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ANALYTIC SHEAF COHOMOLOGY GROUPS OF DIMENSION n OF n-DIMENSIONAL NONCOMPACT COMPLEX MANIFOLDS

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In this paper the following question is considered: if X is a σ-compact noncompact complex manifold of dimension n and $\mathscr{F}$ is a coherent analytic sheaf on X, does $H^n(X, \mathscr{F})$ always vanish? The answer is in the affirmative.

This question was first proposed by Malgrange in [6] and in the same paper he gave the affirmative answer for the special case when $\mathscr{F}$ is locally free.

THEOREM. *If X is an n-dimensional σ-compact noncompact complex manifold and $\mathscr{F}$ is a coherent analytic sheaf on X, then $H^n(X, \mathscr{F}) = 0$.*

Proof. I. For $0 \leqq p \leqq n$ let $\mathscr{A}^{(0,p)}$ denote the sheaf of germs of C^∞ $(0, p)$-forms on X and $\mathscr{O}$ denote the structure-sheaf of X. Since at a point in a complex number space the ring of C^∞ function-germs as a module over the ring of holomorphic function-germs is flat ([7], Ths. 1 and 2 bis), the sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{A}^{(0,0)} \bigotimes_{\mathscr{O}} \mathscr{F} \xrightarrow{\bar\partial'} \cdots$$
$$\xrightarrow{\bar\partial'} \mathscr{A}^{(0,n-1)} \bigotimes_{\mathscr{O}} \mathscr{F} \xrightarrow{\bar\partial'} \mathscr{A}^{(0,n)} \bigotimes_{\mathscr{O}} \mathscr{F} \longrightarrow 0$$

obtained by tensoring

$$0 \longrightarrow \mathscr{O} \longrightarrow \mathscr{A}^{(0,0)} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{A}^{(0,n-1)} \xrightarrow{\bar\partial} \mathscr{A}^{(0,n)} \longrightarrow 0$$

with $\mathscr{F}$ over $\mathscr{O}$ is exact (cf. [8], Th. 3).

The theorem follows if we can prove that

$$\beta_X : \Gamma(X, \mathscr{A}^{(0,n-1)} \bigotimes_{\mathscr{O}} \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{A}^{(0,n)} \bigotimes_{\mathscr{O}} \mathscr{F})$$

induced from

$$\bar\partial' : \mathscr{A}^{(0,n-1)} \bigotimes_{\mathscr{O}} \mathscr{F} \longrightarrow \mathscr{A}^{(0,n)} \bigotimes_{\mathscr{O}} \mathscr{F}$$

is surjective.

II. Suppose $0 \leqq p \leqq n$ and

$$\mathscr{O}^r \xrightarrow{\phi} \mathscr{O}^s \xrightarrow{\psi} \mathscr{F} \longrightarrow 0$$

is an exact sequence of sheaf-homomorphisms on an open subset U of

X which is biholomorphic to an open subset of $\mathbf{C}^n$. Tensoring the sequence with $\mathscr{A}^{(0,p)}$ over $\mathscr{O}$, we obtain an exact sequence

$$(\mathscr{A}^{(0,p)})^r \xrightarrow{\phi'} (\mathscr{A}^{(0,p)})^s \xrightarrow{\psi'} \mathscr{A}^{(0,p)} \underset{\mathscr{O}}{\otimes} \mathscr{F} \longrightarrow 0 .$$

Since $\operatorname{Im}\phi'$ and $\operatorname{Ker}\phi'$ are fine sheaves,

$$\Gamma(U, (\mathscr{A}^{(0,p)})^r) \xrightarrow{\tilde{\phi}} \Gamma(U, \mathscr{A}^{(0,p)})^s) \xrightarrow{\tilde{\psi}} \Gamma(U, \mathscr{A}^{(0,p)} \underset{\mathscr{O}}{\otimes} \mathscr{F}) \longrightarrow 0$$

is exact. $\Gamma(U, (\mathscr{A}^{(0,p)})^s)$ is a Fréchet space if it is given the topology of uniform convergence of derivatives of coefficients on compact subsets. Since $\tilde{\phi}$ is defined by a matrix of holomorphic functions, by paragraph 1 of [7], $\operatorname{Im}\tilde{\phi}$ is a closed subspace of $\Gamma(U, (\mathscr{A}^{(0,p)})^s)$ (cf. [8], Th. 5). We give $\Gamma(U, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$ the quotient topology and it becomes a Fréchet space.

