi_1(t) ,$$

and therefore,

$$\begin{aligned} \frac{\pi}{2} \sum_n n^{-1} | u_n^1 | &= \sum_n \left| \int_0^\pi \left\{ t g^1(n, t) - \int_0^t g^1(n, u) du \right\} d\varphi_1(t) \right| \\ &\leqq \int_0^\pi | d\varphi_1(t) | \left\{ t \sum_n | g^1(n, t) | + \sum_n \left| \int_0^t g^1(n, u) du \right| \right\} , \end{aligned}$$

since by hypothesis, $\int_0^\pi |d\varphi_1(t)| \leqq K$, it suffices for our purpose to show that, uniformly in $0 < t \leqq \pi$,

$$t \sum_n |g^1(n, t)| \leqq K , \tag{5.2}$$

and

$$\sum_n \left| \int_0^t g^1(n, u) du \right| \leqq K . \tag{5.3}$$

In order to establish (5.2), we prove that uniformly in $0 < t \leqq \pi$

$$t \sum_n \frac{1}{n(n+1)} \left| \sum_{\nu=1}^n \frac{\nu}{P_\nu P_{\nu-1}} \sum_{k=0}^{\nu-1} (P_\nu p_k - p_\nu P_k) \exp\{i(\nu - k)t\} \right| \leqq K . \tag{5.4}$$

Now, we have

$$\begin{aligned} & t \sum_{n=1}^\infty \frac{1}{n(n+1)} \left| \sum_{\nu=1}^n \frac{\nu}{P_\nu P_{\nu-1}} \sum_{k=0}^{\nu-1} (P_\nu p_k - p_\nu P_k) \exp\{i(\nu - k)t\} \right| \\ & \leqq t \sum_{n=1}^\infty \frac{1}{n(n+1)} \left| \sum_{\nu=1}^n \frac{1}{P_\nu P_{\nu-1}} \sum_{k=0}^{\nu-1} \{kP_\nu p_k - \nu p_\nu P_k\} \exp\{i(\nu - k)t\} \right| \\ & \quad + t \sum_{n=1}^\infty \frac{1}{n(n+1)} \left| \sum_{\nu=1}^n \frac{1}{P_\nu P_{\nu-1}} \sum_{k=0}^{\nu-1} \{\nu P_\nu p_k - kP_\nu p_k\} \exp\{i(\nu - k)t\} \right| \\ & \leqq t \sum_{n=1}^\infty \frac{1}{n(n+1)} \left| \sum_{\nu=1}^n \frac{1}{P_{\nu-1}} \sum_{k=0}^{\nu-1} (R_k - R_\nu) P_k \exp\{i(\nu - k)t\} \right| \\ & \quad + t \sum_{n \leqq \tau} \frac{1}{n(n+1)} \left| \sum_{\nu=1}^n \frac{1}{P_{\nu-1}} \sum_{k=0}^{\nu-1} (\nu - k) p_k \exp\{i(\nu - k)t\} \right| \\ & \quad + t \sum_{n > \tau} \frac{1}{n(n+1)} \left| \sum_{\nu=1}^n \frac{1}{P_{\nu-1}} \sum_{k=0}^{\nu-1} (\nu - k) p_k \exp\{i(\nu - k)t\} \right| \\ & = \Sigma_1 + \Sigma_2 + \Sigma_3 , \end{aligned} \tag{5.5}$$

say.

