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# EXTREMAL STRUCTURE OF STAR-SHAPED SETS

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**It is shown that the convex kernel of a compact star-shaped subset $S$ of a finite-dimensional linear topological space $L_n$ is determined by the $(n-1)$-extreme points of $S$. The cardinality of the set of $k$-extreme points is determined for compact star-shaped sets of dimension greater than two. Also given is the result that any compact star-shaped subset $S$ of $L_n$ contains a countable set of $(n-1)$-extreme points which determines the convex kernel of $S$. Another result is that a compact nonconvex star-shaped set $S$ in a locally convex space $L$ is determined by the convex kernel of $S$ and the subset of points that are extreme in $S$ relative to the convex kernel of $S$.**

The convex kernel of a star-shaped set $S$ will be denoted by $ckS$, the line segment $\{\alpha x + (1-\alpha)y \colon \alpha \in [0, 1]\}$ will be denoted by $xy$, the ray $\{\beta y + (1-\beta)x \colon \beta \geqq 1\}$ will be denoted by $xy^\infty$ and $L(x, y)$ will denote the line containing $x$ and $y, x \neq y$. The convex hull of a set $S$ will be denoted by conv $S$. The notation intv $S$ will denote the interior of $S$ relative to the minimal flat that contains $S$. The set $\{x \colon f(x) = \alpha\}$, where $f$ is a linear functional, will be denoted $[f \colon \alpha]$. Set-theoretic difference will be denoted by $\backslash$, and the closure of a set $S$ will be denoted by cl $S$.

The concept of $k$-extreme point was introduced by Asplund [1].

DEFINITION 1. If $S$ is a subset of a linear space $L$, a point $x \in S$ is a $k$-extreme point of $S$ if no $k$-simplex $\Delta$ exists such that $x \in \text{intv } \Delta \subset S$.

For a subset $S$ of a linear space $L$, $S_x$ will denote the $x$-star of $S$ determined by the point $x \in S$; that is, the set of points $y$ such that $xy \subset S$. If $S$ is a closed (compact) subset of a linear topological space $L$, then for any $x \in S$, $S_x$ is a closed (compact) set. If $T \subset S$, let

$$S_T = \bigcap_{x \in T} S_x \, .$$

A point $p$ belongs to the convex kernel of $S$ if, and only if, $xp \subset S$ for all $x \in S$, which is true if, and only if, $p \in S_x$ for all $x \in S$. Thus $ckS = S_S$, which motivates the following definition.

DEFINITION 2. In a linear space $L$ a subset $T$ of a star-shaped set $S$ is said to star-generate the convex kernel of $S$ if $ckS = S_T$. Such a subset $T$ is said to be a star-generating set for $ckS$.

THEOREM 1. *Let $S$ be a compact star-shaped subset of $L_{k+1}$. Then the set $S(k)$ of $k$-extreme points of $S$ is a star-generating set for $ckS$.*

*Proof.* Without loss of generality, suppose that $0 \in ckS$. If $S = ckS$, then $S$ is convex and $S_x = S$ for each $x \in S$ and the result follows since $\varnothing \neq S(1) \subset S(k)$. Let $p \in S \backslash ckS$. Then there exists a point $y \in S$ such that $py \not\subset S$. Since $S$ is compact, $y$ can be chosen such that $S \cap \text{intv } py^\infty = \varnothing$. Since $py \not\subset S$, there exists a point $z \in (\text{intv } py) \backslash S$. If $y \in S(k)$, then $p \notin S_y$ implies $p \notin S_{S(k)}$. If $y \notin S(k)$ there exists a $k$-simplex $\Delta$ such that $y \in \text{intv } \Delta \subset S$. Consider the convex cone $C = \{\beta y + (\lambda - \beta + 1)\ z : \beta, \lambda \geqq 0\}$, which has vertex $z$ and is contained in the subspace $L'$ with basis $\{p, y\}$. Since $S \cap \text{intv } py^\infty = \varnothing$, $\Delta$ must intersect $L'$ in some line other than $L(p, y)$; thus, $S \cap \text{intv } C \neq \varnothing$. There exists a linear functional $f$ defined on $L_{k+1}$ such that $f(q) = 1$ for every $q \in L(p, y)$; clearly $0 \notin L(p, y)$ since $py \not\subset S$ and $0 \in ckS$. The continuous linear functional $f_1$, the restriction of $f$ to $L'$, attains a maximum on the compact set $C \cap S$ at some point $w \in \text{intv } C$. Let $H = [f : f(w)]$. Since $H \cap C \cap S$ is a compact subset of the 1-dimensional set $H \cap L'$, there exists a minimal closed line segment in intv $C$ which contains $H \cap C \cap S$. Each endpoint of this segment, which may be degenerate, must be a point in $S(k)$. Let $v$ be one of these endpoints. The points $p, y, z$ and $v$ are in $L'$. If $pv \subset S$, then the fact that $0 \in ckS$ implies that $z \in \text{conv } \{0, p, v\} \subset S$, a contradiction. Hence, $pv \not\subset S$ and $p \notin S_{S(k)}$. Therefore, $S \backslash ckS \subset S \backslash S_{S(k)}$, which gives the desired equality, since clearly $ckS \subset S_{S(k)}$.

