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ALGEBRAIC PROPERTIES OF CERTAIN RINGS OF CONTINUOUS FUNCTIONS

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Let X and Y be any subsets of E^n, and (X', d_1) and (Y', d_2) be any metric spaces. Let $C^m(X)$, $0 \leqq m \leqq \infty$, denote the ring of m-differentiable functions on X, and $L_c(X')$ be the ring of the functions which are Lipschitzian on each compact subset of X', and $L(X')$ be the ring of the bounded Lipschitzian functions on X'. The relations between algebraic properties of $C^m(X)$, (resp. $L_c(X')$ or $L(X')$ and the topological properties of X (resp. X') are studied. It is proved that if X and Y, (resp. (X', d_1) and (Y', d_2)) are m-realcompact, (resp. L_c-real-compact or compact) then $C^m(X) \cong C^m(Y)$ (resp. $L_c(X') \cong L_c(Y')$ or $L(X') \cong L(Y')$ if and only if X and Y are C^m-diffeomorphic (resp. (X', d_1) and (Y', d_2) are L_c or L-homeomorphic).

During the last twenty years, the relations between the algebraic properties of $C^m(X)$ and $C^m(Y)$ and the topological properties of X and Y have been investigated by Hewitt [4], Myers [9], Pursell [11], Nakai [10], and Gillman and Jerison [3], where m is a positive integer, zero or infinite. In 1963, Sherbert [12] studied the ring $L(X)$. Recently, Magill, [6] has obtained the algebraic condition relating $C(X)$ and $C(Y)$ (i. e., $m = 0$) which are both necessary and sufficient for embedding Y in X, where X and Y are two realcompact spaces.

This work is to utilize the method of Gillman and Jerison [3] for studying the algebraic properties of $C^m(X)$ and $L_c(X_1)$ (§§ 2–5), and how they are related with topological properties of X and X_1 respectively. In view of [8, Cor. 1.32], we will restrict X in $C^m(X)$ to a subset of E^n. The results of Magill are also true in $C^m(X)$ and $L_c(X)$ with some modification. In the last section, § 6, we observe some other cases.

2. Rings and ideals. Let X be an arbitrary subset of E^n, an n-dimensional euclidean space, and $C^m(X)$ be the set of all real-valued functions of class C^m in the sense of Whitney [14, § 3], where m will always refer to an arbitrary integer such that $0 \leqq m \leqq \infty$. By [15, Th. 4], we know that $C^m(X)$ forms a ring with the identity u, the constant function of value 1, and zero element θ, the constant function of value 0. Let $C^{m*}(X) = \{f \in C^m(X) : f \text{ is bounded}\}$. It is clear that $C^{m*}(X)$ is a subring of $C^m(X)$ with u and θ. Let X be a metric space, and $L_c(X)$ be the set of all real-valued functions satisfying Lipschitz condition on each compact subset of X [2, p. 354]. We can easily show that $L_c(X)$ is a ring with u and θ. Let $L(X) =$

$\{f \in L_c(X) : f$ is bounded and Lipschitzian on entire $X\}$, $L_c^*(X) = \{f \in L_c(X) : f$ is bounded$\}$. Then, both $L(X)$ and $L_c^*(X)$ are the subrings of $L_c(X)$ with u and θ.

Since the properties of $C^m(X)$ (resp. $C^{m*}(X)$) and those of $L_c(X)$ (resp. $L_c^*(X)$ and $L(X)$) are almost all the same, we will use $\mathfrak{A}$ and $\mathfrak{A}'$ to denote $C^m(X)$ (resp. $L_c(X)$) and $C^m(Y)$ (resp. $L_c(Y)$), and $\mathfrak{B}$ and $\mathfrak{B}'$ to denote $C^{m*}(X)$ (resp. $L_c^*(X)$, and $L(X)$) and $C^{m*}(Y)$ (resp. $L_c^*(Y)$ and $L(Y)$) respectively, where X, and Y are appropriately the subsets of E^n or metric space. Also " a- " and " b- " will mean m- (or C^m-) (resp. L_c) and C^{m*}- (resp. L_c^* and L) respectively according as $\mathfrak{A}$ is $C^m(X)$ (resp. $L_c(X)$) and $\mathfrak{B}$ is $C^{m*}(X)$ (resp. $L_c^*(X)$, and $L(X)$).

The unit element of an $f \in \mathfrak{A}$ or $\mathfrak{B}$ is defined as usual. For $f \in \mathfrak{A}$, $Z(f) = \{x \in X : f(x) = 0\}$ is said to be the zero-set of f. $Z(\mathfrak{A}) = \{Z(f) : f \in \mathfrak{A}\}$. It is then clear that $f \in \mathfrak{A}$ is a unit if and only if $Z(f) = \emptyset$. (For $C^m(X)$ see [15, Th. 4].) Likewise, if $f \in \mathfrak{B}$ is a unit, then $Z(f) = \emptyset$. But the converse need not hold, for the multiplicative inverse $1/f$ of f in $\mathfrak{A}$ may not be a bounded function. For example: let $X = E^1$, and $f(x) = e^{-x^2} \in C^{m*}(E^1)$ and $Z(f) = \emptyset$. But $1/f = e^{x^2} \notin C^{m*}(E^1)$.

A z-filter of $Z(\mathfrak{A})$ is the same as in [3,2.2]. It is obvious that $Z[I] = \{Z(f) : f \in I\}$ is a z-filter on X if I is a proper ideal in $\mathfrak{A}$, and $Z^{-1}[\mathscr{F}] = \{f \in \mathfrak{A} : Z(f) \in \mathscr{F}\}$ is a proper ideal if $\mathscr{F}$ is a z-filter on X. Note that it may be false that a proper ideal $I \subset \mathfrak{B}$ implies that $Z[I]$ is a z-filter. For example: let us consider $\mathfrak{B} = C^{m*}(E^1)$ and let $f(x) = 1/(1 + x^2)$, and $I = (f)$ be the ideal generated by f in $\mathfrak{B}$. Then it is clear that $\emptyset \in Z[I]$.