We write

$$\begin{aligned} \Sigma_1 & = t \sum_{n=1}^\infty \frac{1}{n(n+1)} \left| \sum_{\nu=1}^n \frac{1}{P_{\nu-1}} \sum_{k=0}^{\nu-1} P_k \exp\{i(\nu - k)t\} \sum_{\mu=k}^{\nu-1} \Delta R_\mu \right| \\ & = t \sum_{n=1}^\infty \frac{1}{n(n+1)} \left| \sum_{\nu=1}^n \frac{1}{P_{\nu-1}} \sum_{\mu=0}^{\nu-1} \Delta R_\mu \sum_{k=0}^{\mu} P_k \exp\{i(\nu - k)t\} \right| \\ & \leqq K \frac{t}{|1 - \exp it|} \sum_{n=1}^\infty \frac{1}{n(n+1)} \sum_{\nu=1}^n \frac{1}{P_{\nu-1}} \sum_{\mu=0}^{\nu-1} |\Delta R_\mu| P_\mu \\ & \qquad \text{by Abel's Lemma,} \\ & \leqq K \sum_{\nu=1}^\infty \frac{1}{P_{\nu-1}} \sum_{\mu=0}^{\nu-1} |\Delta R_\mu| P_\mu \sum_{n=\nu}^\infty \frac{1}{n(n+1)} \end{aligned} \tag{5.6}$$

$$\leqq K \sum_{\nu=1}^{\infty} \frac{1}{\nu P_{\nu-1}} \sum_{\mu=0}^{\nu-1} |\Delta R_\mu| P_\mu$$

$$= K \sum_{\mu=0}^{\infty} |\Delta R_\mu| P_\mu \sum_{\nu=\mu+1}^{\infty} \frac{1}{\nu P_{\nu-1}}$$

$$\leqq K \sum_{\mu=0}^{\infty} |\Delta R_\mu| \leqq K ,$$

by virtue of the hypotheses (i) and (ii).

Next, since $|\exp\{i(n-k)t\}| \leqq 1$, therefore

$$\Sigma_2 \leqq t \sum_{n \leqq \tau} \frac{1}{n(n+1)} \sum_{\nu=1}^{n} \frac{\nu}{P_{\nu-1}} \sum_{k=0}^{\nu-1} p_k \leqq Kt \sum_{n \leqq \tau} 1 \leqq K . \tag{5.7}$$

In Σ_3 changing the order of summation of the inner sums, we get

$$\begin{aligned} \Sigma_3 &= t \sum_{n>\tau} \frac{1}{n(n+1)} \left| \sum_{\nu=0}^{n-1} \frac{1}{P_\nu} \sum_{k=0}^{\nu} (\nu - k + 1) p_k \exp\{i(\nu - k + 1)t\} \right| \\ &= t \sum_{n>\tau} \frac{1}{n(n+1)} |\exp it| \left| \sum_{k=0}^{n-1} p_k \sum_{\nu=k}^{n-1} \frac{\nu - k + 1}{P_\nu} \exp\{i(\nu - k)t\} \right| . \end{aligned} \tag{5.8}$$

Now, by Abel's transformation

$$\begin{aligned} &\sum_{\nu=k}^{n-1} \frac{\nu - k + 1}{P_\nu} \exp\{i(\nu - k)t\} \\ &= \sum_{\nu=k}^{n-1} \Delta_\nu\left(\frac{\nu - k + 1}{P_\nu}\right) \sum_{\mu=k}^{\nu} \exp\{i(\mu - k)t\} \\ &\quad + \frac{n - k + 1}{P_n} \sum_{\mu=k}^{n-1} \exp\{i(\mu - k)t\} \\ &= (1 - \exp it)^{-1} \Bigg[\sum_{\nu=k}^{n-1} \Delta_\nu\left(\frac{\nu - k + 1}{P_\nu}\right) [1 - \exp\{i(\nu - k + 1)t\}] \\ &\quad + \frac{n - k + 1}{P_n} [1 - \exp\{i(n - k)t\}] \Bigg] \\ &= (1 - \exp it)^{-1} \Bigg[- \sum_{\nu=k}^{n-1} \frac{p_{\nu+1}}{P_\nu P_{\nu+1}} (\nu - k + 1) \exp\{i(\nu - k + 1)t\} \\ &\quad + \sum_{\nu=k}^{n-1} \frac{1}{P_{\nu+1}} \exp\{i(\nu - k + 1)t\} \\ &\quad - \frac{n - k + 1}{P_n} \exp\{i(n - k)t\} + \frac{1}{P_k} \Bigg] . \end{aligned}$$

