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ON THE JOIN OF SUBNORMAL ELEMENTS IN A LATTICE

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Of fundamental importance to the study of subnormal subgroups is the following result of Wielandt:

Let A and B be subnormal subgroups of a group G such that A is normal in $A \cup B$. Then $A \cup B$ is subnormal in G.

The usual proof of Wielandt's result depends on the construction by conjugation of a special subnormal series from A to G. It would be of interest to obtain a proof which uses only the given subnormal series, without explicit dependence on conjugation, and valid in algebraic systems other than groups.

This note presents, in the more general context of a lattice with the normality relation introduced by R. A. Dean, a proof of the analogous result in case either A or B has defect three or less.

We begin with the definition of a lattice normality relation from [1].

DEFINITION. A reflexive relation $\lhd$ on a lattice $\mathfrak{L}$ is called a *normality relation* if, for all $a, b, c, d \in \mathfrak{L}$:

(1) $a \lhd b$ implies $a \leqq b$,

(2) $a \lhd b, c \lhd d$ implies $a \cap c \lhd b \cap d$,

(3) $a \lhd b, a \lhd c$ implies $a \lhd b \cup c$,

(4) $a \lhd b, c \lhd d$ implies $a \cup c \lhd a \cup c \cup (b \cap d)$,

(5) $a \leqq b$ and either $a \lhd a \cup c$ or $c \lhd a \cup c$ implies

$$a \cup (b \cap c) = b \cap (a \cup c) .$$

An element a of a lattice $\mathfrak{L}$ is called *subnormal* in $b \in \mathfrak{L}$, denoted $a \lhd \lhd b$, if there exists a chain of elements $a_i \in \mathfrak{L}$, $i = 0, 1, \cdots, n$, such that

$$a = a_n \lhd a_{n-1} \lhd \cdots \lhd a_0 = b .$$

The length of the shortest such chain is called the *defect* of a in b.

Suppose $a \lhd \lhd u$ and $b_3 \lhd b_2 \lhd b_1 \lhd u$. We shall prove:

THEOREM 1. *If $b_3 \lhd a \cup b_3$, then $a \cup b_3 \lhd \lhd u$.*

THEOREM 2. *If $a \lhd a \cup b_3$, then $a \cup b_3 \lhd \lhd u$.*

The following results will be needed in the proofs.

LEMMA A. *If $x \lhd \lhd u$, $y \lhd \lhd u$, and x has defect 2 or less in u, then $x \cup y \lhd \lhd u$.*

LEMMA B. *If $a \leqq x \leqq b$ and $a \lhd b$, then $a \lhd x$.*

Lemma A is proved in [1], while Lemma B is an immediate consequence of (2).

Proof of Theorem 1. Since $b_3 \lhd a \cup b_3$ and $b_3 \lhd b_2$, by (3),

$$b_3 \lhd (a \cup b_3) \cup b_2 = a \cup b_2 \,.$$

By intersection of subnormal chains $a \lhd \lhd a \cup b_2$. Then, by Lemma A, $a \cup b_3 \lhd \lhd a \cup b_2$, and $a \cup b_2 \lhd \lhd u$. Thus $a \cup b_3 \lhd \lhd u$.

Proof of Theorem 2. Let the given subnormal chain from a to u be

$$a = a_n \lhd a_{n-1} \lhd \cdots \lhd a_0 = u \,.$$

Define, for $m = 0, 1, \cdots, n$,

$$x_m = a \cup b_3 \cup (a_m \cap b_2) \,.$$

By a finite induction it will be shown that $x_m \lhd \lhd x_{m-1}$, $1 \leqq m \leqq n$. But $x_n = a \cup b_3$, and $x_0 = a \cup b_2$, so, by Lemma A, $x_0 \lhd \lhd u$. $a \cup b_3 \lhd \lhd u$ thus follows from transitivity of subnormality. Since the relation $a \cup (a_0 \cap b_2) = a_0 \cap x_0$ is trivial, the proof of Theorem 2 will be complete upon verification of the induction step:

LEMMA C. *Suppose $a \cup (a_{m-1} \cap b_2) = a_{m-1} \cap x_{m-1}$. Then $x_m \lhd \lhd x_{m-1}$ and $a \cup (a_m \cap b_2) = a_m \cap x_m$.*

