

UNIQUENESS OF THE GRAPH OF A RANK THREE GROUP

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Let G be a transitive permutation group on a set Ω . Let $\{a\}$ be an element of Ω and $\Delta(a)$ an orbit of the stabilizer G_a of $\{a\}$ in G . A graph on the points of Ω can be constructed by joining $\{a\}$ to $\Delta(a)$ and joining each image of $\{a\}$ by an element of G to the corresponding image of $\Delta(a)$. The permutation group G is said to be of rank 3 if G_a has three orbits including $\{a\}$. In this paper, conditions are discussed which ensure that the graph of a rank three permutation group is unique in the following sense. Suppose G is a rank three permutation group and G_a has orbits $\{a\}$, $\Delta(a)$, and $\Gamma(a)$. Is the graph uniquely determined once the permutation representation of G_a on $\Delta(a)$ and on $\Gamma(a)$ is known? Conditions involving orbit lengths of G_a acting on $\Delta(a)$ and on $\Gamma(a)$ are found which ensure the graph is uniquely determined. The conditions specify how the graph is to be drawn. They give, therefore, a prescription for constructing rank three groups. They shed no light however on the difficult problem of the transitivity of the group of automorphisms of the graph.

A few applications are given at the end to illustrate the method. They are not intended to be exhaustive. The same ideas can be used to show the nonexistence of certain rank three extensions.

2. Construction of the graph from the orbits. We assume G is a transitive rank three permutation group on a set Ω containing n points. The notation is the same as in [2]. For $\{a\}$ in Ω let the orbits of G_a be $\Delta(a)$ and $\Gamma(a)$. We assume $(\Delta(a))^g = \Delta(a^g)$ for all elements g in G . A graph is defined by joining $\{a\}$ to the points of $\Delta(a)$ for each $\{a\}$ in Ω . The group G is then a subgroup of the full automorphism group of the graph. We assume that the permutation representation is primitive and that G has even order. We set $|\Delta(a)| = k$, $|\Gamma(a)| = l$.

We now give a description of the graph derived from a consideration of the orbits $\{a\}$, $\Delta(a)$, and $\Gamma(a)$. Here $\{a\}$ is a specific point in Ω . By definition, $\{a\}$ is joined to each of the points in $\Delta(a)$. Let $\{b\}$ be a point in $\Delta(a)$ and $\{c\}$ a point in $\Gamma(a)$. We will describe the points which are joined to $\{b\}$ and $\{c\}$ in terms of unions of orbits of G_{ab} and G_{ac} . Here G_{ab} is the stabilizer of $\{a\}$ and $\{b\}$ and G_{ac} is the stabilizer of $\{a\}$ and $\{c\}$.

The parameters λ and μ are integers defined as follows, [2, §2].

If $b \in \Delta(a)$, $|\Delta(a) \cap \Delta(b)| = \lambda$ and if $c \in \Gamma(a)$, $|\Delta(a) \cap \Delta(c)| = \mu$. These values are independent of the particular element $\{b\}$ chosen in $\Delta(a)$ and of the particular element $\{c\}$ chosen in $\Gamma(a)$. As G has even order, the orbits are self paired by [4, 16.5]. This means that $\{b\}$ is joined to $\{a\}$ and that $\{c\}$ is not. As G is primitive $\mu \neq 0$ [2]. In particular then $\lambda \neq k - 1$ by [2, lemma 5].

The point $\{b\}$ is joined to the k points of $\Delta(b)$. There are λ of these points $\Delta(b) \cap \Delta(a)$ in $\Delta(a)$ and so $k - \lambda$ outside $\Delta(a)$. As $\{b\}$ is joined to $\{a\}$, $k - \lambda - 1$ of these points are in $\Gamma(a)$. These points are of course $\Gamma(a) \cap \Delta(b)$. The point $\{c\}$ is joined to the μ points $\Delta(c) \cap \Delta(a)$ in $\Delta(a)$ and to the $k - \mu$ points $\Delta(c) \cap \Gamma(a)$ in $\Gamma(a)$. We have the following table.