Thus, from (5.8), we get

$$\begin{aligned} \Sigma_3 \leqq K &\sum_{n>\tau} \frac{1}{n(n+1)} \left| \sum_{k=0}^{n-1} p_k \sum_{\nu=k}^{n-1} \frac{p_{\nu+1}}{P_\nu P_{\nu+1}} (\nu - k + 1) \exp\{i(\nu - k + 1)t\} \right| \\ + K &\sum_{n>\tau} \frac{1}{n(n+1)} \left| \sum_{k=0}^{n-1} p_k \sum_{\nu=k}^{n-1} \frac{1}{P_{\nu+1}} \exp\{i(\nu - k + 1)t\} \right| \end{aligned}$$

$$(5.9) \qquad + K \sum_{n>\tau} \frac{1}{n(n+1)P_n} \left| \sum_{k=0}^{n-1} p_k(n-k+1) \exp\{i(n-k)t\} \right|$$

$$+ K \sum_{n>\tau} \frac{1}{n(n+1)} \sum_{k=0}^{n-1} \frac{p_k}{P_k}$$

$$= \Sigma_{31} + \Sigma_{32} + \Sigma_{33} + \Sigma_{34} \, ,$$

say.

First changing the order of summation of the inner sums and then breaking the range of ν, we get

$$\Sigma_{31} = K \sum_{n>\tau} \frac{1}{n(n+1)} \left| \sum_{\nu=0}^{n-1} \frac{p_{\nu+1}}{P_\nu P_{\nu+1}} \sum_{k=0}^{\nu} (\nu - k + 1) p_k \exp\{i(\nu - k + 1)t\} \right|$$

$$\leqq K \sum_{n>\tau} \frac{1}{n(n+1)} \sum_{\nu=0}^{\tau-1} \frac{R_{\nu+1}}{P_\nu} \sum_{k=0}^{\nu} p_k$$

$$+ K \sum_{n>\tau} \frac{1}{n(n+1)} \sum_{\nu=\tau}^{n-1} \frac{R_{\nu+1}}{P_\nu} \max_{0 \leqq \rho \leqq \nu} \left| \sum_{k=0}^{\rho} p_k \exp\{i(\nu - k + 1)t\} \right| \, ,$$

(5.10) by Abel's Lemma,

$$\leqq K \sum_{n>\tau} \frac{1}{n(n+1)} \sum_{\nu=0}^{\tau-1} 1 + K P_\tau \sum_{n>\tau} \frac{1}{n(n+1)} \sum_{\nu=\tau}^{n-1} \frac{1}{P_\nu} \, ,$$

by virtue of hypothesis (i) and Lemma 1,

$$\leqq K\tau \sum_{n>\tau} \frac{1}{n(n+1)} + K P_\tau \sum_{\nu=\tau}^{\infty} \frac{1}{P_\nu} \sum_{n=\nu+1}^{\infty} \frac{1}{n(n+1)}$$

$$\leqq K + K P_\tau \sum_{\nu=\tau}^{\infty} \frac{1}{P_\nu(\nu+1)} \leqq K \, ,$$

by the hypothesis (ii).