Hereafter, we will always use " ideal " to mean the proper ideal, unless the contrary is mentioned.

Accordingly, every z-filter is of the form $Z[I]$, for some ideal I in $\mathfrak{A}$. That $Z^{-1}[Z[I]] \supset I$ is also clear. The inclusion may be proper.

For instance, consider $\mathfrak{A} = C^m(E^1)$. (a) For any positive integer m, let $i(x) = x$ for all $x \in E^1$, and $I = (i)$. Then

$$Z^{-1}[Z[I]] = M_0 = \{f \in C^m(E) : f(0) = 0\} .$$

However, $i^{(3m+1)/3} \in M_0 - I$. (b) In case $m = \infty$, let $f_1(x) = e^{-1/x^2}$ for $x \in E^1$ and $I_1 = (f_1)$. Then $M_0 = Z^{-1}[Z[I]]$ contains an element $i \notin I_1$. Note that M_0 is a maximal fixed ideal. Now, as for $L_c(X)$, we may consider (X, d) to be a bounded metric space, and $f_0(x) = (f_p(x))^2 = (d(p, x))^2$. Then $f_0 \in L_c(X)$. Let $I_0 = (f_0)$. Then

$$Z^{-1}[Z[I_0]] = \{f \in L_c(X) : f(p) = 0\} = M_p$$

is clear. However, $f_p(x) = d(p, x) \in M_p - I_0$.

A z-ultrafilter on X is a maximal z-filter [3,2.5]. We know that every subfamily of $Z(\mathfrak{A})$ with the finite intersection property, by Zorn's Lemma, is contained in some z-ultrafilter on X.

The proofs of following propositions are obvious.

PROPOSITION 2.1. If M is a maximal ideal in $\mathfrak{A}$, then $Z[M]$ is a z-ultrafilter on X.

PROPOSITION 2.2. If $\mathscr{A}$ is a z-ultrafilter on X, then $Z^{-1}[\mathscr{A}]$ is a maximal ideal in $\mathfrak{A}$.

It follows from Propositions (2.1) and (2.2) that the mapping Z is one-one from the set of all maximal ideals in $\mathfrak{A}$ onto the set of all z-ultrafilters on X.

PROPOSITION 2.3. Let M be a maximal ideal in $\mathfrak{A}$. If $Z(f)$ meets every member of $Z[M]$, then $f \in M$.

PROPOSITION 2.4. Let $\mathscr{A}$ be a z-ultrafilter on X. If a zero-set Z meets every member of $\mathscr{A}$, then $Z \in \mathscr{A}$.

An ideal I in $\mathfrak{A}$ is z-ideal if $Z(f) \in Z[I]$ implies $f \in I$. That is, $I = Z^{-1}[Z[I]]$, [3,2.7]. It is obvious that every maximal ideal is a z-ideal. A prime ideal is defined in the usual sense. The following theorem is only true for $L_c(X)$, $L_c^*(X)$ or $L(X)$. For we can show that these are lattice-ordered rings; while $C^m(X)$ and $C^{m*}(X)$ are not.

THEOREM 2.5. *For any z-ideal I in $L_c(X)$ ($L_c^*(X)$ or $L(X)$) the following are equivalent*:

(1) *I is prime.*

(2) *I contains a prime ideal.*

(3) *For all $g, h \in L_c(X)$ ($L_c^*(X)$ or $L(X)$), $g \cdot h = \theta$, then $g \in I$ or $h \in I$.*

(4) *For every $f \in L_c(X)$ ($L_c^*(X)$ or $L(X)$), there is a zero-set in $Z[I]$ on which f does not change sign.*

Proof is similar to [3,2.9].

3. Zero-set, α-completely regular and α-normal spaces. We know from the proof of Lemma 25 [16, p. 669] that each closed subset F of E^n, there is an $f \in C^m(X)$ such that $Z(f) = F$.

PROPOSITION 3.1. For each closed subset A of (X, d), there is $f \in L_c(X)$ (in fact $f \in L(X)$) such that $Z(f) = A$.

Proof. Let $g(x) = d(A, x)$ and $f = g \wedge u$[1]. Then $f \in L(X)$ and $Z(f) = A$.

DEFINITION 3.2. Let X be a topological space. X is said to be a-completely regular if and only if for each closed subset F of X and $x \notin F$, there is an $f \in \mathfrak{A}$ such that $f(x) = 1$, and $f[F] = \{0\}$.

THEOREM 3.3. *A topological space is a-completely regular if and only if the family* $Z(\mathfrak{A}) = \{Z(f) : f \in \mathfrak{A}\}$ *is a base for the closed subsets of* X.

Proof is similar to [3, 3.2].

DEFINITION 3.4. A topological space is said to be a-normal if for any disjoint closed subsets F_1 and F_2, there is an $f \in \mathfrak{A}$ such that $f[F_1] = \{0\}$ and $f[F_2] = \{1\}$.

PROPOSITION 3.5. Every subset X of E^n is m-normal. Hence is m-completely regular.

Proof. Let F_1 and F_2 be any two disjoint closed subsets of X. We know that there are closed subsets F_1' and F_2' of E^n such that $F_i = F_i' \cap X$, $i = 1, 2$. We know that there are $f_i \in C^m(E^n)$ with $Z(f_i) = F_i'$. Let $g_i = f_i \mid X$, and $f = g_1^2/(g_1^2 + g_2^2)$. Then $f \in C^m(X)$ and $Z(f) = Z(g_1) = F_1$, $f[F_2] = \{1\}$. The last part is obvious.

PROPOSITION 3.5′. Every metric space (X, d) is L-completely regular; and every compact metric space (X, d) is L-normal.

Proof. Let F be a closed subset of X and $p \in X - F$. Then $d(F, p) \neq 0$. Let f be defined as follows: $f[F] = \{1\}$, and $f(p) = 0$. Then f is bounded by 1 satisfies a Lipschitz condition with constant $K = (d(F, p))^{-1}$ on $F \cup \{p\}$. We know that there exists $f_0 \in L(X)$ such that $f_0 \mid F \cup \{p\} = f$, (by [7, p. 97]). Hence the first assertion follows. The proof of the second part is the same.