It is not always sufficient to consider only the set of familiar extreme points $S(1)$ as a star-generating set for $ckS$. For example, in $E_3$ let $S$ be the union of three closed faces of a 3-simplex. In some cases, proper subsets of $S(k)$ exist which will star-generate $ckS$. However, characterizing such subsets may be very difficult, as indicated by the following example.

EXAMPLE 1. In the plane $E_2$ let $B_u$ be the upper closed unit half-disc, $B_r$ the right closed unit half-disc. Let

$$\begin{aligned} T_1 &= \text{conv}\,[\{-2e_1\} \cup (B_r + (2e_1 + e_2))]\ , \\ T_2 &= \text{conv}\,[\{-2e_2\} \cup (B_u + (2e_2 - e_1))]\ , \\ S &= T_1 \cup T_2 \cup (-T_1) \cup (-T_2)\ . \end{aligned}$$

Then any star-generating subset of $S(1)$ must contain four distinct sequences of carefully chosen extreme points.

THEOREM 2. *If $S$ is a compact star-shaped set in $L_n$, and* dim $(S) \geqq 3$, *then $S(n-1)$ is an uncountable set.*

*Proof.* Without loss of generality, it can be assumed that $0 \in ckS$. Since $\dim(S) \geqq 3$ there exists some point $x \in S, x \neq 0$. If $\beta x \in S(n-1)$ for every $\beta \in (0,1)$, then $S(n-1)$ is uncountable. Otherwise, consider some $w = \beta x$ such that $w \notin S(n-1)$. Then there exists an $(n-1)$-simplex $\Delta$ such that $w \in \operatorname{intv} \Delta \subset S$. Since $n - 1 \geqq 2$ there exists a nondegenerate line segment $zw \subset \Delta$ such that $zw \cap 0x = \{w\}$. There exists a linear functional $f$ on $L_n$ such that

$$f(w) = f(z) = 1 .$$

There exists a point $y \in [f:0]$ such that the set $\{y, z, w\}$ is linearly independent. For each $\lambda \in [0,1]$ consider the subspace $L(\lambda)$ of $L_n$ with basis $\{y, \lambda z + (1-\lambda)w\}$. Let $f_\lambda$ be the restriction of $f$ to $L(\lambda)$. The set $L(\lambda) \cap S$ is compact; hence; $f_\lambda$ attains a maximum on $L(\lambda) \cap S$ at some point $u, f_\lambda(u) \geqq 1$. Since $\dim(L(\lambda) \cap [f: f(u)]) = 1$ and

$$L(\lambda) \cap S \cap [f: f(u)]$$

is compact, there exists a minimal closed line segment in $L(\lambda)$ which contains $L(\lambda) \cap [f: f(u)] \cap S$. This line segment must have at least one endpoint, which must belong to $S(n-1)$. For each pair of distinct real numbers $\lambda, \mu$ in $[0,1]$, $L(\lambda) \cap L(\mu) \subset [f;0]$. There exists points $p_\lambda \in L(\lambda) \cap S(n-1)$, $p_\mu \in L(\mu) \cap S(n-1)$ such that $f(p_\lambda) \geqq 1, f(p_\mu) \geqq 1$, which implies that $p_\lambda \neq p_\mu$. Thus, the set $S(n-1)$ is uncountable.

THEOREM 3. *Let $S$ be a closed subset of a linear topological space $L$ and let $T$ be a subset of $S$ that star-generates $ckS$, which may be empty. If $M$ is a dense subset of $T$, then $M$ star-generates $ckS$.*

*Proof.* Since $M \subset T$ then clearly $S_T \subset S_M$. Suppose that $M$ is a proper subset of $T$ and $ckS$ is a proper subset of $S_M$. Then there exists a point $q \in S_M \backslash S_T$. But $S_T = S_M \cap S_{T \backslash M}$; thus $q \notin S_{T\backslash M}$. This implies that $q \notin S_x$ for some $x \in T\backslash M$. Since $q \in S_M$, $M \subset S_q$, which is closed. Hence, $x \in T \subset \operatorname{cl} M \subset S_q$, which implies that $xq \subset S$ and that $q \in S_x$, a contradiction. Therefore, $ckS = S_M$.

