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QF-3 RINGS WITH ZERO SINGULAR IDEAL

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Let R be a ring with identity. R is left QF-3 if R has a minimal faithful (left) module, i.e., a faithful (left) module, which is (isomorphic to) a summand of every faithful (left) module. We show that left QF-3 rings are characterized by the existence of a faithful projective-injective left ideal with an essential socle which is a finite sum of simple modules. The main result is a structure theorem for left and right QF-3 rings with zero left singular ideal. This theorem gives several descriptions of this class of rings. Among these is that the above rings are exactly the orders (containing units) with essential left and right socles in semi-simple two-sided (complete) quotient rings.

Let R be a ring with identity. R is *left* QF-3 if R has a minimal faithful (left) module, i.e., a faithful (left) module which is (isomorphic to) a summand of every faithful (left) module. Finite dimensional algebras with this property were introduced by R. M. Thrall as a generalization of quasi-Frobenius algebras, and other authors have considered Artinian and semi-primary QF-3 rings. For a semi-primary ring, being left QF-3 is equivalent to the existence of a faithful projective-injective left ideal. The first theorem shows that to characterize left QF-3 rings in general one must assume that some faithful projective-injective left ideal has an essential socle which is a finite sum of simple modules. The rest of this paper concerns rings with zero singular ideal which are either QF-3 or have faithful projective-injective one-sided ideals. The main result is a structure theorem for left and right QF-3 rings with zero left singular ideal. This theorem gives several descriptions of this class of rings. Among these is that the above rings are exactly the orders (containing units) with essential left and right socles in semi-simple[1] two-sided (complete) quotient rings. This theorem extends and unifies result of M. Harada [5, 6], J. P. Jans [8], and H. Mochizuki [12].

RESULTS. A submodule N of a module M is *essential* in M if every nonzero submodule of M meets N nontrivially. The *singular submodule* $Z(M) = \{x \in M \mid Ix = 0$ for some essential left ideal I of $R\}$. $Z({}_RR)$ is an ideal of R called the left singular ideal of R. An R-module M is called *uniform* if every nonzero submodule of M is essential in M. If M and N are R-modules with M uniform and

[1] Throughout this paper, *semi-simple* means *semi-simple Artinian.*

$Z(N) = 0$, then every nonzero homomorphism of M into N is a monomorphism (see Goldie [4]). In particular, if M is injective and N is indecomposable, every such homomorphism is an isomorphism.

The structure of a minimal faithful R-module is given by the following theorem. We denote the injective envelope of an R-module M by $E(M)$ (see [1] or [11]).

THEOREM 1. *The following are equivalent.*

(1) *R is left QF-3.*

(2) *There exist (nonisomorphic) simple (left) R-modules $S_1, \cdots, S_n$ such that $E(\bigoplus_{i=1}^n S_i)$ is a faithful, projective module.*

(3) *There exist (nonisomorphic) minimal left ideals $M_1, \cdots, M_k$ of R such that $E(\bigoplus_{i=1}^k M_i)$ is a faithful left ideal of R.*

Proof. It is clear that (3) implies (2). Thus it suffices to show (1) implies (3) and (2) implies (1).

Assume (1). Let M be the minimal faithful module. Since ${}_R R$ is faithful, M is isomorphic to a summand of ${}_R R$ and so is projective and cyclic. Let $\{S_\alpha : \alpha \in A\}$ be a complete set of representatives for the distinct isomorphic classes of simple left R-modules. Then $\bigoplus_\alpha E(S_\alpha)$ is faithful (see [14]) and so M is isomorphic to a summand of $\bigoplus_\alpha E(S_\alpha)$. Since M is cyclic, its image is contained in a finite number of the $E(S_\alpha)$. Thus M is injective and has an essential socle which is the direct sum of a finite number of simple modules.

Assume (2). Then since $E(\bigoplus_{i=1}^n S_i) \cong \bigoplus_{i=1}^n E(S_i)$ each $E(S_i)$ is an indecomposable projective-injective module and so is isomorphic to a left ideal L_i of R (see [2]) which must contain a unique minimal left ideal M_i. One now shows that $\bigoplus_{i=1}^n L_i$ is a minimal faithful module as in [8].

LEMMA 2. *Suppose that Re is a faithful projective-injective left ideal of R where $e^2 = e \in R$. If either* (a) *eRe is semi-simple Artinian or* (b) *$Z(R) = 0$ and R contains no infinite set of orthogonal idempotents, then*

(1) *ReR is the right socle of R and contains only a finite number of isomorphism classes of simple right R-modules.*

(2) *ReR is an essential right ideal of R.*

(3) *the right singular ideal of R is zero.*

Proof. Since Re is injective and eRe is the endomorphism ring of Re, in either case we have $Re = Re_1 \oplus \cdots \oplus Re_n$ where the e_i's are orthogonal idempotents and each Re_i is indecomposable and injective. Hence each e_iRe_i is a local ring. In case (a), the radical of e_iRe_i is