Suppose G is an open subset of X. We can find a countable Stein open cover $\{U_k\}_{k=1}^{\infty}$ of G such that U_k is biholomorphic to an open subset of $\mathbf{C}^n$ and on U_k we have an exact sequence of sheaf-homomorphisms

$$\mathscr{O}^{r_k} \xrightarrow{\phi_k} \mathscr{O}^{s_k} \xrightarrow{\psi_k} \mathscr{F} \longrightarrow 0 .$$

We give $\Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$ the smallest topology that makes every restriction map $\Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F}) \to \Gamma(U_k, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$ continuous. This topology of $\Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$ is independent of the choices of $\{U_k\}$, $\{\phi_k\}$, and $\{\psi_k\}$. $\Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$ is a Fréchet space.

$$\beta_G\colon \Gamma(G, \mathscr{A}^{(0,n-1)} \underset{\mathscr{O}}{\otimes} \mathscr{F}) \longrightarrow \Gamma(G, \mathscr{A}^{(0,n)} \underset{\mathscr{O}}{\otimes} \mathscr{F})$$

induced from

$$\bar{\partial}'\colon \mathscr{A}^{(0,n-1)} \underset{\mathscr{O}}{\otimes} \mathscr{F} \longrightarrow \mathscr{A}^{(0,n)} \underset{\mathscr{O}}{\otimes} \mathscr{F}$$

is continuous (cf. [8], pp. 21–24).

III. Suppose G is an open subset of X. Denote the strong dual of $\Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$ by $(\Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F}))^*$, $0 \leqq p \leqq n$. Suppose $T \in (\Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F}))^*$. The support of T, denoted by $\operatorname{Supp} T$, is defined as the complement in G of the largest open subset H such that, if $a \in \Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$ and $\operatorname{Supp} a \subset H$, then $T(a) = 0$. $\operatorname{Supp} T$ is well-defined, because H exists by partition of unity. Observe that, if $a_k \in \Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$ and for every compact subset K of G $(\bigcup_{k=m}^{\infty} \operatorname{Supp} a_k) \cap K = \emptyset$ for some m depending on K, then $a_k \to 0$ in $\Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$. We have:

(1) If V is a bounded subset of $(\Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F}))^*$, then there is a compact subset K of G such that $\operatorname{Supp} T \subset K$ for $T \in V$.

IV. Suppose G is an open subset of X. Fix

$$T \in (\Gamma(G, \mathscr{A}^{(0,n)} \underset{\mathscr{O}}{\otimes} \mathscr{F}))^*$$

and let $\operatorname{Supp} T\beta_G = K$. Let $\hat{K}$ denote the union of K together with all the components of $G - K$ relatively compact in G. We are going to prove that $\operatorname{Supp} T \subset \hat{K}$. Let L be a component of $G - K$ not relatively compact in G. We need only prove that $L \cap \operatorname{Supp} T = \emptyset$. Suppose the contrary. Since L is not relatively compact in $G, L \not\subset \operatorname{Supp} T$ ($\operatorname{Supp} T$ is compact by (1)). $\operatorname{Supp} T$ has a boundary point x_0 in L. We would have a contradiction if we can prove:

(2) Every boundary point x of $\operatorname{Supp} T$ is a boundary point of $\operatorname{Supp} T\beta_G$.