TABLE 1. Points connected to b .

	<i>location</i>	<i>points</i>	<i>number</i>
(i)	$\{a\}$	a	1
(ii)	$\Delta(a)$	$\Delta(a) \cap \Delta(b)$	λ
(iii)	$\Gamma(a)$	$\Gamma(a) \cap \Delta(b)$	$k - \lambda - 1$.

Points connected to c .

	<i>location</i>	<i>points</i>	<i>number</i>
(iv)	$\Delta(a)$	$\Delta(a) \cap \Delta(c)$	μ
(v)	$\Gamma(a)$	$\Gamma(a) \cap \Delta(c)$	$k - \mu$.

Once the subsets in the various categories are found for all points $b \in \Delta(a)$ and $c \in \Gamma(a)$ the graph can be completely drawn.

The points in category (iii) or (iv) determine all lines joining points in $\Delta(a)$ to points in $\Gamma(a)$. These lines can be determined once the sets in (iii) are known for each $b \in \Delta(a)$ or once the sets in (iv) are known for each c in $\Gamma(a)$. Counting these lines each way gives again [2, Lemma 5] $\mu l = k(k - \lambda - 1)$. The graph can be drawn once the sets in category (ii), (v) and either (iii) or (iv) are determined for every b in $\Delta(a)$ and c in $\Gamma(a)$.

It is clear that G_{ab} leaves invariant $\Delta(a)$, $\Gamma(a)$, and $\Delta(b)$. Therefore $\Delta(a) \cap \Delta(b)$ and $\Gamma(a) \cap \Delta(b)$ are invariant under G_{ab} . It follows that these two sets are unions of orbits of G_{ab} . Similarly, the sets $\Delta(a) \cap \Delta(c)$ and $\Gamma(a) \cap \Delta(c)$ are unions of orbits of G_{ac} . If the unions of orbits in the sets in category (ii), (v), and either of (iii) or (iv) can be chosen uniquely, the graph can be drawn in only one way.

We know that $\Delta(a) \cap \Delta(b)$ is invariant under G_{ab} . Also $\Delta(a) \cap \Gamma(b) = \Delta(a) - \{a\} - \Delta(b)$ is invariant under G_{ab} . Also $|\Delta(a) \cap \Gamma(b)| = k - \lambda - 1$. There are now two sets of length $k - \lambda - 1$, $\Delta(a) \cap \Gamma(b)$ and $\Gamma(a) \cap \Delta(b)$ invariant under G_{ab} . We will show they are permutation isomorphic.

This will provide a further restriction on the choice of either $\Delta(a) \cap \Delta(b)$ or $\Gamma(a) \cap \Delta(b)$. Similarly $\Delta(a) \cap \Gamma(c)$ and $\Gamma(a) \cap \Delta(c)$ are permutation isomorphic G_{ac} invariant subsets.

It follows from [4, 16.4] that there is an element τ in G which interchanges $\{a\}$ and $\{b\}$ and an element τ_1 in G which interchanges $\{a\}$ and $\{c\}$. Here the orbits are self paired as G has even order. Clearly, τ normalizes G_{ab} and τ_1 normalizes G_{ac} . We will show that τ maps $\Delta(a) \cap \Gamma(b)$ to $\Gamma(a) \cap \Delta(b)$ and that τ_1 maps $\Delta(a) \cap \Gamma(c)$ to $\Gamma(a) \cap \Delta(c)$. If $X \in \Delta(a) \cap \Gamma(b)$ and $g \in G_{ab}$, then $(X)^{g\tau} = (X^\tau)^{\tau^{-1}g\tau}$. This means that the map of $\Delta(a) \cap \Gamma(b)$ to $\Gamma(a) \cap \Delta(b)$ by τ together with the inner automorphism of G_{ab} to itself given by τ provides a permutation isomorphism between the action of G_{ab} on $\Delta(a) \cap \Gamma(b)$ and an action of G_{ab} on $\Gamma(a) \cap \Delta(b)$. Similarly, τ_1 provides a permutation isomorphism between the action of G_{ac} on $\Delta(a) \cap \Gamma(c)$ and an action of G_{ac} on $\Gamma(a) \cap \Delta(c)$.