Note that the compactness in (3.5)′ cannot be omitted. For instance: let $X = E^2$,

$$F = \{(x, y) \in E^2 : xy = 1\} \quad \text{and} \quad F' = \{(x, y) \in E^2 : xy = -1\} .$$

Then F and F' are two disjoint closed sets in E^2. However, it is clear that there is no $f \in L(E^2)$ such that $f[F] = \{1\}$ and $f[F'] = \{0\}$.

[1] u stands for the constant function of value 1.

Having done these, we can show the characterization of fixed maximal ideal of $\mathfrak{A}$ and $\mathscr{B}$ and how they are related to a compact space are the same as in [3, (4.6), (4.8), (4.9) (a), (4.10) Lemma, and (4.11)].

4. Real ideals, α-realcompact space. In 1948, E. Hewitt defined real maximal ideals and realcompact space (Q-spaces) (see [4, § 7] and [3, Ch. 5]). He also contributed many interesting properties about real maximal ideals and realcompact spaces. Unfortunately, those properties can only be carried to the rings $L_c(X)$, $L_c^*(X)$ and $L(X)$, but not to $C^m(X)$, since $C^m(X)$ is not a lattice-ordered ring. (As $f \in C^m(X)$ implies $|f| \in C^m(X)$ is not always true.) Recently (1965), Rudolphe Bkouche has shown that every paracompact Hausdorff differentiable n-manifold is m-realcompact (see (4.2), and [1, Th. 2]). Here will show that every closed subset of E^n is m-realcompact.

We can show easily that every residue class field of $\mathfrak{A}$ or $\mathscr{B}$ module a maximal ideal contains canonical copy of real field $\boldsymbol{R}$. We can also show, by using [3, (5.1) to (5.4)] or [13], that L_c/M, L_c^*/M, and $L(X)/M$ are totally ordered for each maximal ideal M. We will show that $L_c^*(X)/M$ (resp. $L(X)/M) \cong \boldsymbol{R}$ if M is maximal in L_c^* (resp. $L(X)$). The real and hyper-real ideals are defined in [3, 5.9].

LEMMA 4.1. *Let M be a maximal ideal in $L_c^*(X)$, (resp. $L(X)$), and $L_c^*(X)$, (resp. $L(X)$) be normed by the* sup *norm $\|\cdot\|_\infty$. Then M is closed in $L_c^*(X)$ (resp. $L(X)$) under $\|\cdot\|_\infty$.*

Proof. In view of [3, 2 Ml], cl M is either a proper ideal of $L_c^*(X)$ or $L_c^*(X)$ itself. Suppose $\operatorname{cl} M = L_c^*(X)$. Then $u \in \operatorname{cl} M$, and for any neighborhood of u, $N_\varepsilon(u)$, $N_\varepsilon(u) \cap M \neq \varnothing$. Take $\varepsilon = 1/2$. Then $N_{1/2}(u) \cap M \neq \varnothing$. That is, there is an $f \in M$ such that

$$\|u - f\|_\infty < \frac{1}{2}\,.$$

This implies $|f(x)| > 1/2$ for each $x \in X$. We can easily show that $1/f \in L_c^*(X)$. That is, M has a unit so that $M = L_c^*(X)$. This is a contradiction. Hence cl M is a proper ideal containing M so that $M = \operatorname{cl} M$. The proof for M in $L(X)$ is similar.

PROPOSITION 4.2. For each maximal ideal M in $L_c^*(X)$, (resp. $L(X)$), $L_c^*(X)/M$ (resp. $L(X)/M) \cong \boldsymbol{R}$.

Proof. It is enough to show that for any positive nonconstant residue class $M(f)$, simply denoted by f, there is a positive integer n such that $f - 1/n$ is positive. (See [3, 5.6] and [13].) Suppose that there does not exist such a positive integer. Then we would have $f - 1/n$ is negative for all $n \in N$. That is, $(f - 1/n) + |f - 1/n| \in M$ for all n. Consider now the sequence $\{g_n = (f - 1/n) + f - 1/n : n \in N\}$ which has $f + |f|$ as the limit under the norm $\|\cdot\|_\infty$. By (4.1), $f + |f| \in \operatorname{cl} M = M$. This shows that $-f \equiv |f| \pmod{M}$. This is a contradiction.

DEFINITION 4.3. A topological space X is said to be a-realcompact if every real maximal ideal in $\mathfrak{A}$ is fixed.

It is clear that if X is compact, then X is a-realcompact.

LEMMA 4.4. *An ideal in $\mathfrak{A}$ is free if and only if for every compact subset A of X there exists an $f \in I$ having no zero in A.*

Proof. Suppose I is free and A is any compact subset of X. If for each $f \in I$, $Z(f) \cap A \neq \varnothing$, then $\mathscr{F} = \{Z(f) \cap A$: for some $f \in I\}$ has the finite intersection property. Since A is compact, $\cap \mathscr{F} \neq \varnothing$. Hence $\cap Z[I] \supseteq \cap \mathscr{F} \neq \varnothing$, which is impossible.

The sufficiency is clear.

PROPOSITION 4.5. Let X be a closed subspace of E^n. Then X is m-realcompact as well as L_c-realcompact.

