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ON INDETERMINATE HAMBURGER MOMENT PROBLEMS

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This paper is concerned with the moment problems associated with a sequence of orthogonal polynominals defined by a recurrence formula. The principle interest centers on the question of the determinacy of the Stieltjes moment problem in the case where the corresponding Hamburger moment problem is indeterminate. Necessary and sufficient conditions expressed in terms of the recurrence formula are obtained for an indeterminate Hamburger moment problem to be a determined Stieltjes moment problem. Using this result, various criteria concerning the determinacy of the moment problems are obtained. It is also shown that if an indeterminate Hamburger moment problem has at least one solution whose spectrum is bounded below, then there is an extremal solution ψ_* such that every substantially different solution has at least one spectral point smaller than the least spectral point of ψ_*.

Let $\{P_n(x)\}$ be a sequence of monic polynomials defined by a recurrence

$$(1.1)\qquad \begin{aligned} &P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), && n = 1, 2, 3, \cdots \\ &P_{-1}(x) = 0,\ P_0(x) = 1;\ c_n \text{ real},\ \lambda_{n+1} > 0 && (n \geqq 1)\,. \end{aligned}$$

Then it is a classical result that the $P_n(x)$ are orthogonal with respect to some mass distribution $d\psi(x)$ on the real line, ψ being a bounded, nondecreasing function with an infinite spectrum.

In [2, Th. 1], it was shown that a necessary and sufficient condition that there is at least one such distribution function ψ whose spectrum is a subset of $[0, \infty)$ is that $c_n > 0$ $(n \geqq 1)$ and that $\{\lambda_{n+1}/c_n c_{n+1}\}_{n=1}^{\infty}$ is a chain sequence. After an arbitrary choice of $\mu_0 > 0$, the orthogonality of the $P_n(x)$ determines a Hamburger moment sequence $\{\mu_n\}_{n=0}^{\infty}$. With the above chain sequence condition, this is also a Stieltjes moment sequence.

In [3], we gave some conditions on the coefficients in (1.1) by which the determinacy or indeterminacy of the Hamburger moment problem (HMP) can be decided. In the event the HMP is indeterminate, it is natural to inquire about the determinacy of the corresponding Stieltjes moment problem (SMP). In what follows, we will pursue this inquiry and apply our results and methods to a study of the uniqueness of the distributions obtained by Al-Salam and Carlitz [1] for a certain class of orthogonal "q-polynomials."

2. Preliminaries. For the sake of definiteness (and without loss of generality), we will take $\mu_0 = 1$ for the moment sequence determined by (1.1). Thus we will speak of "the" moment problem associated with (1.1).

With the recurrence (1.1), we have the associated J-fraction

$$\frac{\lambda_1 |}{| x - c_1} - \frac{\lambda_2 |}{| x - c_2} - \frac{\lambda_3 |}{| x - c_3} - \cdots \qquad (\lambda_1 = \mu_0 = 1) . \tag{2.1}$$

The nth approximant of (2.1) is the rational function, $P_{n-1}^{(1)}(x)/P_n(x)$, where $\{P_n^{(1)}(x)\}$ is defined by (1.1) after replacing c_n and λ_n by $c_n^{(1)} = c_{n+1}$ and $\lambda_n^{(1)} = \lambda_{n+1}$, respectively.

If $x_{n1} < x_{n2} < \cdots < x_{nn}$ denote the zeros of $P_n(x)$, then it follows from the separation properties of the zeros of $P_n(x)$ and $P_{n+1}(x)$ that $\{x_{ni}\}_{n=i}^{\infty}$ is decreasing while $\{x_{nn}\}$ is increasing. We write

$$\begin{aligned} \xi_i &= \lim_{n\to\infty} x_{ni} \qquad\qquad i = 1, 2, 3, \cdots \\ \xi_\infty &= \lim_{n\to\infty} x_{nn} . \end{aligned} \tag{2.2}$$

Then (ξ_1, ξ_∞) is the so-called "true" interval of orthogonality of $\{P_n(x)\}$ and there is always one solution of the associated HMP whose spectrum is a subset of $[\xi_1, \xi_\infty]$. Also, $\xi_1 \geqq c$ if and only if $c_n - c > 0$ $(n \geqq 1)$ and $\{\lambda_{n+1}/(c_n - c)(c_{n+1} - c)\}$ is a chain sequence [2, Lemma 5].