We show now that τ maps $\Delta(a) \cap \Gamma(b)$ to $\Gamma(a) \cap \Delta(b)$ and τ_1 maps $\Delta(a) \cap \Gamma(c)$ to $\Gamma(a) \cap \Delta(c)$. Now $a^\tau = b$ and $b^\tau = a$ and so $(\Delta(a))^\tau = \Delta(a^\tau) = \Delta(b)$, $(\Gamma(b))^\tau = \Gamma(b^\tau) = \Gamma(a)$. This means $(\Delta(a) \cap \Gamma(b))^\tau = (\Delta(a))^\tau \cap (\Gamma(b))^\tau = \Delta(b) \cap \Gamma(a)$. Similarly as $a^{\tau_1} = c$ and $c^{\tau_1} = a$ we have $(\Delta(a) \cap \Gamma(c))^{\tau_1} = \Gamma(a) \cap \Delta(c)$. We now have proved the following theorem.

THEOREM 1. *Let G be a primitive rank three permutation group of even order. Suppose $\{a\}, \Delta(a), \Gamma(a)$ are the orbits of G_a and the action of G_a on these orbits is known. As well, assume that the parameters λ and μ are determined. Let $\{b\}$ be an element in $\Delta(a)$ and $\{c\}$ an element in $\Gamma(a)$. Then the following union of orbits of G_{ab} and G_{ac} must occur.*

TABLE 2. Union of Orbits of G_{ab} .

	location	points	number of points
(i)	$\Delta(a)$	$\Delta(a) \cap \Delta(b)$	λ
(ii)	$\Delta(a)$	$\Delta(a) \cap \Gamma(b)$	$k - \lambda - 1$
(iii)	$\Gamma(a)$	$\Gamma(a) \cap \Delta(b)$	$k - \lambda - 1$
(iv)	$\Gamma(a)$	$\Gamma(a) \cap \Gamma(b)$	$l - k + \lambda + 1$.

Union of Orbits of G_{ac} .

	location	points	number
(v)	$\Delta(a)$	$\Delta(a) \cap \Delta(c)$	μ
(vi)	$\Delta(a)$	$\Delta(a) \cap \Gamma(c)$	$k - \mu$
(vii)	$\Gamma(a)$	$\Gamma(a) \cap \Delta(c)$	$k - \mu$
(viii)	$\Gamma(a)$	$\Gamma(a) \cap \Gamma(c)$	$l - k + \mu$.

The graph can be drawn once the sets in (i), (vii) and either of (iii) or (v) are determined. The permutation representations of G_{ab} on $\Delta(a) \cap \Gamma(b)$ and $\Gamma(a) \cap \Delta(b)$ are permutation isomorphic as are the permutation representations of G_{ac} on $\Delta(a) \cap \Gamma(c)$ and on $\Gamma(a) \cap \Delta(c)$. When these unions of orbits can be chosen in only one way the graph can be given uniquely.

REMARK 1. Suppose G_{ac} acting on $\Delta(a)$ has a unique union of orbits of length μ and further the representation of G_a on $\Gamma(a)$ is primitive. Then this representation of G_a on $\Gamma(a)$ is the representation given by the distinct images by G_a of these unique μ points.

REMARK 2. The nonexistence of rank three extensions can quite often be determined by showing there is no union of orbits of the appropriate lengths. We do not however give any examples of this.

3. **Applications.** We apply Theorem 1 in several cases. We first consider the Higman-Sims group defined in [3]. Here $G_a \cong M_{22}$ has orbits of length 1, 22, and 77. There is only one permutation group of M_{22} on 22 points. It is doubly transitive. This means λ must be zero. Using [2, Lemma 5] we see $\mu = 6$. In particular, G_{ab} acting on $\Delta(a)$ has a unique orbit of length zero. From the character table of M_{22} it is clear that a transitive permutation representation of degree 77 must be of rank 3 with permutation character degrees 1, 21, and 55. There is therefore a unique orbit of G_{ac} acting on $\Gamma(a)$ of length 16. The restriction of a Sylow 2-group of M_{22} to 22 points has a unique sum of orbits of length six. In fact, the orbit lengths are 2, 4, and 16. A Sylow 2-group must be in the stabilizer of a point on 77-letters. It follows that the stabilizer of these 6 points in M_{22} must be the stabilizer of a point on 77-letters and so the permutation representation on these 77-letters is unique. The conditions of Theorem 1 are met and the uniqueness of the graph is shown. The uniqueness of the Higman-Sims group as a rank three extension of M_{22} is now immediate from the definition in [3].