Proof. Suppose that M is a free maximal ideal and $C^m(X)/M \cong \boldsymbol{R}$. Let $g(x) = 1/(\|x\|^2 + 1)$. Then that $g \in C^m(X)$ and g is a unit is clear. Hence $g \notin M$. i.e., $M(g) \neq 0$. For any positive number r and a sufficiently small number $\varepsilon > 0$, $g < r - \varepsilon$ for all but a compact subset of E^n, say A_ε. Then $B_\varepsilon = A_\varepsilon \cap X$ is compact in X as X is closed. Let $A' = \operatorname{cl}_X(X - B_\varepsilon)$. Then there is an $f \in C^m(X)$ such that $Z(f) = A'$. We will show that $Z(f) \in Z[M]$. We know that B_ε is compact in X. By (4.4), there is an $f_1 \in M$ such that $Z(f_1) \cap B_\varepsilon = \varnothing$. Hence $Z(f_1) \subseteqq X - B_\varepsilon \subseteqq Z(f)$ so that $Z(f) \in Z(M)$. Therefore, $g \leqq r - \varepsilon$ on the zero-set $Z(f)$, and $r - g \geqq \varepsilon$. Let $h_1 = (r - g)^{1/2}$ on $Z(f)$. Then h_1 is C^m on $Z(f)$ which is closed in E^n. By Whitney's Analytic Extension Theorem [14], we have a C^m extension h, i.e., $h \mid Z(f) = h_1$. Hence $h^2 = r - g$ on $Z(f)$. Therefore, $h^2 \equiv r - g \pmod{M}$. In other words, $M(h^2) = M(r - g) = M(r) - M(g) = r - M(g)$. But, since $C^m(X)/M$ is real $M(h^2) \geqq 0$, we have $M(g) \leqq r$. As r is any positive number, $M(g)$ is infinitely small. This is a contradiction. The proof of the last part is similar.

We now will give an example to show that a nonparacompact space may not be an m-realcompact space. However, the existence of non-L_e-realcompact spaces remains as an open question.

Let L be the long line as defined in [5]. Then, L is Hausdorff space satisfying the first axiom of countability. Furthermore we have:

PROPOSITION 4.6. For each $\alpha \in L$, $\alpha \neq 1$, $[1, \alpha]$ is isotonehomeomorphic to unit interval $[0, 1]$. Consequently, each point of L, not the first element, 1, has an open neighborhood which is homeomorphic to an open interval.

Proof. Use transinfinite induction.

PROPOSITION 4.7. L is countably compact but is not paracompact, hence is not a compact space.

Proof. Let A be any countably infinite subset of L. Then, A will be contained in the union of $\{I_\alpha : \alpha \in \Delta \subset W\}$, where W is the set of all ordinal numbers less than the first uncountable ordinal, and Δ has at most countably many elements. Let α_0 be the least upper bound of Δ. Then $[1, \alpha_0]$ is homeomorphic to $[0, 1]$. Hence $A \subset [1, \alpha_0]$, a compact set, must have a limit point in $[1, \alpha_0] \subset L$, so that L is countably compact. In view of (4.6), L is locally metrizable [5, p. 80] but L is not metrizable. By Theorem 2-68 [5, p. 81] L is not paracompact. Hence, it is not compact.

PROPOSITION 4.8. Of any two disjoint closed sets in L, one is bounded.

Proof is similar [3, 5.12 (b)].

By (4.6), we know that L is a 1-dimensional manifold with a boundary point 1. Hence we may define the differentiable function on L.

PROPOSITION 4.9. Every function $f \in C(L)$ is a constant on a tail $L - L(\alpha)$ where α depends on f, and $L(\alpha) = \{\sigma \in L : \sigma < \alpha\}$.

Proof is similar to [3, 5.12(c)].

Let L^* be the union space of L and the point Ω, the first uncountable ordinal. Then, L^* is a compact 1-dimensional manifold. For each $f \in C^m(L)$, we extend f to a function f^* on L^* by defining that $f^*(\Omega)$ is the final constant value of f. Evidently $f^* \in C^m(L^*)$ and is unique. On the other hand, for each $g \in C^m(L^*)$, $g \mid L \in C^m(L)$. Hence $C^m(L)$ is isomorphic with $C^m(L^*)$, under the mapping $f \to f^*$.

Since L^* is compact, every ideal is fixed, and the maximal ideals assume the form $M_\sigma = \{f^* \in C^m(L^*) : f^*(\sigma) = 0\}$, where $\sigma \in L^*$. By virtue of isomorphism of $C^m(L^*)$ with $C^m(L)$, the maximal ideals in $C^m(L)$ are in one-one correspondence with those of $C^m(L^*)$. Moreover, the fixed maximal ideals in $C^m(L)$ correspond to the ideals M_σ in $C^m(L^*)$ for each $\sigma \in L$, leaving just one free maximal ideal in $C^m(L)$, namely, $M_0 = \{f \in C^m(L) : f^* \in M_\Omega\}$, the one that corresponds to M_Ω. Though M_0 is free, it is not hyper-real, for $C^m(L)/M_0 \cong C^m(L^*)/M \cong \boldsymbol{R}$. Hence L is not m-realcompact.

5. Homomorphism, α-mapping and α-homeomorphism.

In this section we will describe the relation between any α-mapping from X into Y and homomorphisms from $\mathfrak{A}'$ to $\mathfrak{A}$.

DEFINITION 5.1. Let $X \subset E^{n_1}$, and $Y \subset E^{n_2}$. A mapping $\tau : X \rightarrow Y$ is said to be a C^m-mapping at a point p, if each component of $\tau(x) = (\tau_1(x_1, \cdots, x_{n_1}), \cdots, \tau_{n_2}(x_1, \cdots, x_{n_1}))$ is C^m [14, § 3] at p. If τ is C^m at each point of X, then τ is said to be a C^m-mapping on X. If τ is a C^m-mapping, one-one, onto Y and its inverse mapping, τ^{-1}, is also a C^m-mapping, then τ is C^m-diffeomorphism. We will say then X and Y are C^m-diffeomorphic.

Note that by (5.1), X and Y are C^m-diffeomorphic implies $n_1 = n_2$.

DIFINITION 5.1.′ A mapping τ from (X, d_1) to (X, d_2) is said to be an L_c-(resp. L-) mapping if, for each compact subset A of X, there is a positive number K_A such that $d_2(\tau(x), \tau(x')) \leqq K_A d_1(x, x')$ for all $x, x' \in A$. (resp. if there is a positive number K such that

$$d_2(\tau(x), \tau(x')) \leqq Kd(x, x')$$

for all $x, x' \in X$). τ is said to be an L_c-(resp. L-) homeomorphism, if τ is one-one, onto Y and both τ and its inverse τ^{-1} are L_c-(resp. L-) mappings.