In case the HMP associated with (2.1) is also a solvable SMP, then (2.1) is the even part of the corresponding S-fraction (e.g. [5, p. 73])

$$\frac{1 |}{| k_1 z} + \frac{1 |}{| k_2} + \frac{1 |}{| k_3 z} + \frac{1 |}{| k_4} + \cdots . \tag{2.3}$$

The coefficients in the two continued fractions are related by the formulas (see [9, § 28])

$$\begin{aligned} c_n &= -(b_{2n-2} + b_{2n-1}) \qquad\qquad n = 1, 2, 3, \cdots \\ \lambda_{n+1} &= b_{2n-1} b_{2n} \end{aligned} \tag{2.4}$$

where $b_0 = 0$ and $b_i = (k_i k_{i+1})^{-1}$ $(i \geqq 1)$ and $k_1 = \lambda_1^{-1} = 1$.

3. Stieltjes' condition. Stieltjes [6] (cf. also [5], Th. 2.20]) showed that a necessary and sufficient condition that the SMP associated with (2.3) (hence also with (2.1)) be determined is

$$\sum_{i=1}^{\infty} | k_i | = \infty . \tag{3.1}$$

From the relations (2.4) and (2.5), we obtain

$$\begin{aligned} &k_1 = 1\,, \quad k_{2n+1} = \frac{b_1 b_3 \cdots b_{2n-1}}{b_2 b_4 \cdots b_{2n}} \\ &k_2 = \frac{1}{b_1}\,, \quad k_{2n+2} = \frac{b_2 b_4 \cdots b_{2n}}{b_1 b_3 \cdots b_{2n+1}}\,, \qquad n = 1, 2, 3, \cdots . \end{aligned} \tag{3.2}$$

On the other hand, from relations in [2, § 2], we know that if $c_n > 0$ and $\alpha = \{\lambda_{n+1}/c_n c_{n+1}\}$ is a chain sequence, then the associated moment problem is a solvable SMP and we have

$$\begin{aligned} c_n &= \gamma_{2n-1} + \gamma_{2n} \\ \lambda_{n+1} &= \gamma_{2n}\gamma_{2n+1} \end{aligned} \qquad \gamma_1 = 0,\ \gamma_n > 0\ (n \geqq 2) \tag{3.3}$$

where $\gamma_{2k-1} = m_{k-1}c_k$ and $\{m_k\}_0^\infty$ is the minimal parameter sequence for α.

Comparing (2.4) and (3.3), we thus obtain

$$\begin{aligned} k_1 = 1\,, \quad k_{2n+1} &= \frac{c_1(1-m_1)\cdots(1-m_{n-1})}{m_1 m_2 \cdots m_n c_{n+1}} \\ k_2 = -\frac{1}{c_1}\,, \quad k_{2n+2} &= -\frac{m_1 m_2 \cdots m_n}{(1-m_1)\cdots(1-m_n)}\,. \end{aligned}$$

If we write $p_n(x) = (\lambda_2 \cdots \lambda_{n+1})^{-1/2} P_n(x)$, then [3, p. 708][1]

$$[p_n(0)]^2 = \frac{c_1(1-m_1)\cdots(1-m_{n-1})}{m_1 m_2 \cdots m_n c_{n+1}}\,.$$

Thus Stieltjes' (3.1) is equivalent to the divergence of at least one of the two series

$$\sum_{n=1}^\infty \frac{m_1 m_2 \cdots m_n}{(1-m_1)\cdots(1-m_n)} \quad \text{and} \quad \sum_{n=1}^\infty [p_n(0)]^2\,. \tag{3.4}$$

But by a theorem of Wall [9, Th. 19.3], a necessary and sufficient condition that α determine its parameters uniquely is that the first series in (3.4) diverge. Thus we can state the transformed criterion of Stieltjes as follows:

THEOREM 1. *Let $c_n > 0$ and let $\alpha = \{\lambda_{n+1}/c_n c_{n+1}\}$ be a chain sequence (so that the corresponding* SMP *has a solution). Then a necessary and sufficient condition that the* SMP *be determined is that either $\sum [p_n(0)]^2 = \infty$ or α determines its parameters uniquely.*

Throughout the remainder of this paper, we will maintain the notation, $\alpha = \{\lambda_{n+1}/c_n c_{n+1}\}$.

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[1] There is a factor of c_1 missing from the pertinent formula in the citation. This omission is carried through to a number of the following formulas but does not affect any of the conclusions drawn.