To prove (2) we suppose that x is a boundary point of $\operatorname{Supp} T$ and x is not a boundary point of $\operatorname{Supp} T\beta_G$. Since $\operatorname{Supp} T\beta_G \subset \operatorname{Supp} T$, $x \in X - \operatorname{Supp} T\beta_G$. On some connected open neighborhood D of x in $X - \operatorname{Supp} T\beta_G$ we have a sheaf-epimorphism $\theta\colon \mathscr{O}^s \to \mathscr{F}$. Tensoring it with $\mathscr{A}^{(0,p)}$ over $\mathscr{O}$, we obtain a sheaf-epimorphism $\theta'_p\colon (\mathscr{A}^{(0,p)})^s \to \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F}$. $\tilde{\theta}_p\colon \Gamma(D, (\mathscr{A}^{(0,p)})^s) \to \Gamma(D, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$ induced by θ'_p is surjective.

Let $\{N_k\}_{k=1}^{\infty}$ be a sequence of compact subsets of D such that $N_k \subset \operatorname{Int} N_{k+1}$ and $\bigcup_{k=1}^{\infty} N_k = D$. Let $\Gamma_{N_k}(D, (\mathscr{A}^{(0,p)})^s)$ be the set of all elements of $\Gamma(D, (\mathscr{A}^{(0,p)})^s)$ having supports contained in N_k. Give $\Gamma_{N_k}(D, \mathscr{A}^{(0,p)})^s)$ the topology induced from $\Gamma(D, (\mathscr{A}^{(0,p)})^s)$. Give $\Gamma_*(D, (\mathscr{A}^{(0,p)})^s) = \bigcup_{k=1}^{\infty} \Gamma_{N_k}(D, (\mathscr{A}^{(0,p)})^s)$ the topology as the strict inductive limit of $\{\Gamma_{N_k}(D, \mathscr{A}^{(0,p)})^s)\}$. $\Gamma_*(D, (\mathscr{A}^{(0,p)})^s)$ and its topology are independent of the choice of $\{N_k\}$.

For $a \in \Gamma_*(D, (\mathscr{A}^{(0,p)})^s)$, since $\operatorname{Supp} \tilde{\theta}_p(a) \subset D$ is compact, $\tilde{\theta}_p(a)$ can be trivially extended to an element $(\tilde{\theta}_p(a))' \in \Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$. The map $\xi_p\colon \Gamma_*(D, (\mathscr{A}^{(0,p)})^s) \to \Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$ defined by $\xi_p(a) = (\tilde{\theta}_p(a)))'$ is a continuous linear map.

(3) If $b \in \Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$ and $\operatorname{Supp} b$ is a compact subset of D, then $b \in \operatorname{Im} \xi_p$.

The following diagram is commutative:

$$\begin{array}{ccc}
\Gamma_*(D, (\mathscr{A}^{(0,n-1)})^s) & \xrightarrow{\bar{\partial}} & \Gamma_*(D, (\mathscr{A}^{(0,n)})^s) \\
\Big\downarrow{\scriptstyle \xi_{n-1}} & & \Big\downarrow{\scriptstyle \xi_n} \\
\Gamma(G, \mathscr{A}^{(0,n-1)} \underset{\mathscr{O}}{\otimes} \mathscr{F}) & \xrightarrow{\beta_G} & \Gamma(G, \mathscr{A}^{(0,n)} \underset{\mathscr{O}}{\otimes} \mathscr{F}) .
\end{array}$$

Since $\operatorname{Supp}(T\beta_G) \cap D = \emptyset$, $T\xi_n\bar{\partial} = T\beta_G\xi_{n-1} = 0$. $T\xi_n$ can be represented by an s-tuple of distribution-$(n, 0)$-forms on D (cf. the argument on p. 42, [2]). $T\xi_n\bar{\partial} = 0$ implies that $T\xi_n$ can be represented by an s-tuple of holomorphic $(n, 0)$-forms on D. Since $\operatorname{Supp} T\xi_n \subset \operatorname{Supp} T$ and $D \not\subset \operatorname{Supp} T$, the s-tuple of holomorphic forms representing $T\xi_n$ must be identically zero. Hence $T\xi_n = 0$. By (3) $\operatorname{Supp} T$ is disjoint from all compact subsets of D. x is not a boundary point of $\operatorname{Supp} T$.