Similarly,

$$\Sigma_{32} = K \sum_{n>\tau} \frac{1}{n(n+1)} \left| \sum_{\nu=0}^{n-1} \frac{1}{P_{\nu+1}} \sum_{k=0}^{\nu} p_k \exp\{i(\nu - k + 1)t\} \right|$$

$$\leqq K \sum_{n>\tau} \frac{1}{n(n+1)} \sum_{\nu=0}^{\tau-1} \frac{P_\nu}{P_{\nu+1}}$$

$$(5.11) \qquad + K \sum_{n>\tau} \frac{1}{n(n+1)} \sum_{\nu=\tau}^{n-1} \frac{1}{P_{\nu+1}} \left| \sum_{k=0}^{\nu} p_k \exp\{i(\nu - k + 1)t\} \right|$$

$$\leqq K\tau \sum_{n>\tau} \frac{1}{n(n+1)} + K \sum_{n>\tau} \frac{1}{n(n+1)} \sum_{\nu=\tau}^{n-1} \frac{P_\tau}{P_{\nu+1}}$$

$$\leqq K \, ,$$

by the technique used in showing (5.10).

Applying Abel's Lemma, we get

$$\Sigma_{33} = K \sum_{n>\tau} \frac{1}{(n+1)nP_n} \left| \sum_{k=0}^{n-1} p_k(n-k+1) \exp\{i(n-k)t\} \right|$$

$$\leqq K \sum_{n>\tau} \frac{1}{nP_n} \max_{0 \leqq \rho \leqq n-1} \left| \sum_{k=0}^{\rho} p_k \exp\{i(n-k)t\} \right| \tag{5.12}$$

$$\leqq KP_\tau \sum_{n>\tau} \frac{1}{nP_n} \leqq K ,$$

by virtue of Lemma 1 and the hypothesis (ii).

Finally,

$$\Sigma_{34} \leqq K \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{k=0}^{n-1} \frac{p_k}{P_k}$$

$$= K \sum_{k=0}^{\infty} \frac{p_k}{P_k} \sum_{n=k+1}^{\infty} \frac{1}{n(n+1)} \tag{5.13}$$

$$\leqq K \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} \leqq K ,$$

since $(k+1)p_k \leqq P_k$.

Combining (5.9)-(5.13), we get that $\Sigma_3 \leqq K$. This result combined with (5.6) and (5.7) proves (5.4) and *a fortiori* (5.2).

Lastly, in order to establish (5.3) we have to show that uniformly in $0 < t \leqq \pi$,

$$\Sigma' \equiv \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left| \sum_{\nu=1}^{n} \frac{\nu}{P_\nu P_{\nu-1}} \sum_{k=0}^{\nu-1} (P_\nu p_k - p_\nu P_k) \frac{\sin(\nu-k)t}{(\nu-k)} \right| \leqq K .$$

Now

$$\Sigma' \leqq \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{\nu=1}^{n} \frac{\nu}{P_\nu P_{\nu-1}} \left| \sum_{k=0}^{\nu-1} (P_\nu p_k - p_\nu P_k) \frac{\sin(\nu-k)t}{\nu-k} \right|$$

$$= \sum_{\nu=1}^{\infty} \frac{\nu}{P_\nu P_{\nu-1}} \left| \sum_{k=0}^{\nu-1} (P_\nu p_k - p_\nu P_k) \frac{\sin(\nu-k)t}{\nu-k} \right| \sum_{n=\nu}^{\infty} \frac{1}{n(n+1)}$$

$$\leqq \sum_{\nu=1}^{\infty} \frac{1}{P_\nu P_{\nu-1}} \left| \sum_{k=0}^{\nu-1} (P_\nu p_k - p_\nu P_k) \frac{\sin(\nu-k)t}{\nu-k} \right|$$

$$\leqq K ,$$

by virtue of Lemma 2.

This completes the proof of Theorem 1.