LEMMA 1. *Let $c_n > 0$ and let α be a chain sequence. Then*

$$\lim_{n\to\infty} \frac{P_{n+1}(0)}{P_n^{(1)}(0)} = -c_1 M_0$$

where M_0 denotes the "0th" maximal parameter for α.

Proof. By a formula of Wall [9, (19.6)],

$$M_0 = 1 - \frac{\alpha_1|}{|1} - \frac{\alpha_2|}{|1} - \frac{\alpha_3|}{|1} - \cdots, \qquad \alpha_n = \frac{\lambda_{n+1}}{c_n c_{n+1}}.$$

Comparing this with (2.1) (with $x = 0$), we see that

$$(-c_1 M_0)^{-1} = \frac{\lambda_1|}{|-c_1} - \frac{\lambda_2|}{|-c_2} - \frac{\lambda_3|}{|-c_3} - \cdots = \lim_{n\to\infty} \frac{P_n^{(1)}(0)}{P_{n+1}(0)}.$$

Lemma 1 is really needed in the next section but it is of some interest here in view of a theorem of Hamburger [5, Th. 2.17] that a necessary and sufficient condition for a HMP to be determined is the divergence of at least one of the series, $\sum [p_n(0)]^2$ and $\sum [p_n^{(1)}(0)]^2$, where $p_n^{(1)}(x) = (\lambda_2^{(1)} \cdots \lambda_{n+1}^{(1)})^{-1/2} P_n^{(1)}(x)$ (cf. also [3, Th. 4.1]).

THEOREM 2. *Let $c_n > 0$ and let α be a chain sequence.*

(A) *If α does not determine its parameters uniquely, then the* HMP *and* SMP *are both determined or are both indeterminate according as $\sum [p_n(0)]^2$ diverges or converges.*

(B) *The case of a determined* SMP *which is an indeterminate* HMP *occurs if and only if α determines its parameters uniquely and $\sum [p_n^{(1)}(0)]^2 < \infty$.*

Proof. (A) If $\sum [p_n(0)]^2 = \infty$, then by Hamburger's theorem, the HMP (and hence the SMP) is determined. Conversely, if the series converges and α does not determine its parameters uniquely, then by Theorem 1, the SMP (hence the HMP) is indeterminate.

(B) It follows that if the SMP is determined while the HMP is indeterminate, then α must determine its parameters uniquely. By Hamburger's theorem, we must also have $\sum [p_n^{(1)}(0)]^2 < \infty$. Conversely, if the latter series converges, so does $\sum [p_n(0)]^2$ by Lemma 1. Thus the HMP is indeterminate by Hamburger's theorem while the SMP will be determined according to Theorem 1 if α determines its parameters uniquely.

4. The case of an indeterminate HMP. We consider the polynomials $B_n(z)$ and $D_n(z)$ defined by

$$\lambda_2 \cdots \lambda_{n+1} B_{n+1}(z) = P_{n+1}(z) P_{n-1}^{(1)}(0) - P_n(z) P_n^{(1)}(0)$$
$$\lambda_2 \cdots \lambda_{n+1} D_{n+1}(z) = P_{n+1}(z) P_n(0) - P_n(z) P_{n+1}(0) .$$

Using the fact that $B_n(0) = -1$ [5, (2.46)], we obtain

$$P_n(z) = P_{n-1}^{(1)}(0) D_{n+1}(z) - P_n(0) B_{n+1}(z) .$$

(Note that we write λ_k, $P_n(z)$ and $P_n^{(1)}(z)$ for β_{k-1}, $Q_n(z)$ and $P_{n+1}(z)$, respectively, in [5].)

If we now assume that the HMP associated with (1.1) is indeterminate, then $B_n(z)$ and $D_n(z)$ converge uniformly on bounded subsets of the complex plane to entire functions $B(z)$ and $D(z)$ [5, Th. 2.11]. Hence if we now also assume that $c_n > 0$ and α is a chain sequence, then by Lemma 1.

$$\lim_{n\to\infty} \frac{P_n(z)}{P_{n-1}^{(1)}(0)} = D(z) + c_1 M_0 B(z) \tag{4.1}$$

uniformly on bounded subsets of the complex plane.

Now by a theorem of Nevanlinna [5, Th. 2.13], the zeros of the entire function $E(z) = D(z) + c_1 M_0 B(z)$ are real, simple and coincide with the spectrum of one of the extremal solutions, ψ_*, of the HMP.