We will use "α-mapping" to mean C^m-mapping, L_c-or L-mapping, and "α-homeomorphism" to mean C^m-diffeomorphism, L_c- or L-homeomorphism according as $\mathfrak{A}$ is $C^m(X)$, $L_c(X)$ or $\mathscr{B} = L(X)$.

DEFINITION 5.2. An $f \in C^m(X)$ is said to be a local i-th projection at a point p if there exists a neighborhood U of p such that $f \mid U = i$, where i always denotes the i-th projection of the space E^n or $X \subset E^n$.

LEMMA 5.3. *Let X be any subset of E^n. For each $p \in X$ and $r > 0$, there are h_i, $(1 \leqq i \leqq n)$, $h_i \in C^{\infty *}(X)$ such that $h_i(x) = x_i$ for all $x \in \mathrm{cl}_X B_r(p)$. We call h_i, $(1 \leqq i \leqq n)$ the i-th bounded local projection at p.*

Proof. Choose $r' > r$. It is well-known that there exists $g \in C^{\infty *}(E^n)$ such that

$$g(x) = \begin{cases} 1 \text{ if } x \in \mathrm{cl}_{E^n} B_r(p) \\ 0 \text{ if } x \in E^n - B_{r'}(p) \\ 0 < g(x) < 1, \text{ elsewhere} . \end{cases}$$

Set $h_i(x) = i(x) \cdot g(x)$.

Let C_0^m be a subset of $C^m(Y)$ (resp. $C^{m*}(Y)$), and τ be a mapping from X to Y. Then we will see what C_0^m should be in order that $g \cdot \tau \in C^m(X)$ (resp. $C^{m*}(X)$) for all $g \in C_0^m$ implies τ is a C^m-mapping from X into Y.

THEOREM 5.4. *Let τ be a mapping from X to Y and C_0^m be a subset of $C^m(Y)$.*

(1) *τ is a C^m-mapping implies $g \cdot \tau \in C^m(x)$ for all $g \in C_0^m$.*

(2) *If $g \cdot \tau \in C^m(X)$ for each $g \in C_0^m$, and C_0^m includes all projections of X, then τ is a C^m-mapping on X.*

Proof. (1) It is obvious. (2) Since $g \cdot \tau \in C^m(X)$ for each $g \in C_0^m$ which includes all projections on X, we have, in particular, $i \cdot \tau(x) = \tau_i(x) \in C^m(X)$ for $1 \leqq i \leqq n_2$. Hence, by (5.1) τ is a C^m-mapping.

THEOREM 5.4.* *Let τ be a mapping from X to Y and C_0^m be a subset of $C^{m*}(Y)$.*

(1) *τ is C^m-mapping implies $g \cdot \tau \in C^{m*}(X)$ for all $g \in C_0^m$,*

(2) *If $g \cdot \tau \in C^m(X)$ for each $g \in C_0^m$, and C_0^m includes all local projections, then τ is a C^m-mapping on X.*

The proof is similar to (5.4)

THEOREM 5.4.′ *Let τ be a mapping from a metric space (X, d_1) to another metric space (Y, d_2).*

(1) *If τ is an L_c-mapping, then $f \cdot \tau \in L_c(X)$ for all $f \in L_c(Y)$.*

(2) *If $f \cdot \tau \in L_c(X)$ for all $f \in L_c(Y)$, then τ is an L_c-mapping of (X, d_1) into (Y, d_2).*

Proof. (1) is clear. (2) Consider any compact subset $A \neq \varnothing$ of X. We will show that τ is an L-mapping on A. By [3, 3.8] we know τ is continuous. Hence $\tau[A]$ is compact. Let Φ be a mapping from $L_c(Y)$ to $L_c(X)$ defined by $\Phi(f) = f \cdot \tau$ for all $f \in L_c(Y)$. Then, it is obvious that Φ is a homeomorphism of $L_c(Y)$ into $L_c(X)$. We restrict Φ to $L_c(Y) \mid \tau[A] = \{f \mid \tau[A] : f \in L_c(Y)\}$, then Φ is into

$$L_c(X) \mid A = \{g \mid A : g \in L_c(X)\} .$$

By compactness of A and the fact that every function which is Lipschitzian on a nonempty subset of a space can be extended to the whole space [7], we can show that $L_c(X) \mid A = L(A)$, and

$$L_c(Y) \mid \tau[A] = L(\tau[A]) .$$

By [12, 5.1] τ is an L-mapping on A. Since A is arbitrary, τ is an L_c-mapping.

The induced mapping is defined in [3, 10.2]. We are concerned with an a-mapping τ of X into Y, where the role of D in [3, 10.2] is taken by E^1. The appropriate subset of E^{1^Y} will be $\mathfrak{A}'$ or $\mathscr{B}'$. Evidently, the induced mapping τ', defined by $\tau'(g) = g\cdot\tau \in \mathfrak{A}$ for each $g \in \mathfrak{A}'$ (resp. $\mathscr{B}'$) is a homomorphism from $\mathfrak{A}'$ to $\mathfrak{A}$ (resp. $\mathscr{B}'$ into $\mathscr{B}$), and τ carries the constant functions onto the constant functions identically. Moreover, τ' determines the mapping τ uniquely.

We now examine the duality relation between τ and τ'.

Definition 5.5. A subset A of $X \subset E^n$ is C^m (resp. C^{m*})-embedded in X if for each $f \in C^m(A)$ (resp. $C^{m*}(A)$) there is $g \in C^m(X)$ (resp. $C^{m*}(X)$) such that $g \mid A = f$.

Definition 5.5.′ A subset A of a metric space (X, d) is L_c (resp. L_c^*, or L)-embedded in X if for each $f \in L_c(A)$ (resp. $L_c^*(A)$, or $L(A)$), there is $g \in L_c(X)$ (resp. $L_c^*(X)$, or $L(X)$) such that $g \mid A = f$.