6. Proof of Theorem 2. As in Pati [11], for the conjugate series (1.6), we have

$$t_n - t_{n-1} = \frac{2}{\pi} \int_0^\pi \psi(t) \left\{ \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k) \sin(n-k)t \right\} dt$$

and therefore,

$$n^{-1}u_n^1 = \frac{2}{\pi}\int_0^\pi \psi(t)\Big\{\frac{1}{n(n+1)}\sum_{\nu=1}^{n}\frac{\nu}{P_\nu P_{\nu-1}}\sum_{k=0}^{\nu-1}(P_\nu p_k - p_\nu P_k)\sin(\nu-k)t\Big\}dt .$$

Thus

$$\begin{aligned}&\sum_n n^{-1}|u_n^1| \\ &= \frac{2}{\pi}\sum_n\Big|\int_0^\pi \psi(t)\Big\{\frac{1}{n(n+1)}\sum_{\nu=1}^{n}\frac{\nu}{P_\nu P_{\nu-1}}\sum_{k=0}^{\nu-1}(P_\nu p_k - p_\nu P_k)\sin(\nu-k)t\Big\}dt\Big| \\ &\leqq \int_0^\pi \frac{|\psi(t)|}{t}\Big\{t\sum_n \frac{1}{n(n+1)}\Big|\sum_{\nu=1}^{n}\frac{\nu}{P_\nu P_{\nu-1}} \\ &\quad\times\sum_{k=0}^{\nu-1}(P_\nu p_k - p_\nu P_k)\sin(\nu-k)t\Big|\Big\}dt .\end{aligned}$$

Hence by virtue of the condition that $\int_0^\pi t^{-1}|\psi(t)|\,dt \leqq K$, it is sufficient to show that uniformly in $0 < t \leqq \pi$

$$t\sum_n \frac{1}{n(n+1)}\Big|\sum_{\nu=1}^{n}\frac{\nu}{P_\nu P_{\nu-1}}\sum_{k=0}^{\nu-1}(P_\nu p_k - p_\nu P_k)\sin(\nu-k)t\Big| \leqq K ,$$

in order to prove the theorem.

This follows directly from (5.4), which has been proved in the previous section. This completes the proof of the theorem.

7. Proof of Theorem 3. Since

$$nB_n(x) = \frac{2}{\pi}\int_0^\pi \psi(t)n\sin nt\,dt ,$$

therefore for the sequence $\{nB_n(x)\}$, we have

$$t_n - t_{n-1} = \frac{2}{\pi}\sum_{k=0}^{n}\Big(\frac{p_k}{P_n} - \frac{p_{k-1}}{P_{n-1}}\Big)\int_0^\pi \psi(t)(n-k)\sin(n-k)t\,dt .$$

Hence

$$\begin{aligned}\sum_n n^{-1}|u_n^1| = \frac{2}{\pi}\sum_n \frac{1}{n(n+1)}&\Big|\sum_{\nu=1}^{n}\nu\sum_{k=0}^{\nu}\Big(\frac{p_k}{P_\nu} - \frac{p_{k-1}}{P_{\nu-1}}\Big) \\ &\times\int_0^\pi \psi(t)(\nu-k)\sin(\nu-k)t\,dt\Big| .\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}\sum_n n^{-1}|u_n^1| &= \frac{2}{\pi}\sum_n \frac{1}{n(n+1)}\Big|\sum_{\nu=1}^{n}\nu\sum_{k=0}^{\nu}\Big(\frac{p_k}{P_\nu} - \frac{p_{k-1}}{P_{\nu-1}}\Big) \\ &\qquad\times\int_0^\pi\{\cos(\nu-k)t - 1\}d\psi(t)\Big| \\ &\leqq \int_0^\pi\Big\{\sum_n \frac{1}{n(n+1)}\Big|\sum_{\nu=1}^{n}\nu\sum_{k=0}^{\nu}\Big(\frac{p_k}{P_\nu} - \frac{p_{k-1}}{P_{\nu-1}}\Big) \\ &\qquad\times\{\cos(\nu-k)t - 1\}\Big|\Big\}|d\psi(t)| .\end{aligned}$$