Hence (2) is proved. We have:

(4) $\operatorname{Supp} T \subset (\widehat{\operatorname{Supp} T\beta_G})$ for $T \in (\Gamma(G, \mathscr{A}^{(0,n)} \otimes_{\mathscr{O}} \mathscr{F}))^*$.

Denote the transpose of β_G by $(\beta_G)^*$. (4) implies that

(5) $(\beta_G)^*$ is injective,

because every component of G is noncompact.

V. By Lemma 3, [6], we have:

(6) For every point x of X there is an open neighborhood U of x in X such that $H^n(W, \mathscr{F}) = 0$ for every open subset W of U.

Suppose K is a compact subset of X. By (6) we can find two finite collections $\mathfrak{A}, \mathfrak{B} = \{B_k\}_{k=1}^m$ of relatively compact open Stein subsets of X such that (i) both $\mathfrak{A}$ and $\mathfrak{B}$ cover K; (ii) intersections of subcollections of $\mathfrak{A}$ and intersections of subcollections of $\mathfrak{B}$ are Stein; (iii) the closure of any member of $\mathfrak{A}$ is contained in some member of $\mathfrak{B}$; and (iv) for any open subset W of any $B_k, 1 \leqq k \leqq m, H^n(W, \mathscr{F}) = 0$.

Let G and H be respectively the union of all the members of $\mathfrak{A}$ and $\mathfrak{B}$. Define inductively $G_0 = G$ and $G_k = G_{k-1} \cup B_k, 1 \leqq k \leqq m$. $H^n(G_k, \mathscr{F}) \to H^n(G_{k-1}, \mathscr{F}) \oplus H^n(B_k, \mathscr{F}) \to H^n(G_{k-1} \cap B_k, \mathscr{F})$ is exact (Part a of § 17, [1]). $H^n(G_{k-1} \cap B_k, \mathscr{F}) = 0$ implies that the restriction map $H^n(G_k, \mathscr{F}) \to H^n(G_{k-1}, \mathscr{F})$ is surjective for $1 \leqq k \leqq m$. Since $H = G_m$, the restriction map $H^n(H, \mathscr{F}) \to H^n(G, \mathscr{F})$ is surjective. $H^n(G, \mathscr{F})$ is finite-dimensional (cf. Proof of Th. 11, § 17, [1]). Since $H^n(G, \mathscr{F}) \approx \operatorname{Coker} \beta_G$, $\operatorname{Im} \beta_G$ is closed. $\operatorname{Im} (\beta_G)^*$ is weakly closed ([5], Préliminaires, § 3, Th. 2). Therefore we have:

(7) Every compact subset K of X has an open neighborhood G in X such that $\operatorname{Im} (\beta_G)^*$ is weakly closed.

VI. By (5) and Th. 2, § 3, Préliminaires, [5], the theorem follows if we can prove that the intersection of $\operatorname{Im} (\beta_X)^*$ with every weakly compact sebset of $(\Gamma(X, \mathscr{A}^{(0,n-1)} \otimes_{\mathscr{O}} \mathscr{F}))^*$ is weakly compact. Suppose V is a weakly compact subset of $(\Gamma(X, \mathscr{A}^{(0,n-1)}) \otimes_{\mathscr{O}} \mathscr{F}))^*$. V is strongly bounded ([3], Th. 3). By (1) there exists a compact subset K of X such that

(8) $\operatorname{Supp} S \subset K$ for $S \in V$.

$\hat{K}$ is compact ([5], Chap. IV, § 3, Lemma 3). By (7) there exists an open neighbourhood G of $\hat{K}$ in X such that $\operatorname{Im} (\beta_G)^*$ is weakly closed. By (4) and (8) we have:

(9) $\operatorname{Supp} T \subset \hat{K}$ if $T \in (\Gamma(X, \mathscr{A}^{(0,n)} \otimes_{\mathscr{O}} \mathscr{F}))^*$ and $T\beta_X \in V$.

Let g be a C^∞ function on G having compact support and being identically one on some neighborhood of $\hat{K}$. Suppose $0 \leqq p \leqq n$. Let σ_p: $\Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F}) \to \Gamma(X, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$ be defined by trivial extension after multiplication by g. σ_p is continuous. Let ρ_p: $\Gamma(X, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F}) \to \Gamma(G, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F})$ be the restriction map.