We will simply say that a subset A of a topological space is a (resp. b)-embedded if A is C^m, or L_c (resp. C^{m*}, L_c^*, or L)-embedded.

Theorem 5.6. *Let τ be an a-mapping from X into Y, and τ' be the induced homomorphism $g \to g\cdot\tau$ from $\mathfrak{A}'$ into $\mathfrak{A}$* (resp. *$\mathscr{B}'$ into $\mathscr{B}$*).

(1) *τ' is an isomorphism (into) if and only if $\tau[X]$ is dense in Y.*

(2) *τ' is onto if and only if τ is an a-homeomorphism whose image is a* (resp. *b*)*-embedded.*

Proof. Having (5.4), (5.4)*, (5.4)′ and (5.3) in hand, the proof is similar to [3, 10.3].

Corollary 5.7. *If τ is an a-homeomorphism from X onto Y, then τ' is an isomorphsim of $\mathfrak{A}'$ onto $\mathfrak{A}$.*

Corollary 5.8. *If τ is an a-homeomorphism of a compact space X to Y, then the induced mapping τ' is onto.*

Proof. Use the theorem of Whitney's analytic extension [14] and the fact that each Lipschitzian function on a subset can be extended to the whole space [7]. The proof is evident.

Next, we examine the inverse problem of determining when a given homomorphism of $\mathfrak{A}'$ into $\mathfrak{A}$ is induced by some a-mapping from X into Y. We shall first consider the homomorphism from $\mathfrak{A}$ into $\boldsymbol{R}$, i.e., the case in which X consists of just one point.

PROPOSITION 5.9. Any nonzero homomorphism Φ from $\mathfrak{A}'$(or $\mathscr{B}'$) into $\boldsymbol{R}$ is onto $\boldsymbol{R}$. In fact $\Phi(r) = r$ for all $r \in \boldsymbol{R}$.

Proof is similar to [3, 10.5(a)].

PROPOSITION 5.10. The correspondence between the homomorphism of $\mathfrak{A}'$(or $\mathscr{B}'$) onto $\boldsymbol{R}$, and the real maximal ideals is one-one.

Proof is similar to [3, 10.5(b)].

PROPOSITION 5.11. Y is a-realcompact if and only if to each nonzero homomorphism Φ from $\mathfrak{A}$ onto $\boldsymbol{R}$, there corresponds a unique point y of Y such that $\Phi(g) = g(y)$ for all $g \in \mathfrak{A}'$.

Proof. Use (5.10) and a-realcompactness.

Our first result about homomorphisms from $\mathfrak{A}'$ into $\mathfrak{A}$ for X is a generalization of (5.11).

THEOREM 5.12. *Let Φ be a homomorphism from $\mathfrak{A}'$ into $\mathfrak{A}$ such that $\Phi(u) = u$. If Y is a-realcompact, then there exists a unique a-mapping τ of X into Y such that $\tau' = \Phi$.*

Notice that the condition $\Phi(u) = u$ is necessary. Proof of the theorem is similar to [3, 10.6].

COROLLARY 5.13. *An a-realcompact space Y contains an image of an a-mapping of X if and only if $\mathfrak{A}$ contains a homomorphic image of $\mathfrak{A}'$ that included the constant functions on X.*

Proof is similar to [3, 10.9(a)].

COROLLARY 5.14. *An a-realcompact space Y contains an image of an a-mapping which is dense in Y if and only if $\mathfrak{A}$ contains an isomorphic image of $\mathfrak{A}'$ that includes the constant functions on X.*

Proof is similar to [3, 10.9(b)].

THE MAIN THEOREM. *Two a-realcompact spaces X and Y are a-homeomorphic if and only if $\mathfrak{A}$ and $\mathfrak{A}'$ are isomorphic.*

Proof. The necessity follows from (5.7).

Sufficiency. Let Φ be an isomorphism of $\mathfrak{A}'$ onto $\mathfrak{A}$. Then Φ^{-1} is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$. By (5.12), there exist unique a-mappings τ and τ_1 from X into Y and from Y into X, respectively, such that $\Phi(g) = g\cdot\tau$, and $\Phi^{-1}(f) = f\cdot\tau_1$, for each $g \in \mathfrak{A}'$ and $f \in \mathfrak{A}$. Then, $g(y) = \Phi^{-1}(g\cdot\tau)(y) = (g\cdot\tau)\cdot\tau_1(y) = g\cdot(\tau\cdot\tau_1)(y)$ for all $y \in Y$. That is, $\tau\cdot\tau_1$ is the identity mapping of Y onto itself. Similarly, $\tau_1\cdot\tau$ is the identity mapping of X onto itself. Thus τ and τ_1 are the inverse mappings of each other. Hence X and Y are a-homeomorphic.

REMARK. S.B. Myers [9], L.E. Pursell [11], and M. Nakai [10] have dealt with C^m-differentiable n-manifolds. This theorem is applicable to any closed subset of E^n.

In spite of the remark made in (5.12), every homomorphism is induced, in essence, by an a-mapping.

THEOREM 5.16. *Let Φ be a homomorphism from $\mathfrak{A}'$* (resp. $\mathscr{B}'$) *into $\mathfrak{A}$* (resp. $\mathscr{B}$), *Y be a-realcompact* (resp. *compact*). *Then the set $E = \{x \in X : \Phi(u)(x) = 1\}$ is open-and-closed in X. Moreover, there exists a unique a-mapping τ from E into Y, such that for any $g \in \mathfrak{A}'$* (resp. $\mathscr{B}'$) *$\Phi(g)(x) = g(\tau(x))$ for all $x \in E$, and $\Phi(g)(x) = 0$ for all $x \in X - E$.*

Proof is similar to [3, 10.8].