Since the convergence in (4.1) is uniform, it follows from (2.2) that each ξ_i is a zero of $E(z)$ and hence also that $\xi_i < \xi_{i+1}$ ($\xi_1 \geqq 0$). It then follows from Hurwitz' theorem on uniform limits of analytic functions [8, Th. 1. 91. 3] that $\{\xi_i \mid i = 1, 2, 3, \cdots\}$ is the spectrum of the extremal solution, ψ_*.

We summarize with a lemma which will be sharpened somewhat subsequently (Theorem 5).

LEMMA 2. *Let $c_n > 0$ and let α be a chain sequence. If the associated* HMP *is indeterminate, then there is an extremal solution ψ_* whose spectrum coincides with the set $\{\xi_i \mid i = 1, 2, 3, \cdots\}$. The spectrum of every solution of the* HMP *which is not substantially equal to ψ_* contains at least one point that does not exceed ξ_1.*

Proof. Only the last assertion requires further comment. (By "substantially equal" solutions of the HMP, we mean, following Shohat and Tamarkin [5], solutions with the same points of continuity whose difference is constant at all points of continuity.) Since the zeros of $P_n(x)$ must lie in the interior of the interval of orthogonality, then ξ_1 must belong to the smallest closed interval containing the support of any $d\psi(x)$ with respect to which the $P_n(x)$ are orthogonal.

THEOREM 3. *If the true interval of orthogonality for* (1.1) *is*

$(0, \infty)$ and if the associated HMP *is indeterminate, then the chain sequence α determines its parameters uniquely.*

Proof. Assume α does not determine its parameters uniquely. Then [2, Th. 1] the $P_n(x)$ are "kernel polynomials"—that is, there is a distribution function φ whose spectrum is a subset of $[0, \infty)$ such that the $P_n(x)$ are orthogonal with respect to $xd\varphi(x)$. Further, by Theorem 2, the corresponding SMP is also indeterminate.

By [2, Th. 2], there is a continuum of indeterminate SMPs whose solutions have the property described for φ above. Consider one of these SMPs. By Lemma 2, this indeterminate SMP—which is also an indeterminate HMP—has an extremal solution φ_* whose spectrum is a well ordered subset of $[0, \infty)$.

It then follows that the spectrum of the distribution function ψ_* defined by $d\psi_*(x) = xd\varphi_*(x)$ does not contain 0, hence must contain a least element $a > 0$. But the $P_n(x)$ are orthogonal with respect to $d\psi_*(x)$ which means the true interval of orthogonality is (a, ∞), contrary to hypothesis.

THEOREM 4. *If the* HMP *associated with* (1.1) *is indeterminate, then the corresponding* SMP *is* (*solvable and*) *determined if and only if the true interval of orthogonality is* $(0, \infty)$.

Proof. If the SMP is solvable and determined, then α is a chain sequence which by Theorem 2 determines its parameters uniquely. According to [2, Th. 1], the $P_n(x)$ are not "kernel polynomials" so the true interval of orthogonality must be $(0, \infty)$.

Conversely, suppose the true interval of orthogonality is $(0, \infty)$. Since the HMP is indeterminate, it follows from Theorem 3 that α must determine its parameters uniquely. Thus by Theorem 1, the SMP is determined.

We can now restate Lemma 2 in stronger form.

THEOREM 5. *Assume there is a real c such that $c_n - c > 0$ and $\{\lambda_{n+1}/(c_n - c)(c_{n+1} - c)\}$ is a chain sequence. Suppose also that the associated* HMP *is indeterminate. Then the* HMP *has an extremal solution ψ_* whose spectrum is the set $\{\xi_i \mid i = 1, 2, 3, \cdots\}$. The spectrum of every solution not substantially equal to ψ_* contains at least one point smaller than ξ_1.*

Proof. By hypothesis, ξ_1 is finite ($\xi_1 \geqq c$). If we put $\pi_n(x) = P_n(x + \xi_1)$, the $\pi_n(x)$ are orthogonal with respect to $d\psi(x + \xi_1)$, where ψ is any solution of the original moment problem, and the corresponding true interval of orthogonality is $(0, \infty)$.

If the original HMP is indeterminate, so is the HMP associated with the $\pi_n(x)$. By Lemma 2, the latter HMP has an extremal solution φ_* whose spectrum coincides with the set $\{\xi_i - \xi_1 \mid i = 1, 2, \cdots\}$ determined by the zeros of $\{\pi_n(x)\}$.