THEOREM 4. *If $S$ is a compact star-shaped subset of a normed linear space $L$, then any subset $T$ of $S$ which star-generates the convex kernel of $S$ contains a countable subset $M$ which also star-generates the convex kernel of $S$.*

*Proof.* The norm of $L$ induces a metric on $L$. The compact set $S$ can be considered as a compact metric space, where space is now used in the topological sense. The compact metric space is separable, which implies that $S$ is second countable [2]. Any nonempty subset $T$ of $S$ is a second countable topological space with the relative topology, which implies that $T$ is separable. There exists a countable subset $M$ of $T$ such that $T \subset \mathrm{cl}\, M$. Theorem 3 implies that $M$ star-generates $ckS$ and the theorem is proved.

COROLLARY. *Let $S$ be a compact star-shaped subset of $L_{k+1}$. Then there exists a countable subset of $S(k)$ which star-generates $ckS$.*

Klee [3] introduced the concept of relative extreme point.

DEFINITION 3. If $S$ and $T$ are subsets of a linear space $L$, then $x \in S$ is said to be extreme in $S$ relative to $T$ if there do not exist points $y \in S, z \in T$ such that $x \in \mathrm{intv}\ yz$.

If $S$ is a star-shaped set, exk $S$ will denote the points of $S$ which are extreme relative to $ckS$, and $E_S = (\mathrm{exk}\ S) \backslash ckS$.

THEOREM 5. *Let $S$ be a compact nonconvex star-shaped set in a locally convex space $L$. Then $C = S$, where*

$$C = \bigcup_{y \in E_S} \mathrm{conv}\,(ckS \cup \{y\}) \,.$$

*Proof.* Since $E_S \subset S$, conv $(ckS \cup \{y\}) \subset S$ for each $y \in E_S$. Thus, $C \subset S$. Consider $z \in ckS \cup \mathrm{exk}\ S$; since $E_S \neq \varnothing$, as shown below, $z \in C$. Let $K = ckS$. Suppose that $z \in S \backslash (ckS \cup \mathrm{exk}\, S)$ and without loss of generality, suppose that $z = 0$. Since $K$ is compact and convex, $K^*$ and $-K^*$ are closed convex cones with vertex 0, where $K^* = \{\lambda x\colon x \in K, \lambda \geqq 0\}$. Since $z \notin \mathrm{exk}\ S$ there exist points $x \in K$ and $w \in S$ such that $0 \in \mathrm{intv}\ xw$. Clearly $w \in -K^* \backslash \{0\}$, $S \cap (-K^* \backslash \{0\}) \neq \varnothing$ and $S \cap (-K^*)$ is compact. Let $u$ be an arbitrary point in $-K^* \backslash \{0\}$; since $L$ is locally convex and $K^*$ is closed and convex, there exists a closed hyperplane $H = [f\colon f(u)]$ such that $u \in H$ and $H \cap K^* = \varnothing$, where $f$ is a continuous linear functional. It can be assumed that $f(K^*) \leqq 0$, which implies that $f(u) > 0$. The functional $f$ then attains a maximum on $S \cap (-K^*)$ at some point $v \in S \cap (-K^*)$. Suppose that $v \notin \mathrm{exk}\ S$. There exist points $p \in K, q \in S$ such that $v \in \mathrm{intv}\ pq$. Since $v \in -K^*, v = -\lambda p', p' \in K, \lambda > 0$, and

$$v = \alpha p + (1 - \alpha) q \,, \qquad 0 < \alpha < 1 \,.$$

Therefore, $v = -\lambda p' = \alpha p + (1 - \alpha) q$ and $q = \tau q'$, where $\tau < 0$ and $q' \in K$. Thus, $q \in S \cap (-K^*)$. But it can be easily shown that

$f(q) > f(v)$, which contradicts the fact that $f(v) \geqq f(x)$ for each $x \in S \cap (-K^*)$. Hence, $v \in (\text{exk}\, S) \cap (-K^*)$ and $0 \in C$, which implies that $S \subset C$. This inclusion, along with the one given earlier, implies that $S = C$.

The following result shows that the set $E_S$ is minimal in its use in Theorem 5.

THEOREM 6. *Let $S$ be a compact nonconvex star-shaped set in a locally convex space $L$. If $T$ is a proper subset of $E_S$ then*

$$C(T) = \bigcup_{y \in T} \text{conv}\,(ckS \cup \{y\})$$

*is a proper subset of $S$.*

*Proof.* Consider any proper subset $T$ of $E_S$; there exists some point $x \in E_S \backslash T$. If $x \in C(T)$ there exists some $y \in T$ such that $x \in \text{conv}\,(ckS \cup \{y\})$. Hence, $x = \lambda z + (1 - \lambda)y$, where $\lambda \in [0, 1]$, $z \in ckS$. But $\lambda \in (0, 1)$ since $x \notin ckS \cup T$. This implies that $x \notin \text{exk}\ S$, a contradiction. Thus, $x \notin C(T)$, which must be a proper subset of $S$.

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