COROLLARY 5.17. *Let Φ be a homomorphism from $\mathfrak{A}'$ into a subring R of $\mathfrak{A}$. If Y is a-realcompact, then there exists a unique closed subset F of Y such that the kernel of Φ is the z-ideal of all functions in R that vanish on F.*

Proof. Let $E = \{x \in X : \Phi(u)(x) = 1\}$. By (5.16), there exists an a-mapping τ from E into Y such that $\Phi(g)(x) = g(\tau(x))$ for all $x \in E$, and $\Phi(g)(x) = 0$ for all $x \in X - E$, for all $g \in \mathfrak{A}'$. Let $F = \mathrm{cl}_Y\tau[E]$, and $I = \{g \in \mathfrak{A}' : Z(g) \supseteq F\}$. We can show easily that $\ker\Phi = I$. The uniqueness of F is clear.

PROPOSITION 5.18. An a-realcompact (resp. compact) space Y contains an image of an a-mapping of X which is a (resp. b)-embedded if and only if $\mathfrak{A}$ (resp. $\mathscr{B}$) is a homomorphism image of $\mathfrak{A}'$ (resp. $\mathscr{B}'$).

Proof is similar to [3, 10.9(c)].

REMARK. With the previous results, one can show without any difficulty that the Theorems (2.1), (2.3) (2.4), (2.5) and (2.6) of [6] are true if $C(X)$ is replaced by $C^m(X)$ (resp. $L_c(X)$) with the condition that $\Phi(C^m(X))$ contains all projections or all local projections of $C^m(Y)$ (resp. $\Phi(L_c(X)) = L_c(Y)$). However, in Theorem (2.2) [6], only the first three statements are equivalent. For Φ is not a lattice homomorphism (see [3, 0.5]) from $C^m(Y)$ to $C^m(X)$ (resp. $L_c(Y)$ to $L_c(X)$).

6. Remarks. We have shown that if X and Y are two α-realcompact spaces, then $\mathfrak{A}$ and $\mathfrak{A}'$ are isomorphic if and only if X and Y are α-homeomorphic. We shall make some observations about other cases.

Let X be a subset of E^n, and S_1 be the set of the projections and the constant functions on X. Let S_2 be the ring generated by S_1, and $\mathscr{R}(X) = \{f/g : f, g \in S_2 \text{ and } Z(g) = \emptyset\}$. Evidently $\mathscr{R}(X)$ is a commutative ring of rational functions on X with unity u and zero element θ. A ring of functions, $A(X)$, is said to satisfy property (6–1), if $\mathscr{R}(X) \subseteqq A(X) \subseteqq C^m(X)$, and if $f \in A(X)$ with $Z(f) = \emptyset$, then $1/f \in A(X)$.

LEMMA 6.1. *If $A(X)$ satisfies property (6–1), then there is an $f \in A(X)$ such that f belongs to no maximal ideal other than $M_a = \{f \in A(X) : f(a) = 0\}$ and $Z(f) = \{a\}$, for each $a \in X$.*

Proof. Take $f(x) = \sum_{i=1}^n (x_i - a_i)^2$, where $(a_1, \cdots, a_n) = a \in X$. Then that $f \in M_a$ and belongs to no other fixed maximal ideal is clear. If f belongs to a free maximal ideal M, then there is $g \in M$ with $g(a) \neq 0$. Let $h = f^2 + g^2$. We have $Z(h) = \emptyset$ so that $1/h \in A(X)$. Hence $u = hh^{-1} \in M$. This is impossible.

LEMMA 6.1.′ *For any metric space (X, d) and $p \in X$, there is $f \in L_c(X)$ such that f belongs to no maximal ideal other than M_p and $Z(f) = \{p\}$.*

Proof. Take $f(x) = d(p, x)$. The proof is quite similar to (6.1).

LEMMA 6.2. *If $A(X)$ and $A(Y)$ satisfy (6–1), and Φ is an isomorphism from $A(X)$ onto $A(Y)$, then for any $M_a \subset A(X)$, $\Phi(M_a)$ is a fixed maximal ideal in $A(Y)$.*

Proof. Consider the image of $f(x) = \sum_{i=1}^{n} (x_i - a_i)^2$, $\Phi(f)$. We can show that $Z(\Phi(f)) = \{b\}$ for some $b \in Y$. The result follows immediately.

LEMMA 6.2.′ *If Φ is an isomorphism of $L_c(X)$ onto $L_c(Y)$, then for $M_p \subset L_c(X)$, $\Phi(M_p)$ is a fixed maximal ideal in $L_c(Y)$.*

Proof is similar to (6.2).

LEMMA 6.3. *Let B_1 and B_2 be subrings of $C(X)$ and $C(Y)$ respectively, which contain all constant functions, Φ be an isomorphism from B_2 to B_1, and X be connected. Then Φ is the identity on the constant functions.*

Proof. It is clear that $\Phi(r) = r$ for all rational constant functions r. If k is an irrational number $k - r \neq 0$, for all rational numbers r. Moreover, $\Phi(k - r)\cdot\Phi(1/(k - r)) = \Phi(u) = u$, we have

$$\Phi\left(\frac{1}{k - r}\right) = \frac{1}{\Phi(k) - r}$$

for any rational number r. Suppose $\Phi(k)$ is not constant. By continuity of $\Phi(k)$ and connectedness of X, we would have $\Phi(1/(k - r))$ is undefined for some r and some point of X. This is a contradiction. Hence $\Phi(k)$ is constant. By [3, 0.22], Φ is the identity on the constant functions.