$e_i(\mathrm{rad}\, eRe)e_i = 0$ so e_iRe_i is a division ring. This also follows if (b) holds since Re_i is uniform and injective and $Z(Re_i) = 0$. Let J denote the radical of R. If (a) holds then $e_iJ = 0$ for each i since Re is faithful. If (b) holds and $e_ix \neq 0$ with $x \in J$, then $re_i \to re_ix$ defines a monomorphism of Re_i into J whose image is a summand of R, a contradiction. Hence e_iR contains no nilpotent right ideals, so e_iR is simple as in ([7], Prop. 1, p. 65). Hence Re_iR is contained in the right socle of R and so ReR is also. Now, if H is a right ideal with $ReR \cap H = 0$, then $HRe \subseteqq ReR \cap H$ so since Re is faithful $H = 0$. Thus ReR is an essential submodule of R_R and so equals the right socle of R. Since ReR is a faithful left R-module the right singular ideal of R is zero.

If R is a subring of a ring Q such that ${}_RR$ is essential in ${}_RQ$, then Q is called a *ring of left quotients* of R.

PROPOSITION 3. *Suppose R contains faithful projective-injective left and right ideals Re and fR, respectively, where e and f are idempotents. Then* $\mathrm{Hom}_{eRe}(Re, fRe) = fR$ *and* $\mathrm{Hom}_{fRf}(fR, fRe) = Re$. *Furthermore,* $Q = \mathrm{Hom}_{eRe}(Re, Re)$ *is a two-sided ring of quotients of R.*

Proof. Let $Q = \mathrm{Hom}_{eRe}(Re, Re)$. Since the map λ of R into Q given by $\lambda(r)(se) = rse$ for $r, s \in R$ is a unital ring monomorphism whose restriction to Re is a Q-monomorphism, we may regard R as a subring of Q such that $Re = Qe$ is a faithful left Q-module. Since Re is faithful fR_R is an essential submodule of fQ_R. Thus, since fR_R is injective, $fQ = fR$. Since $\mathrm{Hom}_{eRe}(Re, fRe) = fQ$ the first assertion holds. Now ${}_QRe$ is faithful and since if $q \in Q$ and $fRq = 0$, then $qRe = 0$, fR_Q is faithful. Thus, if $0 \neq q \in Q$, $qR \cap R \supseteqq qRe \neq 0$ and $Rq \cap R \supseteqq fRq \neq 0$ so Q is a two-sided quotient ring of R.

LEMMA 4. *Suppose that R contains faithful projective-injective left and right ideals Re and fR where e and f are idempotents, and that fRf is semi-simple Artinian. Then Re_{eRe} is not an infinite direct sum of nonzero submodules.*

Proof. Supppose $Re = \bigoplus_j I_j$ where each I_j is a nonzero eRe module. Then $\mathrm{Hom}_{eRe}(Re, fRe) = \prod_j \mathrm{Hom}_{eRe}(I_j, fRe) = fR$. Suppose the direct sum is infinite so there exists $fr \in fR$ such that $fr \notin \bigoplus_j \mathrm{Hom}_{eRe}$ (I_j, fRe). Since ${}_{fRf}fRe$ is faithful and injective and fRf is semi-simple, there exists $se \in Re = \mathrm{Hom}_{fRf}(fR, fRe)$ such that $frse \neq 0$ and $(\bigoplus_j \mathrm{Hom}_{fRf}(I_j, fRe))se = 0$, a contradiction.

An R-module M is *finite dimensional* if M contains no infinite direct sum of nonzero submodules (see [3]). The complete ring of left quotients of R is $\mathrm{Hom}_\Gamma(E(_RR)_\Gamma, E(_RR)_\Gamma)$ where $\Gamma = \mathrm{Hom}_R(E(_RR), E(_RR))$ (see Lambek [10] or [11]). The complete ring of left quotients of R is semi-simple if and only if $_RR$ is finite dimensional and R has zero left singular ideal (Johnson [9]).

If M is an R-module we say that N is a *minimal essential submodule* of M if N is essential in M and no proper submodule of N is essential in M.

We are now ready for the main theorem.

THEOREM 5. *The following are equivalent.*

(1) *R has zero left singular ideal and is left and right QF-3.*

(2) *There exist idempotents e, $f \in R$ such that Re and fR are faithful projective-injective left and right ideals and eRe is semi-simple Artinian.*

(3) *R has zero left singular ideal, contains no infinite set of orthogonal idempotents and has a faithful projective-injective left ideal and a faithful projective-injective right ideal.*

(4) *R is a subring of a semi-simple Artinian ring Q and R contains a left ideal I and a right ideal J such that I and J are, respectively, faithful left and faithful right ideals of Q.*

(5) *R has a two-sided semi-simple Artinian complete ring of quotients and both the left socle and the right socle of R are essential.*

(6) *R has a two-sided semi-simple Artinian complete ring of quotients and $_RR$ and R_R each contain a minimal essential submodule.*

Proof. Conditions (5) and (6) are equivalent since the socle of any module is the intersection of its essential submodules (see Utumi [16]). We complete the proof by showing first that (1) and (2) are equivalent and then that (1) $\rightarrow$ (5) $\rightarrow$ (4) $\rightarrow$ (3) $\rightarrow$ (1).