Proof of lemma. Define

$$\text{(i)} \qquad y = b_1 \cap [a \cup (a_m \cap b_2)] \,.$$

We shall begin by proving

$$\text{(ii)} \qquad b_3 \cup y \lhd x_{m-1} \,.$$

To prove (ii) let us first observe that, by (2),

$$\text{(iii)} \qquad y \lhd a \cup (a_m \cap b_2) \,.$$

From $b_2 \lhd b_1 \geqq y \cup b_2$ Lemma B gives $b_2 \lhd y \cup b_2$. This, with

$$a_m \cap b_2 \leqq y \leqq a_m \,,$$

implies by (5)

(iv) $$y = y \cup (a_m \cap b_2) = a_m \cap (y \cup b_2) .$$

Since $a_m \lhd a_{m-1}$, (2) then gives $y \lhd a_{m-1} \cap (y \cup b_2)$, and (5) implies $a_{m-1} \cap (y \cup b_2) = y \cup (a_{m-1} \cap b_2)$. Next, by (3) let us combine

$$y \lhd y \cup (a_{m-1} \cap b_2)$$

with (iii) to obtain $y \lhd a \cup (a_{m-1} \cap b_2)$. Therefore, by the hypothesis of the lemma,

(v) $$y \lhd a_{m-1} \cap x_{m-1} .$$

Hence, with $b_3 \lhd b_2$, (4) gives

(vi) $$b_3 \cup y \lhd b_3 \cup y \cup (b_2 \cap a_{m-1}) .$$

In addition, $a \lhd a \cup b_3$ implies

$$\begin{aligned} b_3 \cup (a \cap b_1) &= b_1 \cap (a \cup b_3) && \text{by (5)} \\ &\lhd a \cup b_3 && \text{by (2)} . \end{aligned}$$

Since $a \cap b_1 \leqq y$, (4) and (v) imply

$$\begin{aligned} b_3 \cup y &= \{b_3 \cup (a \cap b_1)\} \cup y \\ &\lhd b_3 \cup y \cup [(a \cup b_3) \cap a_{m-1} \cap x_{m-1}] \geqq a , \end{aligned}$$

so Lemma B gives $b_3 \cup y \lhd b_3 \cup y \cup a$. Finally, by (3), let us combine this with (vi) to obtain

$$b_3 \cup y \lhd b_3 \cup y \cup a \cup (a_{m-1} \cap b_2) = x_{m-1} .$$

Thus (ii) is proved.

We next establish $x_m \lhd \lhd x_{m-1}$. From $b_1 \lhd u \geqq a \cup b_1$ Lemma B yields $b_1 \lhd a \cup b_1$. Hence

$$\begin{aligned} x_m &= b_3 \cup a \cup (a_m \cap b_2) \\ &= b_3 \cup \{[a \cup (a_m \cap b_2)] \cap (a \cup b_1)\} && \text{by absorption} \\ &= b_3 \cup \{a \cup \{b_1 \cap [a \cup (a_m \cap b_2)]\}\} && \text{by (5)} \\ &= a \cup b_3 \cup y && \text{by (i)} . \end{aligned}$$

But $b_3 \cup y \lhd x_{m-1}$ and $a \lhd \lhd x_{m-1}$, so Lemma A gives

$$x_m = a \cup (b_3 \cup y) \lhd \lhd x_{m-1} .$$

Finally, we prove $a \cup (a_m \cap b_2) = a_m \cap x_m$. By (ii) $b_3 \cup y \lhd x_{m-1}$, and $a \cup (a_m \cap b_2) \leqq x_m \leqq x_{m-1}$, so Lemma B gives

$$b_3 \cup y \lhd (b_3 \cup y) \cup [a \cup (a_m \cap b_2)] .$$

Thus,

$$
\begin{aligned}
a_m \cap x_m &= a_m \cap \{b_3 \cup a \cup (a_m \cap b_2)\} && \text{by definition of } x_m \\
&= a_m \cap \{(b_3 \cup y) \cup [a \cup (a_m \cap b_2)]\} && \text{since, by (i), } y \leqq a \cup (a_m \cap b_2) \\
&= [a \cup (a_m \cap b_2)] \cup \{a_m \cap (b_3 \cup y)\} && \text{by (5)} \\
&\leqq a \cup [a_m \cap (b_2 \cup y)] && \\
&= a \cup y && \text{by (iv)} \\
&\leqq a \cup (a_m \cap b_2) && \text{by (i)} .
\end{aligned}
$$

The reverse containment is obvious. Thus $a_m \cap x_m = a \cup (a_m \cap b_2)$, and the proof is complete.

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