THEOREM 6.4. *Let X and Y be two arbitrary subsets of E^n. If there are $A(x)$ and $A(Y)$ subrings of $C^m(X)$ and $C^m(Y)$ satisfying (6–1), and an isomorphism, Φ, from $A(Y)$ onto $A(X)$ leaving all constant functions unchanged, then Φ induces a mapping $\tau : X \to Y$ defined by $\Phi(g) = g\cdot\tau$ and τ is a C^m-diffeomorphism.*

Proof. Define τ to be a mapping from X to Y as follows: $\tau(x) = \cap Z[\Phi^{-1}(M_x)]$. By hypothesis and (6.2), $\Phi^{-1}(M_x)$ is a fixed maximal ideal in $A(Y)$. Thus, τ is well-defined. Evidently, $M_{\tau(x)} = \Phi^{-1}(M_x)$, so that τ is one-one. Let y_0 be arbitrary in Y. Then M_{y_0} is a fixed maximal ideal in $A(Y)$, and $\Phi(M_{y_0}) = M_{x_0}$ for some $x_0 \in X$. Thus $y_0 = \cap Z[\Phi^{-1}(M_{x_0})] = \tau(x_0)$. That is, τ is onto. Now, for each $g \in A(Y)$ and each $x \in X$, if $\Phi(g)(x) = r$, then $\Phi(g) - r \in M_x$, $g - \Phi^{-1}(r) \in M_{\tau(x)}$, so that $g(\tau(x)) = (\Phi^{-1}(r))(\tau(x)) = r(\tau(x)) = r = \Phi(g)(x)$. Hence $\Phi(g) = g\cdot\tau$. Similarly, $\Phi^{-1}(f) = f\cdot\tau^{-1}$ where τ^{-1}: $Y \to X$, defined by $\tau^{-1}(y) = \cap Z[\Phi(M_y)]$. Since $f\cdot\tau \in A(X)$ and $g\cdot\tau^{-1} \in A(Y)$ for each $g \in A(X)$ and $f \in A(Y)$, and $A(X)$ and $A(Y)$ contain all projections. By (5.4) τ is

C^m-diffeomorphism.

THEOREM 6.4′. *Let (X, d_1) and (Y, d_2) be any two metric spaces. If there is an isomorphism Φ from $L_c(Y)$ onto $L_c(X)$ leaving all constant functions unchanged, then Φ induces a mapping $\tau: X \to Y$ defined by $\Phi(g) = g \cdot \tau$ and τ is an L_c-homeomorphism.*

Proof is similar to (6.4).

COROLLARY 6.5. *Let X and Y be two connected subsets of E^n. If there are subrings $A(X)$ and $A(Y)$ of $C^m(X)$ and $C^m(Y)$ respectively satisfying* (6–1), *and an isomorphism, Φ, of $A(Y)$ onto $A(Y)$, then Φ induces a C^m-diffeomorphism, τ, from X onto Y such that $\Phi(g) = g \cdot \tau$ for each $g \in A(Y)$.*

Proof. Combine (6.3) and (6.4).

COROLLARY 6.5′. *Let (X, d_1) and (Y, d_2) be two connected metric spaces. If Φ be an isomorphism of $L_c(Y)$ onto $L_c(X)$, then Φ induces an L_c-homeomorphism τ from X on Y such that $\Phi(g) = g \cdot \tau$ for each $g \in L_c(Y)$.*

Proof is similar to (6.5).

REMARK. In (6.4) if $A(X) = \mathscr{R}(X)$, and $A(Y) = \mathscr{R}(Y)$, then τ and τ^{-1} are not only C^m, each of their components is a rational function. We may name this mapping as rational-homeomorphism. We know that there is a nonlinear rational-homeomorphism. Let $X = Y = E^n - (0, \cdots, 0)$, and $\tau(x) = (\tau_1(x), \cdots, \tau_n(x))$ be defined

$$\tau_i(x) = \frac{x_i}{x_1^2 + \cdots + x_n^2} \quad \text{for} \quad 1 \leqq i \leqq n .$$

Then its inverse is known to be $\tau^{-}(y) = (\Phi_1(y), \cdots, \Phi_n(y))$ with

$$\Phi_j(y) = \frac{y_j}{y_1^2 + \cdots + y_n^2}, \qquad 1 \leqq j \leqq n .$$

If the metric spaces are compact subsets of E^n, then we have the same results as (6.1), (6.2), (6.4) and (6.5) with $A(X)$ and $A(Y)$ replaced by $B(X)$ and $B(Y)$ respectively, where $B(X)$ and $B(Y)$ are the subrings of $L(X)$ and $L(Y)$ respectively satisfying the following property: $\mathscr{R}(X) \subset B(X)$, and $f \in B(X)$ with $Z(f) = \Phi$ implies $1/f \in B(X)$. We know that there is such a proper subring $B(X)$. For instance, let $B_0(X) = \{f \in L(X) : f \in C^3(X)\}$. Then $\mathscr{R}(X) \subset B_0(X) \subset L(X)$.

Next, we will see some algebraic properties of the rings of continuous functions which are inapplicable in the rings of C^m-differentiable function, where $1 \leqq m \leqq \infty$.

(1) The rings of C^m-differentiable functions are not lattice-ordered. Let $X = E^1$. Consider $C^m(X)$. We know $i(x) = x, i \in C^m(X)$. But $|i| \in C^m(X)$. Thus, neither $f \wedge \theta$ nor $f \vee \theta$, in general, is in $C^m(X)$.

(2) We know that in the rings of continuous functions, $I(f) \geqq 0$ if there is $g \in C(X)$ such that $g \geqq 0$ and $g \equiv f (\text{mod } I)$. (See [3, 5.2 and 5.4(a)].) In the rings of differentiable functions such a g need not exist. Consider $X = E^1$, and $C^1(X)$. Let $I = \{f \in C^1(X) : Z(f) \supset [0, 1]\}$. Then I is a z-ideal, convex, but not absolutely convex. Let $f_0(x) = x - x^2$. It is clear that $f_0 \geqq 0$ on a zero-set of I. But there is no $g \in C^1(X)$ so that $g \geqq 0$ and g agrees with f_0 on $[0, 1]$.

(3) If I and J are z-ideals in $C(X)$, then $IJ = I \cap J$. This is not true in $C^m(X)$. Let $X = E^1$. Consider in $C^1(E^1)$, $I = J = M_0 = \{f \in C^1(E^1) : f(0) = 0\}$. Then $I \cap J = M_0$. But $i(x) = x, i \in I \cap J$ and $i \notin IJ$. This also shows that the following is not true in $C^m(X)$ or $C^{m*}(X)$. If P and Q are prime ideals in C(or C^*) then $PQ = P \cap Q$. For C^{m*}, we take $X = (-n, n)$.

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DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF OKLAHOMA