We next consider a rank three extension of $\text{PSL}(2, 9)$ on orbits of length 1, 10, and 45 to give $\text{PSL}(3, 4)$. This permutation representation was first mentioned to the authors knowledge by Grant Mitchel in a Princeton University thesis in 1910. The group $\text{PSL}(2, 9) \cong A_6$ has exactly one permutation representation on 10 letters and exactly one on 45 letters. The 10 letters are the cosets of a Sylow 3-normalizer and the 45 letters are the cosets of a Sylow 2-subgroup. These are the projective points over $GF(9)$ and unordered pairs of distinct projective points. The representation on 10 letters is doubly transitive and so $\lambda = 0$. It is easy to check with [2, Lemma 5] that $\mu = 2$. The re-

striction of a Sylow 2-group to 10 letters has orbits of length 2, 4, and 4. There is a unique orbit of length 2. On each of the orbits of length 4 a Sylow 2-group is faithfully represented. The permutation representation on 45 points is the permutation representation on the images of the orbit of length 2. The orbit lengths of the stabilizer of a point are 1, 8, 8, 2, 2, 4, 4, 8, 8. We must associate two orbits of length 4 with this point. This can be done in only one way and so the graph is unique. The uniqueness of $\text{PSL}(3, 4)$ as a rank three extension of $\text{PSL}(2, 9)$ is now easy to show.

We next consider a rank three extension of $U_3(3)$ on orbits of lengths 1, 36, and 63 described in [1]. As shown in [1] the permutation representation on 36 letters is unique. The stabilizer of a point on 36 letters has orbits of length 1, 7, 7, 21. This means $\lambda = 0, 7, 14, 21$, or 28. The only one consistent with [2, Lemmas 6, 7] is $\lambda = 14$, $\mu = 12$. For simplicity we assume the permutation representation on 63 points is the one used in [1] on the cosets of the centralizer of an involution. The stabilizer of a point on 63 points has orbit lengths 12 and 24 on 36 points and so has a unique orbit of length 12. The stabilizer of a point in the 63 orbit has orbit lengths 1, 6, 24, and 32 and so has a unique orbit of length 24. The graph is therefore unique. The uniqueness of such a rank three extension can now be easily shown.

Finally, we consider a rank 3 extension on 416 points of the group HJ defined above. This extension is isomorphic to $G_2(4)$. It was first discovered by Suzuki. He then extended $G_2(4)$ to obtain a new simple group. At any rate, we consider a transitive extension of HJ to 416 points. The subgroups of small index in HJ have indices 100, 280, and 315. The permutation representation of $G_2(4)$ on 416 points is not doubly transitive as $415 \nmid |HJ|$. The representation restricted to any subgroup isomorphic to HJ must have orbits of length 1, 100 and 315. This means there is exactly one conjugate class of subgroups isomorphic to HJ , the stabilizer of a point. The permutation representation is of rank three.

The stabilizer of a point on 100 letters has orbit lengths 1, 36, and 63. Therefore $\lambda = 36$ or 63. It is easy to check using [2, Lemma 5] that $\lambda = 36$, $\mu = 20$. The stabilizer of a point on 100 letters has a unique orbit of length 36. For simplicity we assume the stabilizer of a point on 315 letters is the centralizer of an involution as described in [1, §18]. This group restricted to 100 points has orbit lengths 20 and 80 as shown in [1, §18]. It can be shown that there is a unique orbit of length 80 on the 315 points. This shows uniqueness of the graph.

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