(10) If $R \in (\Gamma(X, \mathscr{A}^{(0,p)} \otimes_{\mathscr{O}} \mathscr{F}))^*$ and $\operatorname{Supp} R \subset \hat{K}$, then $R\sigma_p\rho_p = R$.

To prove that $\operatorname{Im} (\beta_X)^* \cap V$ is weakly compact, it suffices to prove that it is weakly closed. Suppose $\{S_i\}_{i \in I}$ is a net in $\operatorname{Im} (\beta_X)^* \cap V$ converging weakly to $S \in V$. By (8) $\operatorname{Supp} S \subset K$. $S_i = T_i\beta_X$ for some $T_i \in (\Gamma(X, \mathscr{A}^{(0,n)} \otimes_{\mathcal{O}} \mathscr{F}))^*$. By (9) $\operatorname{Supp} T_i \subset \hat{K}$. $\operatorname{Supp} T_i\sigma_n \subset \hat{K}$ and $\operatorname{Supp} S\sigma_{n-1} \subset \hat{K}$. The following diagram is commutative:

$$\begin{array}{ccc}
\Gamma(X, \mathscr{A}^{(0,n-1)} \underset{\mathcal{O}}{\otimes} \mathscr{F}) & \xrightarrow{\beta_X} & \Gamma(X, \mathscr{A}^{(0,n)} \underset{\mathcal{O}}{\otimes} \mathscr{F}) \\
\rho_{n-1}\Big\downarrow & & \rho_n\Big\downarrow \\
\Gamma(G, \mathscr{A}^{(0,n-1)} \underset{\mathcal{O}}{\otimes} \mathscr{F}) & \xrightarrow{\beta_G} & \Gamma(G, \mathscr{A}^{(0,n)} \underset{\mathcal{O}}{\otimes} \mathscr{F})\ .
\end{array}$$

Take $a \in \Gamma(G, \mathscr{A}^{(0,n-1)} \otimes_{\mathcal{O}} \mathscr{F})$. Let $b = \sigma_{n-1}(a) \in \Gamma(X, \mathscr{A}^{(0,n-1)} \otimes_{\mathcal{O}} \mathscr{F})$. Then $\rho_{n-1}(b) = ga$. Since $\hat{K} \cap \operatorname{Supp} \beta_G(a - ga) = \varnothing$,

$$T_i\sigma_n\beta_G(a) = T_i\sigma_n\beta_G(ga) = T_i\sigma_n\beta_G\rho_{n-1}(b) = T_i\sigma_n\rho_n\beta_X(b) = T_i\beta_X(b)$$

by (10). Since $\hat{K} \cap \operatorname{Supp}(a - ga) = \varnothing$,

$$S\sigma_{n-1}(a) = S\sigma_{n-1}(ga) = S\sigma_{n-1}\rho_{n-1}(b) = S(b)\ .$$

Since $T_i\beta_X(b) \to S(b)$, $T_i\sigma_n\beta_G(a) \to S\sigma_{n-1}(a)$. Hence $T_i\sigma_n\beta_G \to S\sigma_{n-1}$ in the weak topology of $(\Gamma(G, \mathscr{A}^{(0,n-1)} \otimes_{\mathcal{O}} \mathscr{F}))^*$. Since $\operatorname{Im}(\beta_G)^*$ is weakly closed, there exists $T' \in (\Gamma(G, \mathscr{A}^{(0,n)} \otimes_{\mathcal{O}} \mathscr{F}))^*$ such that $T'\beta_G = S\sigma_{n-1}$. Let $T = T'\rho_n$. Then

$$T\beta_X = T'\rho_n\beta_X = T'\beta_G\rho_{n-1} = S\sigma_{n-1}\rho_{n-1} = S\ .$$

$S \in \operatorname{Im}(\beta_X)^* \cap V$. $\operatorname{Im}(\beta_X)^* \cap V$ is weakly closed.

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