But since the true interval of orthogonality is $(0, \infty)$, then by Theorem 4, the corresponding SMP is determined. Then φ_* is substantially the only solution of the transformed HMP whose spectrum is a subset of $[0, \infty)$. Thus in terms of the original HMP, $\psi_*(x) = \varphi_*(x - \xi_1)$ has the desired properties.

COROLLARY. *If an indeterminate* HMP *is a determined* SMP, *then the solution of the* SMP *is an extremal solution of the* HMP *whose spectrum contains* 0 *and is a discrete, unbounded subset of* $[0, \infty)$.

Proof. By Theorem 4, the true interval of orthogonality is $(\xi_1, \xi_\infty) = (0, \infty)$.

The preceding shows that—at least from the viewpoint of orthogonal polynomials—when the HMP is indeterminate, the question of the determinacy of the SMP is relatively unimportant since determinacy depends only on an "accident" of translation (whereas determinacy of the HMP is invariant under translation). Stieltjes had observed that in the indeterminate case, $\xi_1 > 0$ [6, p. 441]. However, it does not seem to have been observed previously that there is always substantially only one solution of a SMP whose spectrum is a subset of $[\xi_1, \infty)$.

In a sense, the solution described in Theorem 5 is a "natural" one since every spectral point is a limit point of a sequence of zeros of the $P_n(x)$, a property that is shared by the solution of a determined HMP (Stone [7, Th. 10.42]).

In [2, Th. 8] and [3, Th. 3.1], we have obtained sufficient conditions (in terms of c_n and λ_n) for the spectrum associated with (1.1) to be a discrete, unbounded set. These conditions are accompanied by the hypothesis that the associated HMP is determined but in view of the above, this hypothesis can be dropped if we think in terms of the "natural" solution. With this viewpoint, a recent theorem of Maki on continued fractions [4, Th. 3.2] should be compared with [2, Th. 8].

5. **Examples.** We illustrate various aspects of the preceding theory with two examples.

(A) The Laguerre polynomials. Using the notation of [8], we have

$$P_n(x) = (-1)^n L_n^{(\alpha)}(x) \qquad (\alpha > -1)$$
$$c_n = 2n + \alpha - 1 , \qquad \lambda_{n+1} = n(n + \alpha) \qquad (n \geqq 1) .$$

Then

$$\alpha_n = \frac{\lambda_{n+1}}{c_n c_{n+1}} = \frac{n(n+\alpha)}{(2n+\alpha-1)(2n+\alpha+1)} = \left(1 - \frac{n-1}{2n+\alpha-1}\right)\frac{n}{2n+\alpha+1} .$$

It follows from Wall's criterion (following (3.4)) that $\{\alpha_n\}$ is a chain sequence which determines its parameters, $m_k = k/(2n + \alpha - 1)$ $(k \geqq 0)$, uniquely if and only if $\alpha \leqq 0$.

Then for $-1 < \alpha \leqq 0$, the moment problem is at least a determined SMP (by Theorem 1) and the true interval of orthogonality is $(0, \infty)$. Since the weight function for the Laguerre polynomials is $x^\alpha e^{-x}$ and this does not correspond to an extremal solution, it follows from Theorem 5, corollary, that the HMP must be determined for $-1 < \alpha \leqq 0$. Then, also, the corresponding spectrum is $[0, \infty)$ and the zeros of the polynomials are dense in $(0, \infty)$.

Now the $L_n^{(\alpha+1)}(x)$ are the kernel polynomials corresponding to the $L_n^{(\alpha)}(x)$ so it follows that the true interval of orthogonality must be $(0, \infty)$ for $0 < \alpha \leqq 1$ and hence for all $\alpha > -1$ by induction. Since for $\alpha > 0$, $\{\alpha_n\}$ does not determine its parameters uniquely, Theorem 3 shows that the HMP must be determined for $\alpha > 0$ also.

Thus the HMP is determined for every $\alpha > -1$ and by Riesz' theorem [5, Th. 2.14], the Laguerre polynomials are complete in the corresponding L^2 space (as is well known).