And since by hypothesis $\int_0^\pi |d\psi(t)| \leqq K$, to prove that $\sum_n n^{-1}|u_n^1| \leqq K$, it is enough to show that uniformly in $0 < t \leqq \pi$,

$$\sum_n \frac{1}{n(n+1)}\left|\sum_{\nu=1}^{n}\nu\sum_{k=0}^{\nu}\left(\frac{p_k}{P_\nu}-\frac{p_{k-1}}{P_{\nu-1}}\right)\{\cos(\nu-k)t-1\}\right| \leqq K\,.$$

Applying the Abel's transformation to the inner sum, we get

$$\begin{aligned}\sum_{k=0}^{\nu}&\{\cos(\nu-k)t-1\}\left(\frac{p_k}{P_\nu}-\frac{p_{k-1}}{P_{\nu-1}}\right)\\ &=\sum_{k=0}^{\nu-1}\Delta_k\{\cos(\nu-k)t\}\left(\frac{P_k}{P_\nu}-\frac{P_{k-1}}{P_{\nu-1}}\right)\\ &=-\frac{2\sin(t/2)}{P_\nu P_{\nu-1}}\sum_{k=0}^{\nu-1}(P_\nu p_k-p_\nu P_k)\sin\left(\nu-k-\frac{1}{2}\right)t\,.\end{aligned}$$

Therefore, in order to prove our theorem, it is sufficient to show that uniformly in $0 < t \leqq \pi$,

$$\begin{aligned}|\sin(t/2)|\sum_n\frac{1}{n(n+1)}&\left|\sum_{\nu=1}^{n}\frac{\nu}{P_\nu P_{\nu-1}}\right.\\ &\times\left.\sum_{k=0}^{\nu-1}(P_\nu p_k-p_\nu P_k)\sin\left(\nu-k-\frac{1}{2}\right)t\right| \leqq K\,,\end{aligned}$$

which follows directly from (5.4).

This completes the proof of Theorem 3.

My warmest thanks are due to Professor T. Pati, of the University of Jabalpur, for his kind interest in the preparation of this paper.

References

1. Max Astrachan, *Studies in the summability of Fourier series by Nörlund means*, Duke Math. J. **2** (1936), 543-568.
2. L. S. Bosanquet, *The absolute Cesàro summability of a Fourier series*, Proc. London Math. Soc. **41** (1936), 517-528.
3. L. S. Bosanquet and J. M. Hyslop, *On the absolute summability of the allied series of a Fourier series*, Math. Z. **42** (1937), 489-512.
4. H. P. Dikshit, *A note on a theorem of Astrachan on the $(N, p_n)\cdot(C,1)$ summability of Fourier series*, Math. Student **33** (1964), 77-79.
5. ———, *A generalisation of some recent theorems on the absolute Nörlund summability of a Fourier series*, communicated.
6. E. Kogbetliantz, *Sur les séries absolument sommable par la méthode des moyennes arithmétique*, Bull. des Sc. Math. **49** (1925), 234-256.
7. L. McFadden, *Absolute Nörlund summability*, Duke Math. J. **9** (1942), 168-207.
8. F. M. Mears, *Some multiplication theorems for the Nörlund means*, Bull. Amer. Math. Soc. **41** (1935), 875-880.
9. N. E. Nörlund, *Sur une application des fonctions permutables*, Lunds Universitets Årsskrift (2) **16** (1919), No. 3.

10. T. Pati, *On the absolute Nörlund summability of a Fourier series,* J. London Math. Soc. **34** (1959), 153–160.
11. ———, *On the absolute Nörlund summability of the conjugate series of a Fourier series,* J. London Math. Soc. **38** (1963), 204–213.
12. ———, *On the absolute summability of Fourier series by Nörlund means,* Math. Z. **88** (1965), 244–249.
13. T. Singh, *Absolute Nörlund summability of Fourier series,* Indian J. Math. **6** (1964), 129–136.
14. O. P. Varshney, *On the absolute Nörlund summability of a Fourier series,* Math. Z. **83** (1964), 18–24.

Received June 10, 1967.

UNIVERSITY OF ALLAHABAD
ALLAHABAD, INDIA