PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 3, 1969

# GAMES WITH UNIQUE SOLUTIONS THAT ARE NONCONVEX

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**In 1944 von Neumann and Morgenstern introduced a theory of solutions (stable sets) for $n$-person games in characteristic function form. This paper describes an eight-person game in their model which has a unique solution that is nonconvex. Former results in solution theory had not indicated that the set of all solutions for a game should be of this nature.**

First, the essential definitions for an $n$-person game will be stated. Then, a particular eight-person game is described. Finally, there is a brief discussion on how to construct additional games with unique and nonconvex solutions.

The author [2] has subsequently used some variations of the techniques described in this paper to find a ten-person game which has no solution; thus providing a counterexample to the conjecture that every $n$-person game has a solution in the sense of von Neumann and Morgenstern.

**2. Definitions.** An *$n$-person game* is a pair $(N, v)$ where $N = \{1, 2, \cdots, n\}$ and $v$ is a real valued characteristic function on $2^N$, that is, $v$ assigns the real number $v(S)$ to each subset $S$ of $N$ and $v(\varphi) = 0$. The set of all *imputations* is

$$A = \left\{x: \sum_{i \in N} x_i = v(N) \text{ and } x_i \geqq v(\{i\}) \text{ for all } i \in N\right\}$$

where $x = (x_1, x_2, \cdots, x_n)$ is a vector with real components. If $x$ and $y$ are in $A$ and $S$ is a nonempty subset of $N$, then $x \operatorname{dom}_S y$ means $\sum_{i \in S} x_i \leqq v(S)$ and $x_i > y_i$ for all $i \in S$. For $B \subset A$ let $\operatorname{Dom}_S B = \{y \in A$: there exists $x \in B$ such that $x \operatorname{dom}_S y\}$ and let $\operatorname{Dom} B = \bigcup_{S \subset N} \operatorname{Dom}_S B$. A subset $K$ of $A$ is a *solution* if $K \cap \operatorname{Dom} K = \varphi$ and $K \cup \operatorname{Dom} K = A$. The *core* of a game is

$$C = \left\{x \in A: \sum_{i \in S} x_i \geqq v(S) \text{ for all } S \subset N\right\}.$$

The core consists of those imputations which are maximal with respect to all of the relations $\operatorname{dom}_S$, and hence it is contained in every solution.

**3. Example.** Consider the game $(N, v)$ where $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and where $v$ is given by: $v(N) = 4$, $v(\{1, 4, 6, 7\}) = 2$, $v(\{1, 2\}) =$

$v(\{3, 4\}) = v(\{5, 6\}) = v(\{7, 8\}) = 1$, and $v(S) = 0$ for all other $S \subset N$. For this game

$$A = \left\{x: \sum_{i \in N} x_i = 4 \text{ and } x_i \geqq 0 \text{ for all } i \in N\right\}$$

and

$$C = \{x \in A: x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = 1 \text{ and } x_1 + x_4 + x_6 + x_7 \geqq 2\} .$$