(B) Al-Salam and Carlitz [1] have recently considered two interesting classes of orthogonal polynomials involving "q-numbers." One of these satisfies the recurrence

$$V_{n+1}^{(a)}(x) = [x - (1+a)q^{-n}]V_n^{(a)}(x) - aq^{1-2n}(1-q^n)V_{n-1}^{(a)}(x) \qquad (n \geqq 0)$$
$$V_{-1}^{(a)}(x) = 0,\ V_0^{(a)}(x) = 1; \qquad 0 < q < 1, a > 0 .$$

These authors obtain the corresponding moments and construct a solution of the moment problem. (However, their solution is not non-decreasing when $aq > 1$ and does not exist if $aq^k = 1$.) The other system considered by them has bounded coefficients in its recurrence and hence the corresponding HMP is automatically determined. For the $V_n^{(a)}(x)$, however, the HMP may be indeterminate. Since they did not investigate the uniqueness of the solution of the moment problem, such a study will provide an interesting application of our methods.

Relative to (1.1), we have

$$c_n = (1 + a)q^{1-n} , \qquad \lambda_{n+1} = aq^{1-2n}(1 - q^n)$$
$$\alpha_n = \lambda_{n+1}/(c_n c_{n+1}) = a(1 + a)^{-2}(1 - q^n) .$$

Since $\alpha_n < 1/4$, $\{\alpha_n\}$ is a chain sequence [9, Th, 20.1] which does not determine its parameters uniquely [9, Th. 19.6].

Next let $v_n^{(a)}(x) = (\lambda_2 \cdots \lambda_{n+1})^{-1/2} V_n^{(a)}(x)$. With the aid of [1, (4.8)], we find

$$[v_n^{(a)}(0)]^2 = a^{-n}q^n(q)_n^{-1}[H_n(a)]^2 , \qquad (q)_n = (1 - q) \cdots (1 - q^n) ,$$

where

$$\sum_{r=0}^{n} a^r < H_n(a) \equiv \sum_{r=0}^{n} \frac{(q)_n}{(q)_{n-r}(q)_r} a^r < (q)_n^{-1} \sum_{r=0}^{n} a^r .$$

It follows that $\sum [v_n^{(a)}(0)]^2 < \infty$ if and only if $0 < q < a \leqq 1$ or $1 < a < q^{-1}$. Since $\alpha = \{\alpha_n\}$ does not determine its parameters uniquely, Theorem 2 shows that:

(a) both the SMP and HMP are determined if

$$0 < a \leqq q < 1 \quad \text{or} \quad 1 < q^{-1} \leqq a ;$$

(b) both the SMP and HMP are indeterminate if

$$0 < q < a \leqq 1 \quad \text{or} \quad 1 < a < q^{-1} .$$

Except for the cases, $a = q$ and $a = q^{-1}$, the above results can also be obtained using [3, Th. 4.3].

In the indeterminate case, there may some interest in knowing whether the distribution function obtained by Al-Salam and Carlitz is the extremal solution described in Theorem 5.

To this end, translate the minimal spectral point, 1, of the distribution function, $\beta = \beta^a$, obtained in [1] to the origin. That is, consider (1.1) with c_n replaced by $c'_n = c_n - 1 = (1 + a)q^{1-n} - 1$. Then

$$\alpha'_n = \frac{\lambda_{n+1}}{c'_n c'_{n+1}} = \frac{a(1 - q^n)}{(1 + a - q^{n-1})(1 + a - q^n)}$$

and it is readily verified that $\{\alpha'_n\}$ is a chain sequence with minimal parameters $m_k = (1 - q^k)(1 + a - q^k)^{-1}$ $(k \geqq 0)$. Wall's criterion shows that $\{\alpha'_n\}$ determines its parameters uniquely if and only if $0 < a \leqq 1$. By theorem 2, the translated SMP is determined (and the HMP is indeterminate) if and only if $0 < a \leqq 1$. In terms of the original moment problem, this means that the true interval of orthogonality is $(1, \infty)$.

Summarizing:

(i) if $0 < a \leqq q < 1$ or $1 < q^{-1} \leqq a$, the solution of the HMP is substantially unique (and is given by β in [1] when $0 < a \leqq q < 1$);

(ii) if $0 < q < a \leqq 1$, the solution β is not substantially unique but it is substantially the only solution whose spectrum is a subset of $[1, \infty)$;

(iii) If $1 < a < q^{-1}$, there are infinitely many solutions of the moment problem whose minimal spectral point is larger than 1 (that is, the true interval of orthogonality is (c, ∞) where $c > 1$).

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