Assume condition (1). If Re is a minimal faithful left ideal with $e^2 = e$ then Theorem 1 together with $Z(R) = 0$ imply that the endomorphism ring of Re is semi-simple so (2) holds.

Assume condition (2). By Lemma 2 the right singular ideal of R is zero and the right socle ReR of R is essential in R_R. Since there are only a finite number of isomorphism classes of simple right ideals of R, fR contains a minimal faithful module by Theorem 1. Hence R is right QF-3 with zero singular ideal and if we take fR to be a minimal faithful, fRf is semi-simple. Thus by an argument symmetric to the above, R is left QF-3 and has zero left singular ideal. Hence (1) holds.

Assume condition (1). Let Re, $e^2 = e$, and fR, $f^2 = f$ be minimal faithful left and right ideals of R, respectively. Then both the left and the right socles of R are essential in R by Lemma 2. By Proposition 3, $Q = \mathrm{Hom}_{eRe}(Re, Re)$ is a two-sided ring of quotients of R.

That Q is semi-simple is plain from Lemma 4 since both eRe and fRf are semi-simple.

Next assume condition (5) holds. Let Q denote the two-sided semi-simple quotient ring of R and let E denote the left socle of R. Then $EQ = E$ since E is the left socle of ${}_RQ$. Furthermore, since the left singular ideal of R is zero and Q_R is essential over R_R, E is a faithful right Q-module and so we let $J = E$. Similarly, we can let I be the right socle of R. Thus (4) holds.

Assume condition (4). Then Q_R is a right ring of quotients of R since for any $0 \neq q \in Q$, $0 \neq qI \subseteq R$ and hence since Q is semi-simple, it is the complete ring of right quotients of R. Similarly, Q is the complete ring of left quotients of R. Next note that ${}_RI$ is injective. For, if $f\colon L \to I$ where L is a left ideal of R, define $\bar{f}\colon QL \to I = QI$ by

$$\bar{f}(\sum q_i x_i) = \sum q_i f(x_i)\ , \qquad q_i \in Q\ , \quad x_i \in L\ .$$

$\bar{f}$ is well defined since if E is the left socle of R, $y \in E$ and $\sum q_i x_i = 0$ then $y \sum q_i f(x_i) = f(\sum (yq_i x_i)) = 0$ so $Z({}_RQ) = 0$ implies $\sum q_i f(x_i) = 0$. Thus $\bar{f}$ is a Q-homomorphism and injectivity of ${}_RI$ follows from the injectivity of ${}_QI$. Similarly, J is a faithful projective-injective right ideal of R so (3) holds.

Finally, condition (1) follows from condition (3) by Theorem 1 and Lemma 2.

REMARK 6. If R satisfies the hypotheses of Theorem 5 and $Re, e^2 = e$, and $fR, f^2 = f$, are faithful projective-injective left and right ideals of R, respectively, then fRe character modules define a duality between finitely generated right eRe-modules and finitely generated fRf-modules in the sense of Morita [13]. The semi-simplicity of eRe and fRf with the faithfulness of ${}_{fRf}fRe$ and fRe_{eRe} implies that ${}_{fRf}fRe$ and fRe_{eRe} are injective cogenerators. The mapping of fR onto fRe given by right multiplication by e induces an injection of $\mathrm{Hom}_{fRf}(fRe, fRe)$ into $\mathrm{Hom}_{fRf}(fR, fRe) = Re$ which has image eRe.

It is not difficult to show that if R has zero left singular ideal, then R is left QF-3 and finite dimensional if and only if R is left and right QF-3. Thus Theorem 5 extends results of H. Mochizuki [12] for hereditary QF-3 algebras and M. Harada [5, 6] for semi-primary QF-3 and PP rings. This theorem also generalizes results of J. P. Jans [8] for primitive rings with faithful projective-injective minimal one-sided ideals.

A ring is (meet) *irreducible* if the intersection of an two nonzero ideals is nonzero. If R is left QF-3 and the socle of R is homogeneous, then the socle of a minimal faithful left ideal is simple and hence is contained in every nonzero two-sided ideal. The following corollary to Theorem 5 is easily proved.

COROLLARY 7. *Let R be a ring which satisfies the conditions of Theorem 5 and let Q denote its complete ring of quotients. The following are equivalent.*

(1) *R is the direct sum of ideals $R_1, \cdots, R_n$ and Q is the direct sum of ideals $Q_1, \cdots, Q_n$ where each Q_i is a two-sided simple Artinian complete quotient ring of R_i.*

(2) *R is the direct sum of ideals $R_1, \cdots, R_n$ where each R_i is an irreducible ring.*

(3) *R has a decomposition $R = Re_1 \oplus \cdots \oplus Re_m$ into indecomposable left ideals Re_i such that each Re_i has homogeneous socle.*

It is not difficult to show that if (3) above holds, any such decomposition has the stated property. In (2) above the intersection of the left and right socles of R_i is the minimal ideal of R_i.

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