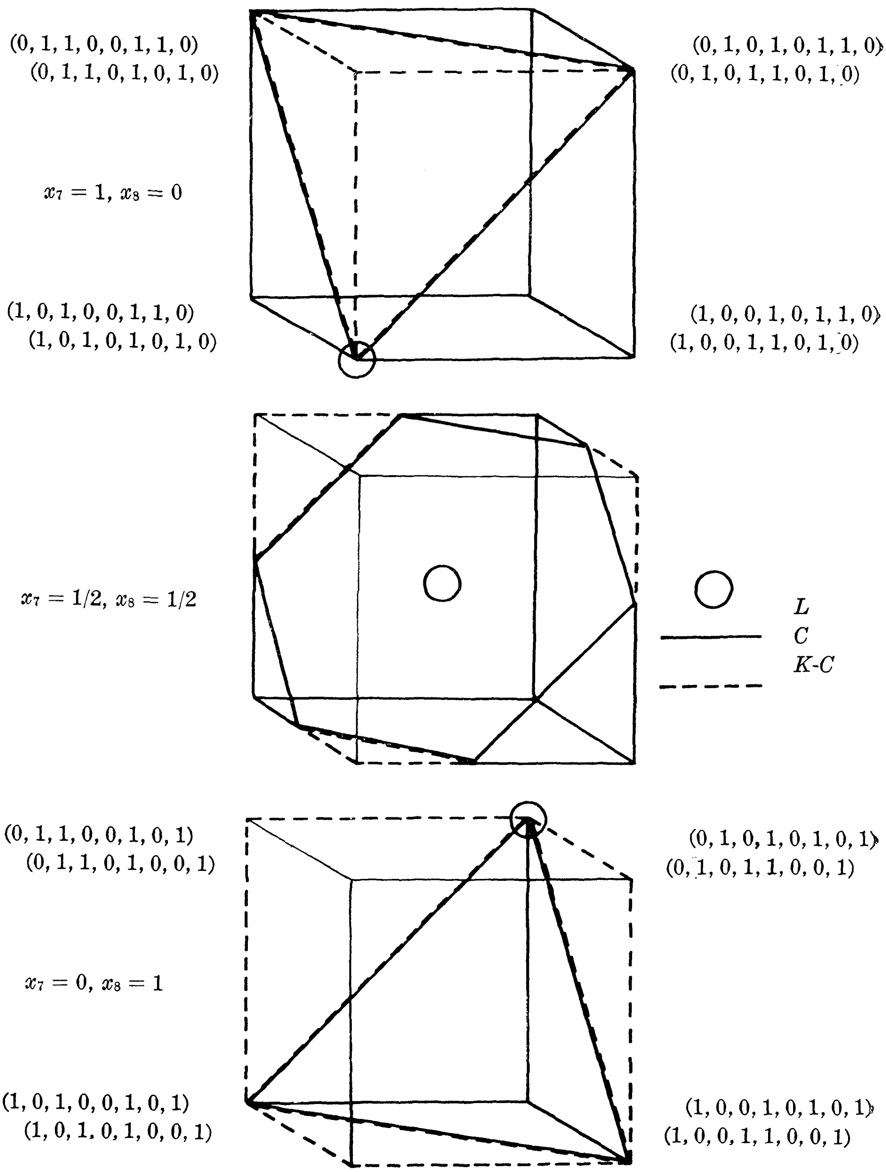


FIG. 1. Traces in $H$ of $L$, $C$ and $K$-$C$

Also define the four-dimensional hypercube

$$H = \{x \in A: x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = 1\} .$$

Three traces of $H$ as well as its 16 vertices are pictured in Fig. 1.

The unique solution for this game is

$$K = C \cup F_1 \cup F_4 \cup F_6 \cup F_7$$

where the cube $F_i$ is the face of $H$ given by

$$F_i = H \cap \{x: x_i = 1\} \quad i = 1, 4, 6, 7 .$$

Each $F_i - C$ is a tetrahedron with one face meeting $C$. In the three traces of $H$ illustrated in Fig. 1, the traces of $C$ are shown in heavy solid lines and the traces of the $F_i - C$ are shown in heavy broken lines.

The proof that $K$ is the unique solution follows readily from two observations. First, $K$ is just those imputations in $H$ which are maximal in $H$ with respect to the relation $\text{dom}_{\{1,4,6,7\}}$. Second, the closed line segment $L$ joining the imputations $(0, 1, 0, 1, 0, 1, 0, 1)$ and $(1, 0, 1, 0, 1, 0, 1, 0)$ has the properties $L \subset C$ and $\bigcup_S \text{Dom}_S L = A - H$ when $S = \{1, 2\}, \{3, 4\}, \{5, 6\}$, and $\{7, 8\}$.

To see that $K$ is nonconvex, note the lower trace

$$F_8 = H \cap \{x: x_8 = 1\}$$

in Fig. 1. The heavy lines (solid and broken) in this trace show $K \cap F_8$, which is clearly not convex. For example, the imputation

$$\tfrac{1}{3}(1, 2, 2, 1, 2, 1, 0, 3) = \tfrac{1}{3}(0, 1, 1, 0, 0, 1, 0, 1) + \tfrac{1}{3}(0, 1, 0, 1, 1, 0, 0, 1) + \tfrac{1}{3}(1, 0, 1, 0, 1, 0, 0, 1)$$

is a linear combination of points in $K$, but it is not itself in $K$.

## 4. Remarks.

The original von Neumann-Morgenstern theory [3] assumed that the characteristic function of a game is superadditive, that is, $v(S_1 \cup S_2) \geqq v(S_1) + v(S_2)$ whenever $S_1$ and $S_2 \subset N$ and $S_1 \cap S_2 = \varphi$. Using the method of Gillies [1, p. 68] this example can be made into a game with a superadditive characteristic function without changing $A, C$, or the unique solution $K$.

The essential idea in the example above is that $\bigcup_S \text{Dom}_S L = A - H$ where $S = \{1, 2\}, \{3, 4\}, \{5, 6\}$, and $\{7, 8\}$. One can generalize this relation in various ways to obtain many games in other dimensions which have a similar property. He can then introduce into these games additional $S \subset N$ with $v(S) > 0$, but in such a way as to maintain the corresponding $L$ as a subset of the core. As a result he will obtain large classes of interesting solutions, many of which are unique and nonconvex.

## REFERENCES

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Received June 26, 1967.

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